Preface

Welcome to *College Algebra*, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

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OpenStax is a nonprofit based at Rice University, and it’s our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 20 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

About OpenStax Resources

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Errata

All OpenStax textbooks undergo a rigorous review process. However, like any professional-grade textbook, errors sometimes occur. Since our books are web based, we can make updates periodically when deemed pedagogically necessary. If you have a correction to suggest, submit it through the link on your book page on openstax.org. Subject matter experts review all errata suggestions. OpenStax is committed to remaining transparent about all updates, so you will also find a list of past errata changes on your book page on openstax.org.

Format

You can access this textbook for free in web view or PDF through openstax.org, and for a low cost in print.

About *College Algebra*

College Algebra provides a comprehensive exploration of algebraic principles and meets scope and sequence requirements for a typical introductory algebra course. The modular approach and richness of content ensure that the book meets the needs of a variety of courses. College Algebra offers a wealth of examples with detailed, conceptual explanations, building a strong foundation in the material before asking students to apply what they've learned.

Coverage and Scope

In determining the concepts, skills, and topics to cover, we engaged dozens of highly experienced instructors with a range of student audiences. The resulting scope and sequence proceeds logically while allowing for a significant amount of flexibility in instruction.

Chapters 1 and 2 provide both a review and foundation for study of functions that begins in Chapter 3. The authors recognize that while some institutions may find this material a prerequisite, other institutions have told us that they have a cohort that need the prerequisite skills built into the course.

Chapter 1: Prerequisites
Chapter 2: Equations and Inequalities

Chapters 3-6: The Algebraic Functions

Chapter 3: Functions
Chapter 4: Linear Functions
Chapter 5: Polynomial and Rational Functions
Chapter 6: Exponential and Logarithm Functions

Chapters 7-9: Further Study in College Algebra

Chapter 7: Systems of Equations and Inequalities
Chapter 8: Analytic Geometry
Chapter 9: Sequences, Probability, and Counting Theory

All chapters are broken down into multiple sections, the titles of which can be viewed in the Table of Contents.

Development Overview

College Algebra is the product of a collaborative effort by a group of dedicated authors, editors, and instructors whose collective passion for this project has resulted in a text that is remarkably unified in purpose and voice. Special thanks is due to our Lead Author, Jay Abramson of Arizona State University, who provided the overall vision for the book and oversaw the development of each and every chapter, drawing up the initial blueprint,

reading numerous drafts, and assimilating field reviews into actionable revision plans for our authors and editors.

The collective experience of our author team allowed us to pinpoint the subtopics, exceptions, and individual connections that give students the most trouble. The textbook is therefore replete with well-designed features and highlights, which help students overcome these barriers. As the students read and practice, they are coached in methods of thinking through problems and internalizing mathematical processes.

Accuracy of the Content

We understand that precision and accuracy are imperatives in mathematics, and undertook a dedicated accuracy program led by experienced faculty.

1. Each chapter’s manuscript underwent rounds of review and revision by a panel of active instructors.
2. Then, prior to publication, a separate team of experts checked all text, examples, and graphics for mathematical accuracy; multiple reviewers were assigned to each chapter to minimize the chances of any error escaping notice.
3. A third team of experts was responsible for the accuracy of the Answer Key, dutifully re-working every solution to eradicate any lingering errors. Finally, the editorial team conducted a multi-round post-production review to ensure the integrity of the content in its final form.

Pedagogical Foundations and Features

Learning Objectives

Each chapter is divided into multiple sections (or modules), each of which is organized around a set of learning objectives. The learning objectives are listed explicitly at the beginning of each section and are the focal point of every instructional element

Narrative text

Narrative text is used to introduce key concepts, terms, and definitions, to provide real-world context, and to provide transitions between topics and examples. Throughout this book, we rely on a few basic conventions to highlight the most important ideas:

- Key terms are boldfaced, typically when first introduced and/or when formally defined.
- Key concepts and definitions are called out in a blue box for easy reference.

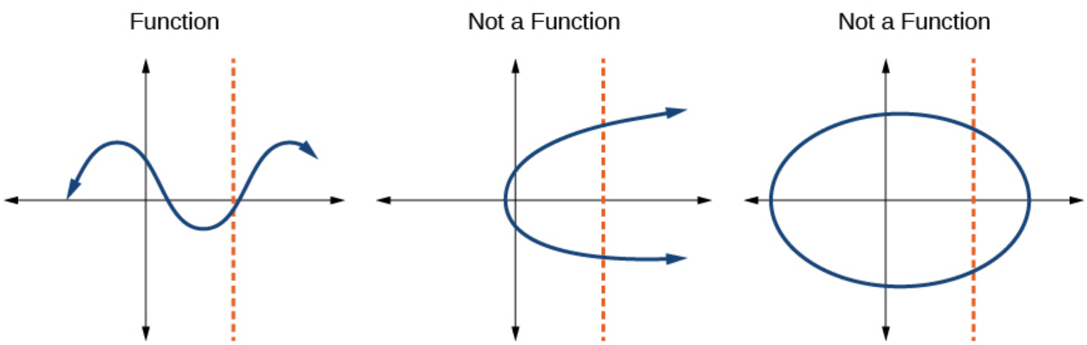
Examples

Each learning objective is supported by one or more worked examples that demonstrate the problem-solving approaches that students must master. The multiple Examples model different approaches to the same type of problem or introduce similar problems of increasing complexity.

All Examples follow a simple two- or three-part format. The question clearly lays out a mathematical problem to solve. The Solution walks through the steps, usually providing context for the approach — in other words, why the instructor is solving the problem in a specific manner. Finally, the Analysis (for select examples) reflects on the broader implications of the Solution just shown. Examples are followed by a “Try It” question, as explained below.

Figures

College Algebra contains many figures and illustrations, the vast majority of which are graphs and diagrams. Art throughout the text adheres to a clear, understated style, drawing the eye to the most important information in each figure while minimizing visual distractions. Color contrast is employed with discretion to distinguish between the different functions or features of a graph.



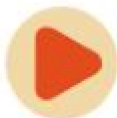
Supporting Features

Four unobtrusive but important features, each marked by a distinctive icon, contribute to and check understanding.

A **How To** is a list of steps necessary to solve a certain type of problem. A How To typically precedes an Example that proceeds to demonstrate the steps in action.

A **Try It** exercise immediately follows an Example or a set of related Examples, providing the student with an immediate opportunity to solve a similar problem. In the Web View version of the text, students can click an Answer link directly below the question to check their understanding. In the PDF, answers to the Try-It exercises are located in the Answer Key.

A **Q&A** may appear at any point in the narrative, but most often follows an Example. This feature pre-empts misconceptions by posing a commonly asked yes/no question, followed by a detailed answer and explanation.



The **Media** icon appears at the conclusion of each section, just prior to the Section Exercises. This icon marks a list of links to online video tutorials that reinforce the concepts and skills introduced in the section.

While we have selected tutorials that closely align to our learning objectives, we did not produce these tutorials, nor were they specifically produced or tailored to accompany *College Algebra*.

Section Exercises

Each section of every chapter concludes with a well-rounded set of exercises that can be assigned as homework or used selectively for guided practice. With over 4600 exercises across the 9 chapters, instructors should have plenty from which to choose.

Section Exercises are organized by question type, and generally appear in the following order:

Verbal questions assess conceptual understanding of key terms and concepts.
Algebraic problems require students to apply algebraic manipulations demonstrated in the section.
Graphical problems assess students' ability to interpret or produce a graph.
Numeric problems require the student to perform calculations or computations.

- **Technology** problems encourage exploration through use of a graphing utility, either to visualize or verify algebraic results or to solve problems via an alternative to the methods demonstrated in the section.
- **Extensions** pose problems more challenging than the Examples demonstrated in the section. They require students to synthesize multiple learning objectives or apply critical thinking to solve complex problems.
- **Real-World Applications** present realistic problem scenarios from fields such as physics, geology, biology, finance, and the social sciences.

Chapter Review Features

Each chapter concludes with a review of the most important takeaways, as well as additional practice problems that students can use to prepare for exams.

- **Key Terms** provides a formal definition for each bold-faced term in the chapter.
- **Key Equations** presents a compilation of formulas, theorems, and standard-form equations.
- **Key Concepts** summarizes the most important ideas introduced in each section, linking back to the relevant Example(s) in case students need to review.
- **Chapter Review Exercises** include 40-80 practice problems that recall the most important concepts from each section.
- **Practice Test** includes 25-50 problems assessing the most important learning objectives from the chapter. Note that the practice test is not organized by section, and may be more heavily weighted toward cumulative objectives as opposed to the foundational objectives covered in the opening sections.
- **Answer Key** includes the answers to all Try It exercises and every other exercise from the Section Exercises, Chapter Review Exercises, and Practice Test.

Additional Resources

Student and Instructor Resources

We've compiled additional resources for both students and instructors, including Getting Started Guides, an instructor solution manual, and PowerPoint slides. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

Partner Resources

OpenStax Partners are our allies in the mission to make high-quality learning materials affordable and accessible to students and instructors everywhere. Their tools integrate seamlessly with our OpenStax titles at a low cost. To access the partner resources for your text, visit your book page on openstax.org.

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Introduction to Prerequisites
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It’s a cold day in Antarctica. In fact, it’s always a cold day in Antarctica. Earth’s southernmost continent, Antarctica experiences the coldest, driest, and windiest conditions known. The coldest temperature ever recorded, over one hundred degrees below zero on the Celsius scale, was recorded by remote satellite. It is no surprise then, that no native human population can survive the harsh conditions. Only explorers and scientists brave the environment for any length of time.

Measuring and recording the characteristics of weather conditions in in Antarctica requires a use of different kinds of numbers. Calculating with them and using them to make predictions requires an understanding of relationships among numbers. In this chapter, we will review sets of numbers and properties of operations used to manipulate numbers. This understanding will serve as prerequisite knowledge throughout our study of algebra and trigonometry.

1.1 Real Numbers: Algebra Essentials
In this section students will:

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

It is often said that mathematics is the language of science. If this is true, then an essential part of the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattlemen, and tradesmen used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a “base state” while counting and used various symbols to represent this null condition. However, it was not until about the fifth century A.D. in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century A.D., negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the **natural numbers**: 1, 2, 3, 4, 5, and so on. We describe them in set notation as $\{1, 2, 3, ...\}$ where the ellipsis (…) indicates that the numbers continue to infinity. The natural numbers are, of course, also called the *counting numbers*. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, ...\}$.

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers: $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.
Equation:

$$\overset{\text{negative integers}}{\ldots, -3, -2, -1,} \quad \overset{\text{zero}}{0,} \quad \overset{\text{positive integers}}{1, 2, 3, \cdots}$$

The set of **rational numbers** is written as $\left\{\frac{m}{n} \;\; m \text{ and } n \text{ are integers and } n \neq 0\right\}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal: $\frac{15}{8} = 1.875$,or
2. a repeating decimal: $\frac{4}{11} = 0.36363636\ldots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

Example:
Exercise:

Problem:
Writing Integers as Rational Numbers

Write each of the following as a rational number.

a. 7
b. 0
c. –8

Solution:

Write a fraction with the integer in the numerator and 1 in the denominator.

a. $7 = \frac{7}{1}$
b. $0 = \frac{0}{1}$
c. $-8 = -\frac{8}{1}$

Note:
Exercise:

Problem: Write each of the following as a rational number.

a. 11
b. 3
c. –4

Solution:

a. $\frac{11}{1}$
b. $\frac{3}{1}$
c. $-\frac{4}{1}$

Example:
Exercise:

Problem:
Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $-\frac{5}{7}$
b. $\frac{15}{5}$
c. $\frac{13}{25}$

Solution:

Write each fraction as a decimal by dividing the numerator by the denominator.

a. $-\frac{5}{7} = -0.\overline{714285}$,a repeating decimal
b. $\frac{15}{5} = 3$ (or 3.0), a terminating decimal
c. $\frac{13}{25} = 0.52$, a terminating decimal

Note:
Exercise:

Problem: Write each of the following rational numbers as either a terminating or repeating decimal.

a. $\frac{68}{17}$
b. $\frac{8}{13}$
c. $-\frac{17}{20}$

Solution:

a. 4 (or 4.0), terminating;
b. 0.615384,repeating;
c. –0.85, terminating

Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even $\frac{3}{2}$,but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3, but still not a rational number. Such numbers are said to be *irrational* because they cannot be written as fractions. These numbers make up the set of **irrational numbers**. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.
Equation:

$$\{h | h \text{ is not a rational number}\}$$

Example:

Exercise:

Problem:
Differentiating Rational and Irrational Numbers

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\sqrt{25}$
b. $\frac{33}{9}$
c. $\sqrt{11}$
d. $\frac{17}{34}$
e. $0.3033033303333\ldots$

Solution:

a. $\sqrt{25}$: This can be simplified as $\sqrt{25} = 5$. Therefore, $\sqrt{25}$ is rational.
b. $\frac{33}{9}$: Because it is a fraction, $\frac{33}{9}$ is a rational number. Next, simplify and divide.
Equation:

$$\frac{33}{9} = \frac{\overset{11}{\cancel{33}}}{\underset{3}{\cancel{9}}} = \frac{11}{3} = 3.6$$

So, $\frac{33}{9}$ is rational and a repeating decimal.
c. $\sqrt{11}$: This cannot be simplified any further. Therefore, $\sqrt{11}$ is an irrational number.
d. $\frac{17}{34}$: Because it is a fraction, $\frac{17}{34}$ is a rational number. Simplify and divide.
Equation:

$$\frac{17}{34} = \frac{\overset{1}{\cancel{17}}}{\underset{2}{\cancel{34}}} = \frac{1}{2} = 0.5$$

So, $\frac{17}{34}$ is rational and a terminating decimal.
e. $0.3033033303333\ldots$ is not a terminating decimal. Also note that there is no repeating pattern because the group of 3s increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

Note:
Exercise:

Problem:

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\frac{7}{77}$
b. $\sqrt{81}$

c. $4.27027002700027\ldots$
d. $\frac{91}{13}$
e. $\sqrt{39}$

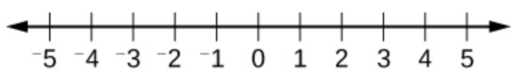
Solution:

a. rational and repeating;
b. rational and terminating;
c. irrational;
d. rational and repeating;
e. irrational

Real Numbers

Given any number *n*, we know that *n* is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of **real numbers**. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or –). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line.The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the **real number line** as shown in [link]**.**



The real number line

Example:
Exercise:

Problem:
Classifying Real Numbers

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $-\frac{10}{3}$
b. $\sqrt{5}$
c. $-\sqrt{289}$
d. -6π
e. $0.615384615384\ldots$

Solution:

a. $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.
b. $\sqrt{5}$ is positive and irrational. It lies to the right of 0.
c. $-\sqrt{289} = -\sqrt{17^2} = -17$ is negative and rational. It lies to the left of 0.
d. -6π is negative and irrational. It lies to the left of 0.
e. $0.615384615384\ldots$ is a repeating decimal so it is rational and positive. It lies to the right of 0.

Note:
Exercise:

Problem:

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

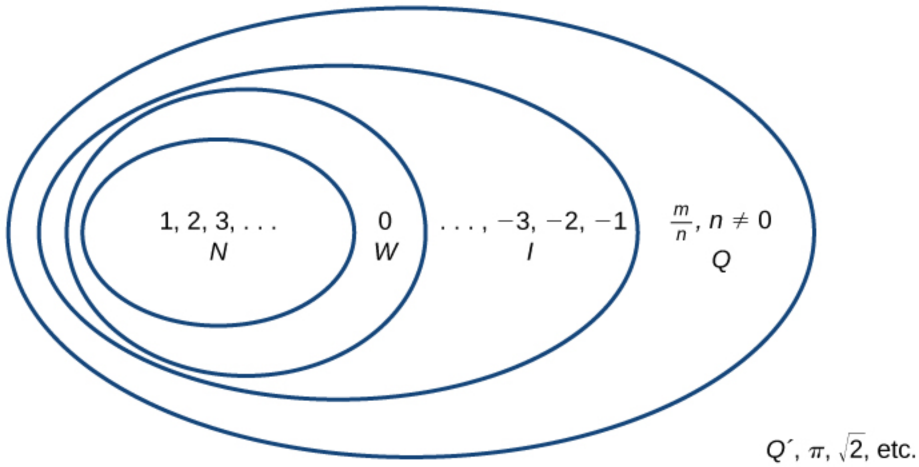
a. $\sqrt{73}$
b. $-11.411411411\ldots$
c. $\frac{47}{19}$
d. $-\frac{\sqrt{5}}{2}$
e. 6.210735

Solution:

a. positive, irrational; right
b. negative, rational; left
c. positive, rational; right
d. negative, irrational; left
e. positive, rational; right

Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram, such as [link].



Sets of numbers
N: the set of natural numbers
W: the set of whole numbers
I: the set of integers
Q: the set of rational numbers
Q´: the set of irrational numbers

Note:
Sets of Numbers
The set of **natural numbers** includes the numbers used for counting: $\{1, 2, 3, ...\}$.
The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, ...\}$.
The set of **integers** adds the negative natural numbers to the set of whole numbers:
$\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.
The set of **rational numbers** includes fractions written as $\left\{\frac{m}{n} \quad m \text{ and } n \text{ are integers and } n \neq 0\right\}$.
The set of **irrational numbers** is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h|h \text{ is not a rational number}\}$.

Example:
Exercise:

Problem:
Differentiating the Sets of Numbers

Classify each number as being a natural number (*N*), whole number (*W*), integer (*I*), rational number (*Q*), and/or irrational number (*Q′*).

a. $\sqrt{36}$
b. $\frac{8}{3}$
c. $\sqrt{73}$
d. -6
e. $3.2121121112\ldots$

Solution:

	N	***W***	***I***	***Q***	***Q′***
a. $\sqrt{36}=6$	X	X	X	X	
b. $\frac{8}{3}=2.\overline{6}$				X	
c. $\sqrt{73}$					X
d. –6			X	X	
e. 3.2121121112...					X

Note:
Exercise:

Problem:

Classify each number as being a natural number (*N*), whole number (*W*), integer (*I*), rational number (*Q*), and/or irrational number (*Q′*).

a. $-\frac{35}{7}$
b. 0
c. $\sqrt{169}$
d. $\sqrt{24}$
e. $4.763763763\ldots$

Solution:

	N	***W***	***I***	***Q***	***Q′***
a. $-\frac{35}{7}$			X	X	
b. 0		X	X	X	
c. $\sqrt{169}$	X	X	X	X	
d. $\sqrt{24}$					X
e. 4.763763763...				X	

Performing Calculations Using the Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4 \cdot 4 = 16$. We can raise any number to any power. In general, the **exponential notation** a^n means that the number or variable a is used as a factor n times.
Equation:

$$a^n = \overbrace{a \cdot a \cdot a \cdot \ldots \cdot a}^{n \text{ factors}}$$

In this notation, a^n is read as the nth power of a, where a is called the **base** and n is called the **exponent.** A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24 + 6 \cdot \frac{2}{3} - 4^2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the **order of operations**. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let’s take a look at the expression provided.
Equation:

$$24 + 6 \cdot \frac{2}{3} - 4^2$$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2, so simplify 4^2 as 16.
Equation:

$$\begin{gathered} 24 + 6 \cdot \tfrac{2}{3} - 4^2 \\ 24 + 6 \cdot \tfrac{2}{3} - 16 \end{gathered}$$

Next, perform multiplication or division, left to right.
Equation:

$$\begin{gathered} 24 + 6 \cdot \tfrac{2}{3} - 16 \\ 24 + 4 - 16 \end{gathered}$$

Lastly, perform addition or subtraction, left to right.
Equation:

$$\begin{gathered} 24 + 4 - 16 \\ 28 - 16 \\ 12 \end{gathered}$$

Therefore, $24 + 6 \cdot \frac{2}{3} - 4^2 = 12$.

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

Note:
Order of Operations
Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym **PEMDAS**:
P(arentheses)
E(xponents)
M(ultiplication) and **D**(ivision)
A(ddition) and **S**(ubtraction)

Note:
Given a mathematical expression, simplify it using the order of operations.

Simplify any expressions within grouping symbols.
Simplify any expressions containing exponents or radicals.
Perform any multiplication and division in order, from left to right.
Perform any addition and subtraction in order, from left to right.

Example:
Exercise:

Problem:
Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

a. $(3 \cdot 2)^2 - 4\,(6 + 2)$
b. $\frac{5^2 - 4}{7} - \sqrt{11 - 2}$
c. $6 - |5 - 8| + 3\,(4 - 1)$
d. $\frac{14 - 3 \cdot 2}{2 \cdot 5 - 3^2}$
e. $7\,(5 \cdot 3) - 2\left[(6 - 3) - 4^2\right] + 1$

Solution:

a.
$$\begin{array}{rcll} (3 \cdot 2)^2 - 4(6 + 2) & = & (6)^2 - 4(8) & \text{Simplify parentheses} \\ & = & 36 - 4(8) & \text{Simplify exponent} \\ & = & 36 - 32 & \text{Simplify multiplication} \\ & = & 4 & \text{Simplify subtraction} \end{array}$$

b.

$$\begin{array}{rcll} \frac{5^2-4}{7} - \sqrt{11-2} & = & \frac{5^2-4}{7} - \sqrt{9} & \text{Simplify grouping symbols (radical)} \\ & = & \frac{5^2-4}{7} - 3 & \text{Simplify radical} \\ & = & \frac{25-4}{7} - 3 & \text{Simplify exponent} \\ & = & \frac{21}{7} - 3 & \text{Simplify subtraction in numerator} \\ & = & 3 - 3 & \text{Simplify division} \\ & = & 0 & \text{Simplify subtraction} \end{array}$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

c.

$$\begin{array}{rcll} 6 - |5-8| + 3(4-1) & = & 6 - |-3| + 3(3) & \text{Simplify inside grouping symbols} \\ & = & 6 - 3 + 3(3) & \text{Simplify absolute value} \\ & = & 6 - 3 + 9 & \text{Simplify multiplication} \\ & = & 3 + 9 & \text{Simplify subtraction} \\ & = & 12 & \text{Simplify addition} \end{array}$$

d.

$$\begin{array}{rcll} \frac{14-3\cdot2}{2\cdot5-3^2} & = & \frac{14-3\cdot2}{2\cdot5-9} & \text{Simplify exponent} \\ & = & \frac{14-6}{10-9} & \text{Simplify products} \\ & = & \frac{8}{1} & \text{Simplify differences} \\ & = & 8 & \text{Simplify quotient} \end{array}$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

e.

$$\begin{array}{rcll} 7\left(5\cdot3\right) - 2\left[\left(6-3\right) - 4^2\right] + 1 & = & 7\left(15\right) - 2\left[\left(3\right) - 4^2\right] + 1 & \text{Simplify inside parentheses} \\ & = & 7\left(15\right) - 2\left(3-16\right) + 1 & \text{Simplify exponent} \\ & = & 7\left(15\right) - 2\left(-13\right) + 1 & \text{Subtract} \\ & = & 105 + 26 + 1 & \text{Multiply} \\ & = & 132 & \text{Add} \end{array}$$

Note:
Exercise:

Problem: Use the order of operations to evaluate each of the following expressions.

a. $\sqrt{5^2-4^2} + 7(5-4)^2$
b. $1 + \frac{7\cdot5-8\cdot4}{9-6}$
c. $|1.8-4.3| + 0.4\sqrt{15+10}$
d. $\frac{1}{2}\left[5\cdot3^2-7^2\right] + \frac{1}{3}\cdot9^2$
e. $\left[(3-8)^2-4\right] - (3-8)$

Solution:

a. 10
b. 2
c. 4.5
d. 25
e. 26

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Commutative Properties

The **commutative property of addition** states that numbers may be added in any order without affecting the sum.
Equation:

$$a+b=b+a$$

We can better see this relationship when using real numbers.
Equation:

$$(-2)+7=5 \quad \text{and} \quad 7+(-2)=5$$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.
Equation:

$$a\cdot b=b\cdot a$$

Again, consider an example with real numbers.
Equation:

$$(-11)\cdot(-4)=44 \quad \text{and} \quad (-4)\cdot(-11)=44$$

It is important to note that neither subtraction nor division is commutative. For example, $17-5$ is not the same as $5-17$. Similarly, $20\div 5\neq 5\div 20$.

Associative Properties

The **associative property of multiplication** tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.
Equation:

$$a\,(bc)=(ab)c$$

Consider this example.
Equation:

$$(3 \cdot 4) \cdot 5 = 60 \quad \text{and} \quad 3 \cdot (4 \cdot 5) = 60$$

The **associative property of addition** tells us that numbers may be grouped differently without affecting the sum.
Equation:

$$a + (b + c) = (a + b) + c$$

This property can be especially helpful when dealing with negative integers. Consider this example.
Equation:

$$[15 + (-9)] + 23 = 29 \quad \text{and} \quad 15 + [(-9) + 23] = 29$$

Are subtraction and division associative? Review these examples.
Equation:

$$\begin{array}{rcl} 8 - (3 - 15) & \stackrel{?}{=} & (8 - 3) - 15 \\ 8 - (-12) & = & 5 - 15 \\ 20 & \neq & 20 - 10 \end{array} \qquad \begin{array}{rcl} 64 \div (8 \div 4) & \stackrel{?}{=} & (64 \div 8) \div 4 \\ 64 \div 2 & \stackrel{?}{=} & 8 \div 4 \\ 32 & \neq & 2 \end{array}$$

As we can see, neither subtraction nor division is associative.

Distributive Property

The **distributive property** states that the product of a factor times a sum is the sum of the factor times each term in the sum.
Equation:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

$$\begin{aligned} 4 \cdot [12 + (-7)] &= 4 \cdot 12 + 4 \cdot (-7) \\ &= 48 + (-28) \\ &= 20 \end{aligned}$$

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by –7, and adding the products.

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.
Equation:

$$\begin{array}{rcl} 6 + (3 \cdot 5) & \stackrel{?}{=} & (6 + 3) \cdot (6 + 5) \\ 6 + (15) & \stackrel{?}{=} & (9) \cdot (11) \\ 21 & \neq & 99 \end{array}$$

A special case of the distributive property occurs when a sum of terms is subtracted.
Equation:

$$a - b = a + (-b)$$

For example, consider the difference $12 - (5 + 3)$. We can rewrite the difference of the two terms 12 and $(5 + 3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5 + 3)$,we add the opposite.
Equation:

$$12 + (-1) \cdot (5 + 3)$$

Now, distribute -1 and simplify the result.
Equation:

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-1) \cdot (5 + 3) \\ &= 12 + [(-1) \cdot 5 + (-1) \cdot 3] \\ &= 12 + (-8) \\ &= 4 \end{aligned}$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.
Equation:

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-5 - 3) \\ &= 12 + (-8) \\ &= 4 \end{aligned}$$

Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.
Equation:

$$a + 0 = a$$

The **identity property of multiplication** states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.
Equation:

$$a \cdot 1 = a$$

For example, we have $(-6) + 0 = -6$ and $23 \cdot 1 = 23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

Inverse Properties

The **inverse property of addition** states that, for every real number *a*, there is a unique number, called the additive inverse (or opposite), denoted−*a*, that, when added to the original number, results in the additive identity, 0.
Equation:

$$a+(-a)=0$$

For example, if $a=-8$,the additive inverse is 8, since $(-8)+8=0$.

The **inverse property of multiplication** holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number *a*, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$,that, when multiplied by the original number, results in the multiplicative identity, 1.
Equation:

$$a\cdot\frac{1}{a}=1$$

For example, if $a=-\frac{2}{3}$,the reciprocal, denoted $\frac{1}{a}$,is $-\frac{3}{2}$ because
Equation:

$$a\cdot\frac{1}{a}=\left(-\frac{2}{3}\right)\cdot\left(-\frac{3}{2}\right)=1$$

Note:
Properties of Real Numbers
The following properties hold for real numbers *a*, *b*, and *c*.

	Addition	Multiplication
Commutative Property	$a+b=b+a$	$a\cdot b=b\cdot a$
Associative Property	$a+(b+c)=(a+b)+c$	$a\,(bc)=(ab)c$
Distributive Property	$a\cdot(b+c)=a\cdot b+a\cdot c$	
Identity Property	There exists a unique real number called the additive identity, 0, such that, for any real number *a* **Equation:** $a+0=a$	There exists a unique real number called the multiplicative identity, 1, such that, for any real number *a* **Equation:** $a\cdot 1=a$
Inverse	Every real number a has an additive	Every nonzero real number *a* has a

Property	inverse, or opposite, denoted –*a*, such that **Equation:** $$a + (-a) = 0$$	multiplicative inverse, or reciprocal, denoted $\frac{1}{a}$,such that **Equation:** $$a \cdot \left(\frac{1}{a}\right) = 1$$

Example:
Exercise:

Problem:
Using Properties of Real Numbers

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $3 \cdot 6 + 3 \cdot 4$
b. $(5 + 8) + (-8)$
c. $6 - (15 + 9)$
d. $\frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right)$
e. $100 \cdot [0.75 + (-2.38)]$

Solution:

a.
$$\begin{array}{lllll} 3 \cdot 6 + 3 \cdot 4 & = & 3 \cdot (6 + 4) & & \text{Distributive property} \\ & = & 3 \cdot 10 & & \text{Simplify} \\ & = & 30 & & \text{Simplify} \end{array}$$
b.
$$\begin{array}{lllll} (5 + 8) + (-8) & = & 5 + [8 + (-8)] & & \text{Associative property of addition} \\ & = & 5 + 0 & & \text{Inverse property of addition} \\ & = & 5 & & \text{Identity property of addition} \end{array}$$
c.
$$\begin{array}{lllll} 6 - (15 + 9) & = & 6 + [(-15) + (-9)] & & \text{Distributive property} \\ & = & 6 + (-24) & & \text{Simplify} \\ & = & -18 & & \text{Simplify} \end{array}$$
d.
$$\begin{array}{lllll} \frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right) & = & \frac{4}{7} \cdot \left(\frac{7}{4} \cdot \frac{2}{3}\right) & & \text{Commutative property of multiplication} \\ & = & \left(\frac{4}{7} \cdot \frac{7}{4}\right) \cdot \frac{2}{3} & & \text{Associative property of multiplication} \\ & = & 1 \cdot \frac{2}{3} & & \text{Inverse property of multiplication} \\ & = & \frac{2}{3} & & \text{Identity property of multiplication} \end{array}$$
e.
$$\begin{array}{lllll} 100 \cdot [0.75 + (-2.38)] & = & 100 \cdot 0.75 + 100 \cdot (-2.38) & & \text{Distributive property} \\ & = & 75 + (-238) & & \text{Simplify} \\ & = & -163 & & \text{Simplify} \end{array}$$

Note:
Exercise:

Problem:

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $\left(-\frac{23}{5}\right) \cdot \left[11 \cdot \left(-\frac{5}{23}\right)\right]$
b. $5 \cdot (6.2 + 0.4)$
c. $18 - (7 - 15)$
d. $\frac{17}{18} + \left[\frac{4}{9} + \left(-\frac{17}{18}\right)\right]$
e. $6 \cdot (-3) + 6 \cdot 3$

Solution:

a. 11, commutative property of multiplication, associative property of multiplication, inverse property of multiplication, identity property of multiplication;
b. 33, distributive property;
c. 26, distributive property;
d. $\frac{4}{9}$, commutative property of addition, associative property of addition, inverse property of addition, identity property of addition;
e. 0, distributive property, inverse property of addition, identity property of addition

Evaluating Algebraic Expressions

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x + 5$, $\frac{4}{3}\pi r^3$,or $\sqrt{2m^3n^2}$. In the expression $x + 5$,5 is called a **constant** because it does not vary and *x* is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.
Equation:

$$\begin{array}{ccccccc}(-3)^5 & = & (-3) \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-3) & & x^5 & = & x \cdot x \cdot x \cdot x \cdot x \\ (2 \cdot 7)^3 & = & (2 \cdot 7) \cdot (2 \cdot 7) \cdot (2 \cdot 7) & & (yz)^3 & = & (yz) \cdot (yz) \cdot (yz)\end{array}$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

Example:

Exercise:

Problem:
Describing Algebraic Expressions

List the constants and variables for each algebraic expression.

a. $x + 5$
b. $\frac{4}{3}\pi r^3$
c. $\sqrt{2m^3n^2}$

Solution:

	Constants	Variables
a. $x + 5$	5	x
b. $\frac{4}{3}\pi r^3$	$\frac{4}{3}, \pi$	r
c. $\sqrt{2m^3n^2}$	2	m, n

Note:
Exercise:

Problem: List the constants and variables for each algebraic expression.

a. $2\pi r\left(r+h\right)$
b. $2(L + W)$
c. $4y^3 + y$

Solution:

	Constants	Variables
a. $2\pi r\left(r+h\right)$	$2, \pi$	r, h
b. 2(L + W)	2	L, W
c. $4y^3 + y$	4	y

Example:
Exercise:

Problem:
Evaluating an Algebraic Expression at Different Values

Evaluate the expression $2x - 7$ for each value for *x*.

a. $x = 0$
b. $x = 1$
c. $x = \frac{1}{2}$
d. $x = -4$

Solution:

a. Substitute 0 for x.
Equation:

$$\begin{aligned} 2x - 7 &= 2(0) - 7 \\ &= 0 - 7 \\ &= -7 \end{aligned}$$

b. Substitute 1 for x.
Equation:

$$\begin{aligned} 2x - 7 &= 2(1) - 7 \\ &= 2 - 7 \\ &= -5 \end{aligned}$$

c. Substitute $\frac{1}{2}$ for x.
Equation:

$$\begin{aligned} 2x - 7 &= 2\left(\frac{1}{2}\right) - 7 \\ &= 1 - 7 \\ &= -6 \end{aligned}$$

d. Substitute -4 for x.
Equation:

$$\begin{aligned} 2x - 7 &= 2(-4) - 7 \\ &= -8 - 7 \\ &= -15 \end{aligned}$$

Note:
Exercise:

Problem: Evaluate the expression $11 - 3y$ for each value for *y*.

a. $y=2$
b. $y=0$
c. $y=\frac{2}{3}$
d. $y=-5$

Solution:

a. 5;
b. 11;
c. 9;
d. 26

Example:
Exercise:

Problem:
Evaluating Algebraic Expressions

Evaluate each expression for the given values.

a. $x+5$ for $x=-5$
b. $\frac{t}{2t-1}$ for $t=10$
c. $\frac{4}{3}\pi r^3$ for $r=5$
d. $a+ab+b$ for $a=11, b=-8$
e. $\sqrt{2m^3n^2}$ for $m=2, n=3$

Solution:

a. Substitute -5 for x.
Equation:

$$\begin{array}{ccc}x+5 & = & (-5)+5 \\ & = & 0\end{array}$$

b. Substitute 10 for t.
Equation:

$$\begin{array}{ccc}\frac{t}{2t-1} & = & \frac{(10)}{2(10)-1} \\ & = & \frac{10}{20-1} \\ & = & \frac{10}{19}\end{array}$$

c. Substitute 5 for r.
Equation:

$$\begin{array}{ccc}\frac{4}{3}\pi r^3 & = & \frac{4}{3}\pi(5)^3 \\ & = & \frac{4}{3}\pi(125) \\ & = & \frac{500}{3}\pi\end{array}$$

d. Substitute 11 for a and –8 for b.

Equation:

$$\begin{aligned} a+ab+b &= (11)+(11)(-8)+(-8) \\ &= 11-88-8 \\ &= -85 \end{aligned}$$

e. Substitute 2 for m and 3 for n.
Equation:

$$\begin{aligned} \sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2} \\ &= \sqrt{2(8)(9)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

Note:
Exercise:

Problem: Evaluate each expression for the given values.

a. $\frac{y+3}{y-3}$ for $y=5$
b. $7-2t$ for $t=-2$
c. $\frac{1}{3}\pi r^2$ for $r=11$
d. $\left(p^2q\right)^3$ for $p=-2, q=3$
e. $4\left(m-n\right)-5\left(n-m\right)$ for $m=\frac{2}{3}, n=\frac{1}{3}$

Solution:

a. 4;
b. 11;
c. $\frac{121}{3}\pi$;
d. 1728;
e. 3

Formulas

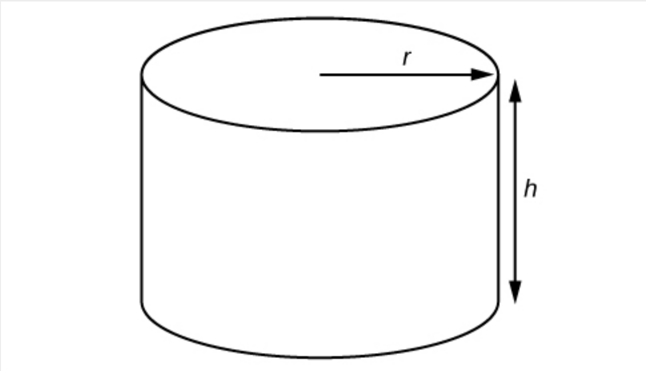
An **equation** is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2x+1=7$ has the unique solution of 3 because when we substitute 3 for x in the equation, we obtain the true statement $2\left(3\right)+1=7$.

A **formula** is an equation expressing a relationship between constant and variable quantities. Very often, the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other quantities. One of the most common examples is the formula for finding the area A of a circle in terms of the radius r of the circle: $A=\pi r^2$. For any value of r, the area A can be found by evaluating the expression πr^2.

Example:
Exercise:

Problem:
Using a Formula

A right circular cylinder with radius r and height h has the surface area S (in square units) given by the formula $S=2\pi r\left(r+h\right).$ See [link]. Find the surface area of a cylinder with radius 6 in. and height 9 in. Leave the answer in terms of π.



Right circular cylinder

Solution:

Evaluate the expression $2\pi r\left(r+h\right)$ for $r=6$ and $h=9$.
Equation:

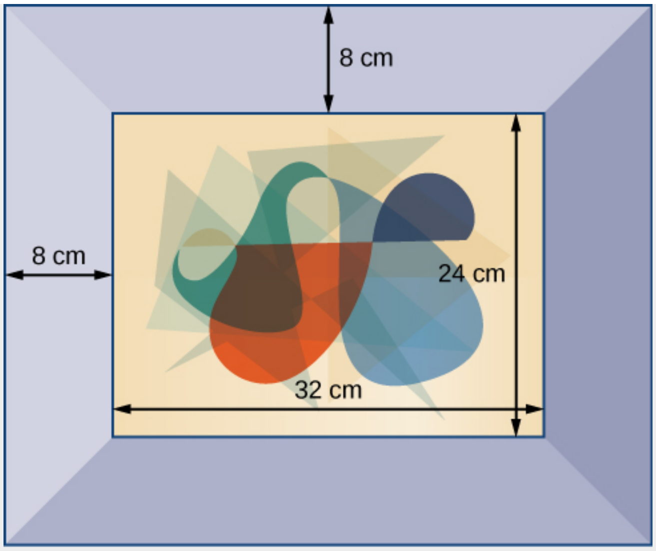
$$\begin{array}{ccc}S& =& 2\pi r(r+h)\\ & =& 2\pi (6)[(6)+(9)]\\ & =& 2\pi (6)(15)\\ & =& 180\pi \end{array}$$

The surface area is 180π square inches.

Note:
Exercise:

Problem:

A photograph with length L and width W is placed in a matte of width 8 centimeters (cm). The area of the matte (in square centimeters, or cm^2) is found to be $A=\left(L+16\right)\left(W+16\right)-L\cdot W.$ See [link]. Find the area of a matte for a photograph with length 32 cm and width 24 cm.



Solution:

1,152 cm^2

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

Example:
Exercise:

Problem:
Simplifying Algebraic Expressions

Simplify each algebraic expression.

a. $3x - 2y + x - 3y - 7$
b. $2r - 5\left(3 - r\right) + 4$
c. $\left(4t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right)$
d. $2mn - 5m + 3mn + n$

Solution:

a.
$$\begin{array}{lllll} 3x - 2y + x - 3y - 7 & = & 3x + x - 2y - 3y - 7 & & \text{Commutative property of additi} \\ & = & 4x - 5y - 7 & & \text{Simplify} \end{array}$$

b.
$$\begin{array}{lllll} 2r - 5(3 - r) + 4 & = & 2r - 15 + 5r + 4 & & \text{Distributive property} \\ & = & 2r + 5r - 15 + 4 & & \text{Commutative property of additi} \\ & = & 7r - 11 & & \text{Simplify} \end{array}$$

c.

$$\begin{array}{rcll} 4t-4\left(t-\frac{5}{4}s\right)-\left(\frac{2}{3}t+2s\right) & = & 4t-\frac{5}{4}s-\frac{2}{3}t-2s & \text{Distributive property} \\ & = & 4t-\frac{2}{3}t-\frac{5}{4}s-2s & \text{Commutative property of additio} \\ & = & \frac{10}{3}t-\frac{13}{4}s & \text{Simplify} \end{array}$$

d.

$$\begin{array}{rcll} mn-5m+3mn+n & = & 2mn+3mn-5m+n & \text{Commutative property of addition} \\ & = & 5mn-5m+n & \text{Simplify} \end{array}$$

Note:
Exercise:

Problem: Simplify each algebraic expression.

a. $\frac{2}{3}y-2\left(\frac{4}{3}y+z\right)$
b. $\frac{5}{t}-2-\frac{3}{t}+1$
c. $4p\left(q-1\right)+q\left(1-p\right)$
d. $9r-\left(s+2r\right)+\left(6-s\right)$

Solution:

a. $-2y–2z$ or $-2\left(y+z\right)$;
b. $\frac{2}{t}–1$;
c. $3pq–4p+q$;
d. $7r–2s+6$

Example:
Exercise:

Problem:
Simplifying a Formula

A rectangle with length L and width W has a perimeter P given by $P=L+W+L+W.$ Simplify this expression.

Solution:
Equation:

$$\begin{array}{rcll} P & = & L+W+L+W & \\ P & = & L+L+W+W & \text{Commutative property of addition} \\ P & = & 2L+2W & \text{Simplify} \\ P & = & 2\left(L+W\right) & \text{Distributive property} \end{array}$$

Note:

Exercise:

Problem:

If the amount P is deposited into an account paying simple interest r for time t,the total value of the deposit A is given by $A=P+Prt.$ Simplify the expression. (This formula will be explored in more detail later in the course.)

Solution:

$A=P\left(1+rt\right)$

Note:
Access these online resources for additional instruction and practice with real numbers.

- Simplify an Expression
- Evaluate an Expression1
- Evaluate an Expression2

Key Concepts

- Rational numbers may be written as fractions or terminating or repeating decimals. See [link] and [link].
- Determine whether a number is rational or irrational by writing it as a decimal. See [link].
- The rational numbers and irrational numbers make up the set of real numbers. See [link]. A number can be classified as natural, whole, integer, rational, or irrational. See [link].
- The order of operations is used to evaluate expressions. See [link].
- The real numbers under the operations of addition and multiplication obey basic rules, known as the properties of real numbers. These are the commutative properties, the associative properties, the distributive property, the identity properties, and the inverse properties. See [link].
- Algebraic expressions are composed of constants and variables that are combined using addition, subtraction, multiplication, and division. See [link]. They take on a numerical value when evaluated by replacing variables with constants. See [link], [link], and [link]
- Formulas are equations in which one quantity is represented in terms of other quantities. They may be simplified or evaluated as any mathematical expression. See [link] and [link].

Verbal

Exercise:

Problem:

Is $\sqrt{2}$ an example of a rational terminating, rational repeating, or irrational number? Tell why it fits that category.

Solution:

irrational number. The square root of two does not terminate, and it does not repeat a pattern. It cannot be written as a quotient of two integers, so it is irrational.

Exercise:

Problem:

What is the order of operations? What acronym is used to describe the order of operations, and what does it stand for?

Exercise:

Problem:

What do the Associative Properties allow us to do when following the order of operations? Explain your answer.

Solution:

The Associative Properties state that the sum or product of multiple numbers can be grouped differently without affecting the result. This is because the same operation is performed (either addition or subtraction), so the terms can be re-ordered.

Numeric

For the following exercises, simplify the given expression.

Exercise:

Problem: $10+2\ \times\ (5-3)$

Exercise:

Problem: $6\div 2-\left(81\div 3^{2}\right)$

Solution:

-6

Exercise:

Problem: $18+(6-8)^{3}$

Exercise:

Problem: $-2\ \times\ \left[16\div (8-4)^{2}\right]^{2}$

Solution:

-2

Exercise:

Problem: $4-6+2\ \times\ 7$

Exercise:

Problem: $3\,(5-8)$

Solution:

-9

Exercise:

Problem:$4+6-10\div 2$

Exercise:

Problem:$12\div (36\div 9)+6$

Solution:

9

Exercise:

Problem:$(4+5)^2\div 3$

Exercise:

Problem:$3-12\ \times\ 2+19$

Solution:

-2

Exercise:

Problem:$2+8\ \times\ 7\div 4$

Exercise:

Problem:$5+(6+4)-11$

Solution:

4

Exercise:

Problem:$9-18\div 3^2$

Exercise:

Problem:$14\ \times\ 3\div 7-6$

Solution:

0

Exercise:

Problem:$9-(3+11)\ \times\ 2$

Exercise:

Problem:$6+2\ \times\ 2-1$

Solution:

9

Exercise:

Problem: $64 \div (8 + 4 \times 2)$

Exercise:

Problem: $9 + 4\left(2^2\right)$

Solution:

25

Exercise:

Problem: $(12 \div 3 \times 3)^2$

Exercise:

Problem: $25 \div 5^2 - 7$

Solution:

-6

Exercise:

Problem: $(15 - 7) \times (3 - 7)$

Exercise:

Problem: $2 \times 4 - 9(-1)$

Solution:

17

Exercise:

Problem: $4^2 - 25 \times \frac{1}{5}$

Exercise:

Problem: $12(3 - 1) \div 6$

Solution:

4

Algebraic

For the following exercises, solve for the variable.

Exercise:

Problem: $8\left(x+3\right)=64$

Exercise:

Problem: $4y+8=2y$

Solution:

-4

Exercise:

Problem: $(11a+3)-18a=-4$

Exercise:

Problem: $4z-2z\left(1+4\right)=36$

Solution:

-6

Exercise:

Problem: $4y(7-2)^{2}=-200$

Exercise:

Problem: $-(2x)^{2}+1=-3$

Solution:

± 1

Exercise:

Problem: $8\left(2+4\right)-15b=b$

Exercise:

Problem: $2\left(11c-4\right)=36$

Solution:

2

Exercise:

Problem: $4\left(3-1\right)x=4$

Exercise:

Problem: $\frac{1}{4}\left(8w-4^{2}\right)=0$

Solution:

2

For the following exercises, simplify the expression.

Exercise:

Problem: $4x + x\left(13 - 7\right)$

Exercise:

Problem: $2y - {\left(4\right)}^{2}y - 11$

Solution:

$-14y - 11$

Exercise:

Problem: $\frac{a}{2^3}\left(64\right) - 12a \div 6$

Exercise:

Problem: $8b - 4b\left(3\right) + 1$

Solution:

$-4b + 1$

Exercise:

Problem: $5l \div 3l \ \times \ \left(9 - 6\right)$

Exercise:

Problem: $7z - 3 + z \ \times \ 6^2$

Solution:

$43z - 3$

Exercise:

Problem: $4 \ \times \ 3 + 18x \div 9 - 12$

Exercise:

Problem: $9\left(y + 8\right) - 27$

Solution:

$9y + 45$

Exercise:

Problem: $\left(\frac{9}{6}t - 4\right)2$

Exercise:

Problem: $6 + 12b - 3 \ \times \ 6b$

Solution:

$-6b+6$

Exercise:

Problem: $18y-2\left(1+7y\right)$

Exercise:

Problem: $\left(\frac{4}{9}\right)^{2}\ \times\ 27x$

Solution:

$\frac{16x}{3}$

Exercise:

Problem: $8\left(3-m\right)+1\left(-8\right)$

Exercise:

Problem: $9x+4x\left(2+3\right)-4\left(2x+3x\right)$

Solution:

$9x$

Exercise:

Problem: $5^{2}-4\left(3x\right)$

Real-World Applications

For the following exercises, consider this scenario: Fred earns \$40 mowing lawns. He spends \$10 on mp3s, puts half of what is left in a savings account, and gets another \$5 for washing his neighbor's car.

Exercise:

Problem:

Write the expression that represents the number of dollars Fred keeps (and does not put in his savings account). Remember the order of operations.

Solution:

$\frac{1}{2}\left(40-10\right)+5$

Exercise:

Problem: How much money does Fred keep?

For the following exercises, solve the given problem.

Exercise:

Problem:

According to the U.S. Mint, the diameter of a quarter is 0.955 inches. The circumference of the quarter would be the diameter multiplied by π. Is the circumference of a quarter a whole number, a rational number, or an irrational number?

Solution:

irrational number

Exercise:

Problem:

Jessica and her roommate, Adriana, have decided to share a change jar for joint expenses. Jessica put her loose change in the jar first, and then Adriana put her change in the jar. We know that it does not matter in which order the change was added to the jar. What property of addition describes this fact?

For the following exercises, consider this scenario: There is a mound of g pounds of gravel in a quarry. Throughout the day, 400 pounds of gravel is added to the mound. Two orders of 600 pounds are sold and the gravel is removed from the mound. At the end of the day, the mound has 1,200 pounds of gravel.

Exercise:

Problem: Write the equation that describes the situation.

Solution:

$g+400-2\left(600\right)=1200$

Exercise:

Problem: Solve for g.

For the following exercise, solve the given problem.

Exercise:

Problem:

Ramon runs the marketing department at his company. His department gets a budget every year, and every year, he must spend the entire budget without going over. If he spends less than the budget, then his department gets a smaller budget the following year. At the beginning of this year, Ramon got $2.5 million for the annual marketing budget. He must spend the budget such that $2{,}500{,}000-x=0$. What property of addition tells us what the value of x must be?

Solution:

inverse property of addition

Technology

For the following exercises, use a graphing calculator to solve for x. Round the answers to the nearest hundredth.

Exercise:

Problem: $0.5{\left(12.3\right)}^{2}-48x=\frac{3}{5}$

Exercise:

Problem: $(0.25 - 0.75)^2 x - 7.2 = 9.9$

Solution:

68.4

Extensions

Exercise:

Problem: If a whole number is not a natural number, what must the number be?

Exercise:

Problem:

Determine whether the statement is true or false: The multiplicative inverse of a rational number is also rational.

Solution:

true

Exercise:

Problem:

Determine whether the statement is true or false: The product of a rational and irrational number is always irrational.

Exercise:

Problem: Determine whether the simplified expression is rational or irrational: $\sqrt{-18 - 4(5)(-1)}$.

Solution:

irrational

Exercise:

Problem: Determine whether the simplified expression is rational or irrational: $\sqrt{-16 + 4(5) + 5}$.

Exercise:

Problem: The division of two whole numbers will always result in what type of number?

Solution:

rational

Exercise:

Problem: What property of real numbers would simplify the following expression: $4 + 7(x - 1)$?

Glossary

algebraic expression
: constants and variables combined using addition, subtraction, multiplication, and division

associative property of addition
: the sum of three numbers may be grouped differently without affecting the result; in symbols, $a+(b+c)=(a+b)+c$

associative property of multiplication
: the product of three numbers may be grouped differently without affecting the result; in symbols, $a\cdot(b\cdot c)=(a\cdot b)\cdot c$

base
: in exponential notation, the expression that is being multiplied

commutative property of addition
: two numbers may be added in either order without affecting the result; in symbols, $a+b=b+a$

commutative property of multiplication
: two numbers may be multiplied in any order without affecting the result; in symbols, $a\cdot b=b\cdot a$

constant
: a quantity that does not change value

distributive property
: the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols, $a\cdot(b+c)=a\cdot b+a\cdot c$

equation
: a mathematical statement indicating that two expressions are equal

exponent
: in exponential notation, the raised number or variable that indicates how many times the base is being multiplied

exponential notation
: a shorthand method of writing products of the same factor

formula
: an equation expressing a relationship between constant and variable quantities

identity property of addition
: there is a unique number, called the additive identity, 0, which, when added to a number, results in the original number; in symbols, $a+0=a$

identity property of multiplication
: there is a unique number, called the multiplicative identity, 1, which, when multiplied by a number, results in the original number; in symbols, $a\cdot 1=a$

integers
: the set consisting of the natural numbers, their opposites, and 0: $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

inverse property of addition
: for every real number a, there is a unique number, called the additive inverse (or opposite), denoted $-a$, which, when added to the original number, results in the additive identity, 0; in symbols, $a+(-a)=0$

inverse property of multiplication

for every non-zero real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, which, when multiplied by the original number, results in the multiplicative identity, 1; in symbols, $a \cdot \frac{1}{a} = 1$

irrational numbers
: the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers

natural numbers
: the set of counting numbers: $\{1, 2, 3, \ldots\}$

order of operations
: a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations

rational numbers
: the set of all numbers of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$. Any rational number may be written as a fraction or a terminating or repeating decimal.

real number line
: a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left.

real numbers
: the sets of rational numbers and irrational numbers taken together

variable
: a quantity that may change value

whole numbers
: the set consisting of 0 plus the natural numbers: $\{0, 1, 2, 3, \ldots\}$

1.2 Exponents and Scientific Notation
In this section students will:

- Use the product rule of exponents.
- Use the quotient rule of exponents.
- Use the power rule of exponents.
- Use the zero exponent rule of exponents.
- Use the negative rule of exponents.
- Find the power of a product and a quotient.
- Simplify exponential expressions.
- Use scientific notation.

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is 2,048 pixels by 1,536 pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour (3,600-second) digital film is then an extremely large number.

Using a calculator, we enter $2{,}048 \times 1{,}536 \times 48 \times 24 \times 3{,}600$ and press ENTER. The calculator displays 1.304596316E13. What does this mean? The "E13" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \cdot x^4$. Both terms have the same base, *x*, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.
Equation:

$$\begin{aligned} x^3 \cdot x^4 &= \overbrace{x \cdot x \cdot x}^{\text{3 factors}} \cdot \overbrace{x \cdot x \cdot x \cdot x}^{\text{4 factors}} \\ &= \overbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}^{\text{7 factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \cdot x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the *product rule of exponents.*
Equation:

$$a^m \cdot a^n = a^{m+n}$$

Now consider an example with real numbers.
Equation:

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product $8 \cdot 16$ equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

Note:
The Product Rule of Exponents
For any real number a and natural numbers m and n,the product rule of exponents states that
Equation:

$$a^m \cdot a^n = a^{m+n}$$

Example:
Exercise:

Problem:
Using the Product Rule

Write each of the following products with a single base. Do not simplify further.

a. $t^5 \cdot t^3$
b. $(-3)^5 \cdot (-3)$
c. $x^2 \cdot x^5 \cdot x^3$

Solution:

Use the product rule to simplify each expression.

a. $t^5 \cdot t^3 = t^{5+3} = t^8$
b. $(-3)^5 \cdot (-3) = (-3)^5 \cdot (-3)^1 = (-3)^{5+1} = (-3)^6$
c. $x^2 \cdot x^5 \cdot x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.
Equation:

$$x^2 \cdot x^5 \cdot x^3 = \left(x^2 \cdot x^5\right) \cdot x^3 = \left(x^{2+5}\right) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.
Equation:

$$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$$

Note:
Exercise:

Problem: Write each of the following products with a single base. Do not simplify further.

a. $k^6 \cdot k^9$
b. $\left(\frac{2}{y}\right)^4 \cdot \left(\frac{2}{y}\right)$
c. $t^3 \cdot t^6 \cdot t^5$

Solution:

a. k^{15}
b. $\left(\frac{2}{y}\right)^5$
c. t^{14}

Using the Quotient Rule of Exponents

The *quotient rule of exponents* allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$,where $m > n.$ Consider the example $\frac{y^9}{y^5}.$ Perform the division by canceling common factors.
Equation:

$$\begin{aligned} \frac{y^9}{y^5} &= \frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\ &= \frac{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot y \cdot y \cdot y \cdot y}{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y}} \\ &= \frac{y \cdot y \cdot y \cdot y}{1} \\ &= y^4 \end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.
Equation:

$$\frac{a^m}{a^n} = a^{m-n}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.
Equation:

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m > n.$ Otherwise, the difference $m - n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

Note:
The Quotient Rule of Exponents
For any real number a and natural numbers m and n,such that $m > n$,the quotient rule of exponents states that
Equation:

$$\frac{a^m}{a^n} = a^{m-n}$$

Example:
Exercise:

Problem:
Using the Quotient Rule

Write each of the following products with a single base. Do not simplify further.

a. $\frac{(-2)^{14}}{(-2)^9}$
b. $\frac{t^{23}}{t^{15}}$
c. $\frac{\left(z\sqrt{2}\right)^5}{z\sqrt{2}}$

Solution:

Use the quotient rule to simplify each expression.

a. $\frac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$
b. $\frac{t^{23}}{t^{15}} = t^{23-15} = t^8$
c. $\frac{\left(z\sqrt{2}\right)^5}{z\sqrt{2}} = \left(z\sqrt{2}\right)^{5-1} = \left(z\sqrt{2}\right)^4$

Note:
Exercise:

Problem: Write each of the following products with a single base. Do not simplify further.

a. $\frac{s^{75}}{s^{68}}$
b. $\frac{(-3)^6}{-3}$
c. $\frac{\left(ef^2\right)^5}{\left(ef^2\right)^3}$

Solution:

a. s^7
b. $(-3)^5$
c. $\left(ef^2\right)^2$

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the *power rule of exponents*. Consider the expression $\left(x^2\right)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

Equation:

$$\begin{aligned}\left(x^2\right)^3 &= \overbrace{(x^2)\cdot(x^2)\cdot(x^2)}^{3\text{ factors}}\\ &= \overbrace{\left(\overbrace{x\cdot x}^{2\text{ factors}}\right)\cdot\left(\overbrace{x\cdot x}^{2\text{ factors}}\right)\cdot\left(\overbrace{x\cdot x}^{2\text{ factors}}\right)}^{3\text{ factors}}\\ &= x\cdot x\cdot x\cdot x\cdot x\cdot x\\ &= x^6\end{aligned}$$

The exponent of the answer is the product of the exponents: $\left(x^2\right)^3 = x^{2\cdot 3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.
Equation:

$$(a^m)^n = a^{m\cdot n}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.
Equation:

$$\begin{array}{ccccccccc} & & \text{Product Rule} & & & & \text{Power Rule} & & \\ 5^3\cdot 5^4 & = & 5^{3+4} & = & 5^7 & \text{but} & \left(5^3\right)^4 & = & 5^{3\cdot 4} & = & 5^{12}\\ x^5\cdot x^2 & = & x^{5+2} & = & x^7 & \text{but} & \left(x^5\right)^2 & = & x^{5\cdot 2} & = & x^{10}\\ (3a)^7\cdot(3a)^{10} & = & (3a)^{7+10} & = & (3a)^{17} & \text{but} & \left((3a)^7\right)^{10} & = & (3a)^{7\cdot 10} & = & (3a)^{70}\end{array}$$

Note:
The Power Rule of Exponents
For any real number a and positive integers m and n,the power rule of exponents states that
Equation:

$$(a^m)^n = a^{m\cdot n}$$

Example:
Exercise:

Problem:
Using the Power Rule

Write each of the following products with a single base. Do not simplify further.

a. $\left(x^2\right)^7$
b. $\left((2t)^5\right)^3$
c. $\left((-3)^5\right)^{11}$

Solution:

Use the power rule to simplify each expression.

a. $\left(x^2\right)^7 = x^{2\cdot 7} = x^{14}$
b. $\left((2t)^5\right)^3 = (2t)^{5\cdot 3} = (2t)^{15}$
c. $\left((-3)^5\right)^{11} = (-3)^{5\cdot 11} = (-3)^{55}$

Note:
Exercise:

Problem: Write each of the following products with a single base. Do not simplify further.

a. $\left((3y)^8\right)^3$
b. $\left(t^5\right)^7$
c. $\left((-g)^4\right)^4$

Solution:

a. $(3y)^{24}$
b. t^{35}
c. $(-g)^{16}$

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m > n$ so that the difference $m - n$ would never be zero or negative. What would happen if $m = n$? In this case, we would use the *zero exponent rule of exponents* to simplify the expression to 1. To see how this is done, let us begin with an example.
Equation:

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

If we were to simplify the original expression using the quotient rule, we would have
Equation:

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number.

Equation:

$$a^0 = 1$$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for now, we will consider the value to be undefined.

Note:
The Zero Exponent Rule of Exponents
For any nonzero real number a, the zero exponent rule of exponents states that
Equation:

$$a^0 = 1$$

Example:
Exercise:

Problem:
Using the Zero Exponent Rule

Simplify each expression using the zero exponent rule of exponents.

a. $\frac{c^3}{c^3}$
b. $\frac{-3x^5}{x^5}$
c. $\frac{\left(j^2k\right)^4}{\left(j^2k\right)\cdot\left(j^2k\right)^3}$
d. $\frac{5\left(rs^2\right)^2}{\left(rs^2\right)^2}$

Solution:

Use the zero exponent and other rules to simplify each expression.

a.
$$\begin{array}{ccc} \frac{c^3}{c^3} & = & c^{3-3} \\ & = & c^0 \\ & = & 1 \end{array}$$

b.
$$\begin{array}{ccc} \frac{-3x^5}{x^5} & = & -3\cdot\frac{x^5}{x^5} \\ & = & -3\cdot x^{5-5} \\ & = & -3\cdot x^0 \\ & = & -3\cdot 1 \\ & = & -3 \end{array}$$

c.

$$\begin{aligned}\frac{\left(j^2k\right)^4}{\left(j^2k\right)\cdot\left(j^2k\right)^3} &= \frac{\left(j^2k\right)^4}{\left(j^2k\right)^{1+3}} && \text{Use the product rule in the denominator.}\\ &= \frac{\left(j^2k\right)^4}{\left(j^2k\right)^4} && \text{Simplify.}\\ &= \left(j^2k\right)^{4-4} && \text{Use the quotient rule.}\\ &= \left(j^2k\right)^0 && \text{Simplify.}\\ &= 1\end{aligned}$$

d.

$$\begin{aligned}\frac{5\left(rs^2\right)^2}{\left(rs^2\right)^2} &= 5\left(rs^2\right)^{2-2} && \text{Use the quotient rule.}\\ &= 5\left(rs^2\right)^0 && \text{Simplify.}\\ &= 5\cdot 1 && \text{Use the zero exponent rule.}\\ &= 5 && \text{Simplify.}\end{aligned}$$

Note:
Exercise:

Problem: Simplify each expression using the zero exponent rule of exponents.

a. $\frac{t^7}{t^7}$
b. $\frac{\left(de^2\right)^{11}}{2\left(de^2\right)^{11}}$
c. $\frac{w^4\cdot w^2}{w^6}$
d. $\frac{t^3\cdot t^4}{t^2\cdot t^5}$

Solution:

a. 1
b. $\frac{1}{2}$
c. 1
d. 1

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m > n$ in the quotient rule even further. For example, can we simplify $\frac{h^3}{h^5}$? When $m < n$—that is, where the difference $m - n$ is negative—we can use the *negative rule of exponents* to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{h^3}{h^5}$.
Equation:

$$\begin{aligned}\frac{h^3}{h^5} &= \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} \\ &= \frac{\cancel{h} \cdot \cancel{h} \cdot \cancel{h}}{\cancel{h} \cdot \cancel{h} \cdot \cancel{h} \cdot h \cdot h} \\ &= \frac{1}{h \cdot h} \\ &= \frac{1}{h^2}\end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

Equation:

$$\begin{aligned}\frac{h^3}{h^5} &= h^{3-5} \\ &= h^{-2}\end{aligned}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}.$ This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

Equation:

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression a^n is defined when n is a natural number, 0, or the negative of a natural number. That means that a^n is defined for any integer $n.$ Also, the product and quotient rules and all of the rules we will look at soon hold for any integer $n.$

Note:

The Negative Rule of Exponents

For any nonzero real number a and natural number n,the negative rule of exponents states that

Equation:

$$a^{-n} = \frac{1}{a^n}$$

Example:

Exercise:

Problem:

Using the Negative Exponent Rule

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{\theta^3}{\theta^{10}}$

b. $\frac{z^2 \cdot z}{z^4}$

c. $\frac{\left(-5t^3\right)^4}{\left(-5t^3\right)^8}$

Solution:

a. $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$
b. $\frac{z^2 \cdot z}{z^4} = \frac{z^{2+1}}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$
c. $\frac{\left(-5t^3\right)^4}{\left(-5t^3\right)^8} = \left(-5t^3\right)^{4-8} = \left(-5t^3\right)^{-4} = \frac{1}{\left(-5t^3\right)^4}$

Note:
Exercise:

Problem:

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{(-3t)^2}{(-3t)^8}$
b. $\frac{f^{47}}{f^{49} \cdot f}$
c. $\frac{2k^4}{5k^7}$

Solution:

a. $\frac{1}{(-3t)^6}$
b. $\frac{1}{f^3}$
c. $\frac{2}{5k^3}$

Example:
Exercise:

Problem:
Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $b^2 \cdot b^{-8}$
b. $(-x)^5 \cdot (-x)^{-5}$
c. $\frac{-7z}{(-7z)^5}$

Solution:

a. $b^2 \cdot b^{-8} = b^{2-8} = b^{-6} = \frac{1}{b^6}$
b. $(-x)^5 \cdot (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$
c. $\frac{-7z}{(-7z)^5} = \frac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \frac{1}{(-7z)^4}$

Note:
Exercise:

Problem:

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $t^{-11} \cdot t^{6}$
b. $\frac{25^{12}}{25^{13}}$

Solution:

a. $t^{-5} = \frac{1}{t^5}$
b. $\frac{1}{25}$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the *power of a product rule of exponents,* which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.
Equation:

$$\begin{aligned} (pq)^3 &= \overbrace{(pq) \cdot (pq) \cdot (pq)}^{3 \text{ factors}} \\ &= p \cdot q \cdot p \cdot q \cdot p \cdot q \\ &= \overbrace{p \cdot p \cdot p}^{3 \text{ factors}} \cdot \overbrace{q \cdot q \cdot q}^{3 \text{ factors}} \\ &= p^3 \cdot q^3 \end{aligned}$$

In other words, $(pq)^3 = p^3 \cdot q^3$.

Note:
The Power of a Product Rule of Exponents
For any real numbers a and b and any integer n, the power of a product rule of exponents states that
Equation:

$$(ab)^n = a^n b^n$$

Example:
Exercise:

Problem:
Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $\left(ab^2\right)^3$
b. $(2t)^{15}$
c. $\left(-2w^3\right)^3$
d. $\frac{1}{(-7z)^4}$
e. $\left(e^{-2}f^2\right)^7$

Solution:

Use the product and quotient rules and the new definitions to simplify each expression.

a. $\left(ab^2\right)^3 = (a)^3 \cdot \left(b^2\right)^3 = a^{1\cdot3} \cdot b^{2\cdot3} = a^3b^6$
b. $(2t)^{15} = (2)^{15} \cdot (t)^{15} = 2^{15}t^{15} = 32,768t^{15}$
c. $\left(-2w^3\right)^3 = (-2)^3 \cdot \left(w^3\right)^3 = -8 \cdot w^{3\cdot3} = -8w^9$
d. $\frac{1}{(-7z)^4} = \frac{1}{(-7)^4\cdot(z)^4} = \frac{1}{2,401z^4}$
e. $\left(e^{-2}f^2\right)^7 = \left(e^{-2}\right)^7 \cdot \left(f^2\right)^7 = e^{-2\cdot7} \cdot f^{2\cdot7} = e^{-14}f^{14} = \frac{f^{14}}{e^{14}}$

Note:
Exercise:

Problem:

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $\left(g^2h^3\right)^5$
b. $(5t)^3$
c. $\left(-3y^5\right)^3$
d. $\frac{1}{\left(a^6b^7\right)^3}$
e. $\left(r^3s^{-2}\right)^4$

Solution:

a. $g^{10}h^{15}$
b. $125t^3$
c. $-27y^{15}$
d. $\frac{1}{a^{18}b^{21}}$
e. $\frac{r^{12}}{s^8}$

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the *power of a quotient rule*, which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let's look at the following example.

Equation:

$$\left(e^{-2}f^{2}\right)^{7}=\frac{f^{14}}{e^{14}}$$

Let's rewrite the original problem differently and look at the result.

Equation:

$$\begin{aligned}\left(e^{-2}f^{2}\right)^{7} &= \left(\frac{f^{2}}{e^{2}}\right)^{7}\\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

Equation:

$$\begin{aligned}\left(e^{-2}f^{2}\right)^{7} &= \left(\frac{f^{2}}{e^{2}}\right)^{7}\\ &= \frac{\left(f^{2}\right)^{7}}{\left(e^{2}\right)^{7}}\\ &= \frac{f^{2\cdot 7}}{e^{2\cdot 7}}\\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

Note:
The Power of a Quotient Rule of Exponents
For any real numbers a and b and any integer n,the power of a quotient rule of exponents states that
Equation:

$$\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}$$

Example:
Exercise:

Problem:
Using the Power of a Quotient Rule

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{4}{z^{11}}\right)^{3}$
b. $\left(\frac{p}{q^{3}}\right)^{6}$
c. $\left(\frac{-1}{t^{2}}\right)^{27}$
d. $\left(j^{3}k^{-2}\right)^{4}$

e. $\left(m^{-2}n^{-2}\right)^3$

Solution:

a. $\left(\frac{4}{z^{11}}\right)^3 = \frac{(4)^3}{(z^{11})^3} = \frac{64}{z^{11\cdot 3}} = \frac{64}{z^{33}}$
b. $\left(\frac{p}{q^3}\right)^6 = \frac{(p)^6}{(q^3)^6} = \frac{p^{1\cdot 6}}{q^{3\cdot 6}} = \frac{p^6}{q^{18}}$
c. $\left(\frac{-1}{t^2}\right)^{27} = \frac{(-1)^{27}}{(t^2)^{27}} = \frac{-1}{t^{2\cdot 27}} = \frac{-1}{t^{54}} = -\frac{1}{t^{54}}$
d. $\left(j^3k^{-2}\right)^4 = \left(\frac{j^3}{k^2}\right)^4 = \frac{(j^3)^4}{(k^2)^4} = \frac{j^{3\cdot 4}}{k^{2\cdot 4}} = \frac{j^{12}}{k^8}$
e. $\left(m^{-2}n^{-2}\right)^3 = \left(\frac{1}{m^2n^2}\right)^3 = \frac{(1)^3}{(m^2n^2)^3} = \frac{1}{(m^2)^3(n^2)^3} = \frac{1}{m^{2\cdot 3}\cdot n^{2\cdot 3}} = \frac{1}{m^6n^6}$

Note:
Exercise:

Problem:

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{b^5}{c}\right)^3$
b. $\left(\frac{5}{u^8}\right)^4$
c. $\left(\frac{-1}{w^3}\right)^{35}$
d. $\left(p^{-4}q^3\right)^8$
e. $\left(c^{-5}d^{-3}\right)^4$

Solution:

a. $\frac{b^{15}}{c^3}$
b. $\frac{625}{u^{32}}$
c. $\frac{-1}{w^{105}}$
d. $\frac{q^{24}}{p^{32}}$
e. $\frac{1}{c^{20}d^{12}}$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

Example:
Exercise:

Problem:
Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

a. $\left(6m^2n^{-1}\right)^3$
b. $17^5 \cdot 17^{-4} \cdot 17^{-3}$
c. $\left(\frac{u^{-1}v}{v^{-1}}\right)^2$
d. $\left(-2a^3b^{-1}\right)\left(5a^{-2}b^2\right)$
e. $\left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4}$
f. $\frac{\left(3w^2\right)^5}{\left(6w^{-2}\right)^2}$

Solution:

a.
$$\begin{aligned}
\left(6m^2n^{-1}\right)^3 &= (6)^3\left(m^2\right)^3\left(n^{-1}\right)^3 && \text{The power of a product rule}\\
&= 6^3m^{2\cdot 3}n^{-1\cdot 3} && \text{The power rule}\\
&= 216m^6n^{-3} && \text{Simplify.}\\
&= \frac{216m^6}{n^3} && \text{The negative exponent rule}
\end{aligned}$$

b.
$$\begin{aligned}
17^5 \cdot 17^{-4} \cdot 17^{-3} &= 17^{5-4-3} && \text{The product rule}\\
&= 17^{-2} && \text{Simplify.}\\
&= \frac{1}{17^2} \text{ or } \frac{1}{289} && \text{The negative exponent rule}
\end{aligned}$$

c.
$$\begin{aligned}
\left(\frac{u^{-1}v}{v^{-1}}\right)^2 &= \frac{\left(u^{-1}v\right)^2}{\left(v^{-1}\right)^2} && \text{The power of a quotient rule}\\
&= \frac{u^{-2}v^2}{v^{-2}} && \text{The power of a product rule}\\
&= u^{-2}v^{2-(-2)} && \text{The quotient rule}\\
&= u^{-2}v^4 && \text{Simplify.}\\
&= \frac{v^4}{u^2} && \text{The negative exponent rule}
\end{aligned}$$

d.
$$\begin{aligned}
\left(-2a^3b^{-1}\right)\left(5a^{-2}b^2\right) &= -2\cdot 5\cdot a^3\cdot a^{-2}\cdot b^{-1}\cdot b^2 && \text{Commutative and associative laws of mu}\\
&= -10\cdot a^{3-2}\cdot b^{-1+2} && \text{The product rule}\\
&= -10ab && \text{Simplify.}
\end{aligned}$$

e.
$$\begin{aligned}
\left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4} &= \left(x^2\sqrt{2}\right)^{4-4} && \text{The product rule}\\
&= \left(x^2\sqrt{2}\right)^0 && \text{Simplify.}\\
&= 1 && \text{The zero exponent rule}
\end{aligned}$$

f.

$$\begin{aligned} \frac{(3w^2)^5}{(6w^{-2})^2} &= \frac{(3)^5 \cdot (w^2)^5}{(6)^2 \cdot (w^{-2})^2} && \text{The power of a product rule} \\ &= \frac{3^5 w^{2 \cdot 5}}{6^2 w^{-2 \cdot 2}} && \text{The power rule} \\ &= \frac{243w^{10}}{36w^{-4}} && \text{Simplify.} \\ &= \frac{27w^{10-(-4)}}{4} && \text{The quotient rule and reduce fraction} \\ &= \frac{27w^{14}}{4} && \text{Simplify.} \end{aligned}$$

Note:
Exercise:

Problem: Simplify each expression and write the answer with positive exponents only.

a. $\left(2uv^{-2}\right)^{-3}$
b. $x^8 \cdot x^{-12} \cdot x$
c. $\left(\frac{e^2 f^{-3}}{f^{-1}}\right)^2$
d. $\left(9r^{-5}s^3\right)\left(3r^6 s^{-4}\right)$
e. $\left(\frac{4}{9}tw^{-2}\right)^{-3}\left(\frac{4}{9}tw^{-2}\right)^3$
f. $\frac{(2h^2k)^4}{(7h^{-1}k^2)^2}$

Solution:

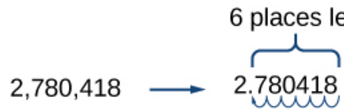
a. $\frac{v^6}{8u^3}$
b. $\frac{1}{x^3}$
c. $\frac{e^4}{f^4}$
d. $\frac{27r}{s}$
e. 1
f. $\frac{16h^{10}}{49}$

Using Scientific Notation

Recall at the beginning of the section that we found the number 1.3×10^{13} when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about 0.00005 m, and the radius of an electron, which is about 0.00000000000047 m. How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called **scientific notation**, in which we express numbers in terms of exponents of 10. To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places *n* that you moved the decimal point. Multiply the decimal number by 10 raised to a power of *n*. If you moved the decimal left as in a very large number, n is positive. If you moved the decimal right as in a small large number, n is negative.

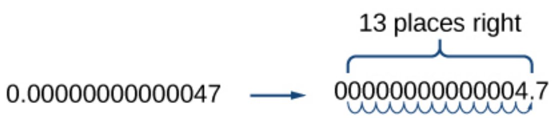
For example, consider the number 2,780,418. Move the decimal left until it is to the right of the first nonzero digit, which is 2.



We obtain 2.780418 by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6, and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.
Equation:

$$2.780418 \times 10^6$$

Working with small numbers is similar. Take, for example, the radius of an electron, 0.00000000000047 m. Perform the same series of steps as above, except move the decimal point to the right.



Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13. The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.
Equation:

$$4.7 \times 10^{-13}$$

Note:
Scientific Notation
A number is written in **scientific notation** if it is written in the form $a \times 10^n$,where $1 \leq |a| < 10$ and n is an integer.

Example:
Exercise:

Problem:
Converting Standard Notation to Scientific Notation

Write each number in scientific notation.

a. Distance to Andromeda Galaxy from Earth: 24,000,000,000,000,000,000,000 m
b. Diameter of Andromeda Galaxy: 1,300,000,000,000,000,000,000 m
c. Number of stars in Andromeda Galaxy: 1,000,000,000,000
d. Diameter of electron: 0.00000000000094 m
e. Probability of being struck by lightning in any single year: 0.00000143

Solution:

a.
$24{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$
$24{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$

$\leftarrow 22 \text{ places}$

$2.4 \times 10^{22} \text{ m}$

b.
$1{,}300{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$
$1{,}300{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$

$\leftarrow 21 \text{ places}$

$1.3 \times 10^{21} \text{ m}$

c.
$1{,}000{,}000{,}000{,}000$
$1{,}000{,}000{,}000{,}000$

$\leftarrow 12 \text{ places}$

1×10^{12}

d.
$0.00000000000094 \text{ m}$
$0.00000000000094 \text{ m}$

$\rightarrow 13 \text{ places}$

$9.4 \times 10^{-13} \text{ m}$

e.
0.00000143
0.00000143

$\rightarrow 6 \text{ places}$

1.43×10^{-6}

Analysis

Observe that, if the given number is greater than 1, as in examples a–c, the exponent of 10 is positive; and if the number is less than 1, as in examples d–e, the exponent is negative.

Note:
Exercise:

Problem: Write each number in scientific notation.

a. U.S. national debt per taxpayer (April 2014): $152,000
b. World population (April 2014): 7,158,000,000
c. World gross national income (April 2014): $85,500,000,000,000
d. Time for light to travel 1 m: 0.00000000334 s
e. Probability of winning lottery (match 6 of 49 possible numbers): 0.0000000715

Solution:

a. $\$1.52 \times 10^5$
b. 7.158×10^9
c. $\$8.55 \times 10^{13}$
d. 3.34×10^{-9}

e. 7.15×10^{-8}

Converting from Scientific to Standard Notation

To convert a number in **scientific notation** to standard notation, simply reverse the process. Move the decimal n places to the right if n is positive or n places to the left if n is negative and add zeros as needed. Remember, if n is positive, the value of the number is greater than 1, and if n is negative, the value of the number is less than one.

Example:
Exercise:

Problem:
Converting Scientific Notation to Standard Notation

Convert each number in scientific notation to standard notation.

a. 3.547×10^{14}
b. -2×10^{6}
c. 7.91×10^{-7}
d. -8.05×10^{-12}

Solution:

a.
$$3.547 \times 10^{14}$$
$$3.54700000000000$$
$$\rightarrow 14 \text{ places}$$
$$354{,}700{,}000{,}000{,}000$$

b.
$$-2 \times 10^{6}$$
$$-2.000000$$
$$\rightarrow 6 \text{ places}$$
$$-2{,}000{,}000$$

c.
$$7.91 \times 10^{-7}$$
$$0000007.91$$
$$\rightarrow 7 \text{ places}$$
$$0.000000791$$

d.
$$-8.05 \times 10^{-12}$$
$$-000000000008.05$$
$$\rightarrow 12 \text{ places}$$
$$-0.00000000000805$$

Note:

Exercise:

Problem: Convert each number in scientific notation to standard notation.

a. 7.03×10^5
b. -8.16×10^{11}
c. -3.9×10^{-13}
d. 8×10^{-6}

Solution:

a. $703,000$
b. $-816,000,000,000$
c. $-0.000\,000\,000\,000\,39$
d. 0.000008

Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents, makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in 1 L of water. Each water molecule contains 3 atoms (2 hydrogen and 1 oxygen). The average drop of water contains around 1.32×10^{21} molecules of water and 1 L of water holds about 1.22×10^4 average drops. Therefore, there are approximately $3 \cdot \left(1.32 \times 10^{21}\right) \cdot \left(1.22 \times 10^4\right) \approx 4.83 \times 10^{25}$ atoms in 1 L of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $\left(7 \times 10^4\right) \cdot \left(5 \times 10^6\right) = 35 \times 10^{10}.$ The answer is not in proper scientific notation because 35 is greater than 10. Consider 35 as $3.5 \times 10.$ That adds a ten to the exponent of the answer.
Equation:

$$(35) \times 10^{10} = (3.5 \times 10) \times 10^{10} = 3.5 \times \left(10 \times 10^{10}\right) = 3.5 \times 10^{11}$$

Example:
Exercise:

Problem:
Using Scientific Notation

Perform the operations and write the answer in scientific notation.

a. $\left(8.14 \times 10^{-7}\right)\left(6.5 \times 10^{10}\right)$
b. $\left(4 \times 10^5\right) \div \left(-1.52 \times 10^9\right)$
c. $\left(2.7 \times 10^5\right)\left(6.04 \times 10^{13}\right)$
d. $\left(1.2 \times 10^8\right) \div \left(9.6 \times 10^5\right)$
e. $\left(3.33 \times 10^4\right)\left(-1.05 \times 10^7\right)\left(5.62 \times 10^5\right)$

Solution:

a.

$$\begin{aligned}\left(8.14 \times 10^{-7}\right)\left(6.5 \times 10^{10}\right) &= (8.14 \times 6.5)\left(10^{-7} \times 10^{10}\right) && \text{Commutative and associative properties of multiplication}\\ &= (52.91)\left(10^{3}\right) && \text{Product rule of exponents}\\ &= 5.291 \times 10^{4} && \text{Scientific notation}\end{aligned}$$

b.

$$\begin{aligned}\left(4 \times 10^{5}\right) \div \left(-1.52 \times 10^{9}\right) &= \left(\frac{4}{-1.52}\right)\left(\frac{10^{5}}{10^{9}}\right) && \text{Commutative and associative properties of multiplication}\\ &\approx (-2.63)\left(10^{-4}\right) && \text{Quotient rule of exponents}\\ &= -2.63 \times 10^{-4} && \text{Scientific notation}\end{aligned}$$

c.

$$\begin{aligned}\left(2.7 \times 10^{5}\right)\left(6.04 \times 10^{13}\right) &= (2.7 \times 6.04)\left(10^{5} \times 10^{13}\right) && \text{Commutative and associative properties of multiplication}\\ &= (16.308)\left(10^{18}\right) && \text{Product rule of exponents}\\ &= 1.6308 \times 10^{19} && \text{Scientific notation}\end{aligned}$$

d.

$$\begin{aligned}\left(1.2 \times 10^{8}\right) \div \left(9.6 \times 10^{5}\right) &= \left(\frac{1.2}{9.6}\right)\left(\frac{10^{8}}{10^{5}}\right) && \text{Commutative and associative properties of multiplication}\\ &= (0.125)\left(10^{3}\right) && \text{Quotient rule of exponents}\\ &= 1.25 \times 10^{2} && \text{Scientific notation}\end{aligned}$$

e.

$$\begin{aligned}\left(3.33 \times 10^{4}\right)\left(-1.05 \times 10^{7}\right)\left(5.62 \times 10^{5}\right) &= [3.33 \times (-1.05) \times 5.62]\left(10^{4} \times 10^{7} \times 10^{5}\right)\\ &\approx (-19.65)\left(10^{16}\right)\\ &= -1.965 \times 10^{17}\end{aligned}$$

Note:
Exercise:

Problem: Perform the operations and write the answer in scientific notation.

a. $\left(-7.5 \times 10^{8}\right)\left(1.13 \times 10^{-2}\right)$
b. $\left(1.24 \times 10^{11}\right) \div \left(1.55 \times 10^{18}\right)$
c. $\left(3.72 \times 10^{9}\right)\left(8 \times 10^{3}\right)$
d. $\left(9.933 \times 10^{23}\right) \div \left(-2.31 \times 10^{17}\right)$
e. $\left(-6.04 \times 10^{9}\right)\left(7.3 \times 10^{2}\right)\left(-2.81 \times 10^{2}\right)$

Solution:

a. -8.475×10^{6}
b. 8×10^{-8}
c. 2.976×10^{13}
d. -4.3×10^{6}
e. $\approx 1.24 \times 10^{15}$

Example:
Exercise:

Problem:
Applying Scientific Notation to Solve Problems

In April 2014, the population of the United States was about 308,000,000 people. The national debt was about $17,547,000,000,000. Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

Solution:

The population was $308{,}000{,}000 = 3.08 \times 10^8$.

The national debt was $\$17{,}547{,}000{,}000{,}000 \approx \$1.75 \times 10^{13}$.

To find the amount of debt per citizen, divide the national debt by the number of citizens.
Equation:

$$\begin{aligned}(1.75 \times 10^{13}) \div (3.08 \times 10^8) &= \left(\frac{1.75}{3.08}\right) \cdot \left(\frac{10^{13}}{10^8}\right) \\ &\approx 0.57 \times 10^5 \\ &= 5.7 \times 10^4\end{aligned}$$

The debt per citizen at the time was about $\$5.7 \times 10^4$,or $57,000.

Note:
Exercise:

Problem:

An average human body contains around 30,000,000,000,000 red blood cells. Each cell measures approximately 0.000008 m long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.

Solution:

Number of cells: 3×10^{13};length of a cell: 8×10^{-6} m; total length: 2.4×10^8 m or $240,000,000$ m.

Note:
Access these online resources for additional instruction and practice with exponents and scientific notation.

- Exponential Notation
- Properties of Exponents
- Zero Exponent
- Simplify Exponent Expressions
- Quotient Rule for Exponents
- Scientific Notation
- Converting to Decimal Notation

Key Equations

Rules of Exponents For nonzero real numbers a and b and integers m and n	
Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \frac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Key Concepts

- Products of exponential expressions with the same base can be simplified by adding exponents. See [link].
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents. See [link].
- Powers of exponential expressions with the same base can be simplified by multiplying exponents. See [link].
- An expression with exponent zero is defined as 1. See [link].
- An expression with a negative exponent is defined as a reciprocal. See [link] and [link].
- The power of a product of factors is the same as the product of the powers of the same factors. See [link].
- The power of a quotient of factors is the same as the quotient of the powers of the same factors. See [link].
- The rules for exponential expressions can be combined to simplify more complicated expressions. See [link].
- Scientific notation uses powers of 10 to simplify very large or very small numbers. See [link] and [link].
- Scientific notation may be used to simplify calculations with very large or very small numbers. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem: Is 2^3 the same as 3^2? Explain.

Solution:

No, the two expressions are not the same. An exponent tells how many times you multiply the base. So 2^3 is the same as $2 \times 2 \times 2$, which is 8. 3^2 is the same as 3×3, which is 9.

Exercise:

Problem: When can you add two exponents?

Exercise:

Problem: What is the purpose of scientific notation?

Solution:

It is a method of writing very small and very large numbers.

Exercise:

Problem: Explain what a negative exponent does.

Numeric

For the following exercises, simplify the given expression. Write answers with positive exponents.

Exercise:

Problem: 9^2

Solution:

81

Exercise:

Problem: 15^{-2}

Exercise:

Problem: $3^2 \times 3^3$

Solution:

243

Exercise:

Problem: $4^4 \div 4$

Exercise:

Problem: $\left(2^2\right)^{-2}$

Solution:

$\frac{1}{16}$

Exercise:

Problem: $(5-8)^0$

Exercise:

Problem:$11^3 \div 11^4$

Solution:

$\frac{1}{11}$

Exercise:

Problem:$6^5 \times 6^{-7}$

Exercise:

Problem:$\left(8^0\right)^2$

Solution:

1

Exercise:

Problem:$5^{-2} \div 5^2$

For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents.

Exercise:

Problem:$4^2 \times 4^3 \div 4^{-4}$

Solution:

4^9

Exercise:

Problem:$\frac{6^{12}}{6^9}$

Exercise:

Problem:$\left(12^3 \times 12\right)^{10}$

Solution:

12^{40}

Exercise:

Problem:$10^6 \div \left(10^{10}\right)^{-2}$

Exercise:

Problem:$7^{-6} \times 7^{-3}$

Solution:

$\frac{1}{7^9}$

Exercise:

Problem: $\left(3^3 \div 3^4\right)^5$

For the following exercises, express the decimal in scientific notation.

Exercise:

Problem: 0.0000314

Solution:

3.14×10^{-5}

Exercise:

Problem: 148,000,000

For the following exercises, convert each number in scientific notation to standard notation.

Exercise:

Problem: 1.6×10^{10}

Solution:

16,000,000,000

Exercise:

Problem: 9.8×10^{-9}

Algebraic

For the following exercises, simplify the given expression. Write answers with positive exponents.

Exercise:

Problem: $\frac{a^3 a^2}{a}$

Solution:

a^4

Exercise:

Problem: $\frac{mn^2}{m^{-2}}$

Exercise:

Problem: $\left(b^3 c^4\right)^2$

Solution:

b^6c^8

Exercise:

Problem: $\left(\frac{x^{-3}}{y^2}\right)^{-5}$

Exercise:

Problem: $ab^2 \div d^{-3}$

Solution:

ab^2d^3

Exercise:

Problem: $\left(w^0x^5\right)^{-1}$

Exercise:

Problem: $\frac{m^4}{n^0}$

Solution:

m^4

Exercise:

Problem: $y^{-4}\left(y^2\right)^2$

Exercise:

Problem: $\frac{p^{-4}q^2}{p^2q^{-3}}$

Solution:

$\frac{q^5}{p^6}$

Exercise:

Problem: $(l \times w)^2$

Exercise:

Problem: $\left(y^7\right)^3 \div x^{14}$

Solution:

$\frac{y^{21}}{x^{14}}$

Exercise:

Problem: $\left(\frac{a}{2^3}\right)^2$

Exercise:

Problem: $5^2m \div 5^0m$

Solution:

25

Exercise:

Problem: $\frac{\left(16\sqrt{x}\right)^2}{y^{-1}}$

Exercise:

Problem: $\frac{2^3}{(3a)^{-2}}$

Solution:

$72a^2$

Exercise:

Problem: $\left(ma^6\right)^2\frac{1}{m^3a^2}$

Exercise:

Problem: $\left(b^{-3}c\right)^3$

Solution:

$\frac{c^3}{b^9}$

Exercise:

Problem: $\left(x^2y^{13} \div y^0\right)^2$

Exercise:

Problem: $\left(9z^3\right)^{-2}y$

Solution:

$\frac{y}{81z^6}$

Real-World Applications

Exercise:

Problem:

To reach escape velocity, a rocket must travel at the rate of 2.2×10^6 ft/min. Rewrite the rate in standard notation.

Exercise:

Problem:

A dime is the thinnest coin in U.S. currency. A dime's thickness measures 1.35×10^{-3} m. Rewrite the number in standard notation.

Solution:

0.00135 m

Exercise:

Problem:

The average distance between Earth and the Sun is 92,960,000 mi. Rewrite the distance using scientific notation.

Exercise:

Problem:A terabyte is made of approximately 1,099,500,000,000 bytes. Rewrite in scientific notation.

Solution:

1.0995×10^{12}

Exercise:

Problem:

The Gross Domestic Product (GDP) for the United States in the first quarter of 2014 was $\$1.71496 \times 10^{13}$. Rewrite the GDP in standard notation.

Exercise:

Problem: One picometer is approximately 3.397×10^{-11} in. Rewrite this length using standard notation.

Solution:

0.00000000003397 in.

Exercise:

Problem:

The value of the services sector of the U.S. economy in the first quarter of 2012 was $10,633.6 billion. Rewrite this amount in scientific notation.

Technology

For the following exercises, use a graphing calculator to simplify. Round the answers to the nearest hundredth.

Exercise:

Problem:$\left(\frac{12^3 m^{33}}{4^{-3}}\right)^2$

Solution:

12,230,590,464 m^{66}

Exercise:

Problem: $17^3 \div 15^2 x^3$

Extensions

For the following exercises, simplify the given expression. Write answers with positive exponents.

Exercise:

Problem: $\left(\frac{3^2}{a^3}\right)^{-2}\left(\frac{a^4}{2^2}\right)^2$

Solution:

$\frac{a^{14}}{1296}$

Exercise:

Problem: $\left(6^2 - 24\right)^2 \div \left(\frac{x}{y}\right)^{-5}$

Exercise:

Problem: $\frac{m^2 n^3}{a^2 c^{-3}} \cdot \frac{a^{-7} n^{-2}}{m^2 c^4}$

Solution:

$\frac{n}{a^9 c}$

Exercise:

Problem: $\left(\frac{x^6 y^3}{x^3 y^{-3}} \cdot \frac{y^{-7}}{x^{-3}}\right)^{10}$

Exercise:

Problem: $\left(\frac{\left(ab^2 c\right)^{-3}}{b^{-3}}\right)^2$

Solution:

$\frac{1}{a^6 b^6 c^6}$

Exercise:

Problem:

Avogadro’s constant is used to calculate the number of particles in a mole. A mole is a basic unit in chemistry to measure the amount of a substance. The constant is 6.0221413×10^{23}. Write Avogadro’s constant in standard notation.

Exercise:

Problem:

Planck’s constant is an important unit of measure in quantum physics. It describes the relationship between energy and frequency. The constant is written as $6.62606957 \times 10^{-34}$. Write Planck’s constant in standard notation.

Solution:

0.000000000000000000000000000000000662606957

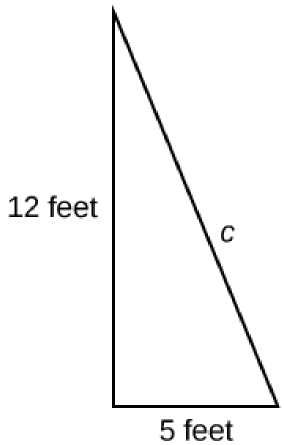
Glossary

scientific notation
: a shorthand notation for writing very large or very small numbers in the form $a \times 10^{n}$ where $1 \leq |a| < 10$ and n is an integer

1.3 Radicals and Rational Exponents
In this section students will:

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

A hardware store sells 16-ft ladders and 24-ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in [link], and use the Pythagorean Theorem.



Equation:

$$\begin{array}{rcl} a^2+b^2 & = & c^2 \\ 5^2+12^2 & = & c^2 \\ 169 & = & c^2 \end{array}$$

Now, we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other words, we need to find a square root. In this section, we will investigate methods of finding solutions to problems such as this one.

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2=16$,the square root of 16 is 4. The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a. The square root could be positive or negative because multiplying two negative numbers gives a positive number. The **principal square root** is the nonnegative

number that when multiplied by itself equals a. The square root obtained using a calculator is the principal square root.

The principal square root of a is written as $\sqrt{a}$. The symbol is called a **radical**, the term under the symbol is called the **radicand**, and the entire expression is called a **radical expression**.

Note:
Principal Square Root
The **principal square root** of a is the nonnegative number that, when multiplied by itself, equals a. It is written as a **radical expression,** with a symbol called a **radical** over the term called the **radicand**: $\sqrt{a}$.

Note:
Does $\sqrt{25} = \pm 5$?
No. Although both 5^2 *and* $(-5)^2$ *are* 25,*the radical symbol implies only a nonnegative root, the principal square root. The principal square root of 25 is* $\sqrt{25} = 5$.

Example:
Exercise:

Problem:
Evaluating Square Roots

Evaluate each expression.

a. $\sqrt{100}$
b. $\sqrt{\sqrt{16}}$
c. $\sqrt{25 + 144}$
d. $\sqrt{49} - \sqrt{81}$

Solution:

a. $\sqrt{100} = 10$ because $10^2 = 100$
b. $\sqrt{\sqrt{16}} = \sqrt{4} = 2$ because $4^2 = 16$ and $2^2 = 4$
c. $\sqrt{25 + 144} = \sqrt{169} = 13$ because $13^2 = 169$

d. $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$ because $7^2 = 49$ and $9^2 = 81$

Note:
For $\sqrt{25+144}$,can we find the square roots before adding?
No. $\sqrt{25} + \sqrt{144} = 5 + 12 = 17$. *This is not equivalent to* $\sqrt{25+144} = 13$. *The order of operations requires us to add the terms in the radicand before finding the square root.*

Note:
Exercise:

Problem: Evaluate each expression.

a. $\sqrt{225}$
b. $\sqrt{\sqrt{81}}$
c. $\sqrt{25-9}$
d. $\sqrt{36} + \sqrt{121}$

Solution:

a. 15
b. 3
c. 4
d. 17

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the *product rule for simplifying square roots,* which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3} \cdot \sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

Note:
The Product Rule for Simplifying Square Roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b.

Equation:

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

Note:

Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

Example:

Exercise:

Problem:

Using the Product Rule to Simplify Square Roots

Simplify the radical expression.

a. $\sqrt{300}$
b. $\sqrt{162a^5b^4}$

Solution:

a.

$\sqrt{100 \cdot 3}$	Factor perfect square from radicand.
$\sqrt{100} \cdot \sqrt{3}$	Write radical expression as product of radical expressions.
$10\sqrt{3}$	Simplify.

b.

$\sqrt{81a^4b^4 \cdot 2a}$	Factor perfect square from radicand.
$\sqrt{81a^4b^4} \cdot \sqrt{2a}$	Write radical expression as product of radical expressions.
$9a^2b^2\sqrt{2a}$	Simplify.

Note:

Exercise:

Problem: Simplify $\sqrt{50x^2y^3z}$.

Solution:

$5\left|x\right|\left|y\right|\sqrt{2yz}$. Notice the absolute value signs around *x* and *y*? That's because their value must be positive!

Note:
Given the product of multiple radical expressions, use the product rule to combine them into one radical expression.

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

Example:
Exercise:

Problem:
Using the Product Rule to Simplify the Product of Multiple Square Roots

Simplify the radical expression.
$\sqrt{12}\cdot\sqrt{3}$

Solution:

$\sqrt{12\cdot 3}$	Express the product as a single radical expression.
$\sqrt{36}$	Simplify.
6	

Note:
Exercise:

Problem: Simplify $\sqrt{50x}\cdot\sqrt{2x}$ assuming $x>0$.

Solution:

$10\left|x\right|$

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the *quotient rule for simplifying square roots.* It can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately. We can rewrite $\sqrt{\frac{5}{2}}$ as $\frac{\sqrt{5}}{\sqrt{2}}$.

Note:
The Quotient Rule for Simplifying Square Roots
The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b,where $b \neq 0$.
Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Note:
Given a radical expression, use the quotient rule to simplify it.

1. Write the radical expression as the quotient of two radical expressions.
2. Simplify the numerator and denominator.

Example:
Exercise:

Problem:
Using the Quotient Rule to Simplify Square Roots

Simplify the radical expression.

$$\sqrt{\frac{5}{36}}$$

Solution:

$\frac{\sqrt{5}}{\sqrt{36}}$	Write as quotient of two radical expressions.
$\frac{\sqrt{5}}{6}$	Simplify denominator.

Note:
Exercise:

Problem: Simplify $\sqrt{\frac{2x^2}{9y^4}}$.

Solution:

$\frac{x\sqrt{2}}{3y^2}$. We do not need the absolute value signs for y^2 because that term will always be nonnegative.

Example:
Exercise:

Problem:
Using the Quotient Rule to Simplify an Expression with Two Square Roots

Simplify the radical expression.

$\frac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}}$

Solution:

$\sqrt{\frac{234x^{11}y}{26x^7y}}$	Combine numerator and denominator into one radical expression.
$\sqrt{9x^4}$	Simplify fraction.
$3x^2$	Simplify square root.

Note:
Exercise:

Problem: Simplify $\frac{\sqrt{9a^5b^{14}}}{\sqrt{3a^4b^5}}$.

Solution:

$b^4\sqrt{3ab}$

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3\sqrt{2}$ is $4\sqrt{2}.$ However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3\sqrt{2}$,so $\sqrt{2}+\sqrt{18}=\sqrt{2}+3\sqrt{2}=4\sqrt{2}.$

Note:
Given a radical expression requiring addition or subtraction of square roots, solve.

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

Example:
Exercise:

Problem:
Adding Square Roots

Add $5\sqrt{12}+2\sqrt{3}.$

Solution:

We can rewrite $5\sqrt{12}$ as $5\sqrt{4\cdot 3}.$ According the product rule, this becomes $5\sqrt{4}\sqrt{3}.$ The square root of $\sqrt{4}$ is 2, so the expression becomes $5\left(2\right)\sqrt{3}$,which is $10\sqrt{3}.$ Now we can the terms have the same radicand so we can add.

$10\sqrt{3}+2\sqrt{3}=12\sqrt{3}$

Note:
Exercise:

Problem: Add $\sqrt{5}+6\sqrt{20}.$

Solution:

$13\sqrt{5}$

Example:
Exercise:

Problem:
Subtracting Square Roots

Subtract $20\sqrt{72a^3b^4c} - 14\sqrt{8a^3b^4c}$.

Solution:

Rewrite each term so they have equal radicands.
Equation:

$$\begin{aligned} 20\sqrt{72a^3b^4c} &= 20\sqrt{9}\sqrt{4}\sqrt{2}\sqrt{a}\sqrt{a^2}\sqrt{\left(b^2\right)^2}\sqrt{c} \\ &= 20(3)(2)\left|a\right|b^2\sqrt{2ac} \\ &= 120\left|a\right|b^2\sqrt{2ac} \end{aligned}$$

Equation:

$$\begin{aligned} 14\sqrt{8a^3b^4c} &= 14\sqrt{2}\sqrt{4}\sqrt{a}\sqrt{a^2}\sqrt{\left(b^2\right)^2}\sqrt{c} \\ &= 14(2)\left|a\right|b^2\sqrt{2ac} \\ &= 28\left|a\right|b^2\sqrt{2ac} \end{aligned}$$

Now the terms have the same radicand so we can subtract.
Equation:

$$120\left|a\right|b^2\sqrt{2ac} - 28\left|a\right|b^2\sqrt{2ac} = 92\left|a\right|b^2\sqrt{2ac}$$

Note:
Exercise:

Problem: Subtract $3\sqrt{80x} - 4\sqrt{45x}$.

Solution:

0

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called *rationalizing the denominator*.

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b\sqrt{c}$,multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

For a denominator containing the sum or difference of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a+b\sqrt{c}$,then the conjugate is $a-b\sqrt{c}$.

Note:
Given an expression with a single square root radical term in the denominator, rationalize the denominator.

a. Multiply the numerator and denominator by the radical in the denominator.
b. Simplify.

Example:
Exercise:

Problem:
Rationalizing a Denominator Containing a Single Term

Write $\frac{2\sqrt{3}}{3\sqrt{10}}$ in simplest form.

Solution:

The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

Equation:

$$\frac{2\sqrt{3}}{3\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$
$$\frac{2\sqrt{30}}{30}$$
$$\frac{\sqrt{30}}{15}$$

Note:
Exercise:

Problem: Write $\frac{12\sqrt{3}}{\sqrt{2}}$ in simplest form.

Solution:

$6\sqrt{6}$

Note:
Given an expression with a radical term and a constant in the denominator, rationalize the denominator.

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

Example:
Exercise:

Problem:
Rationalizing a Denominator Containing Two Terms

Write $\frac{4}{1+\sqrt{5}}$ in simplest form.

Solution:

Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1+\sqrt{5}$ is $1-\sqrt{5}.$ Then multiply the fraction by $\frac{1-\sqrt{5}}{1-\sqrt{5}}$.

Equation:

$\frac{4}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}$	
$\frac{4-4\sqrt{5}}{-4}$	Use the distributive property.
$\sqrt{5}-1$	Simplify.

Note:
Exercise:

Problem: Write $\frac{7}{2+\sqrt{3}}$ in simplest form.

Solution:

$14-7\sqrt{3}$

Using Rational Roots

Although square roots are the most common rational roots, we can also find cube roots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number that, when raised to a certain power, gives a certain number.

Understanding *n*th Roots

Suppose we know that $a^3=8.$ We want to find what number raised to the 3rd power is equal to 8. Since $2^3=8$,we say that 2 is the cube root of 8.

The *n*th root of a is a number that, when raised to the *n*th power, gives $a.$ For example, -3 is the 5th root of -243 because $(-3)^5=-243.$ If a is a real number with at least one *n*th root, then the **principal *n*th root** of a is the number with the same sign as a that, when raised to the *n*th power, equals $a.$

The principal *n*th root of a is written as $\sqrt[n]{a}$,where n is a positive integer greater than or equal to 2. In the radical expression, n is called the **index** of the radical.

Note:
Principal *n*th Root
If a is a real number with at least one *n*th root, then the **principal *n*th root** of a,written as $\sqrt[n]{a}$,is the number with the same sign as a that, when raised to the *n*th power, equals a. The **index** of the radical is n.

Example:
Exercise:

Problem:
Simplifying *n*th Roots

Simplify each of the following:

a. $\sqrt[5]{-32}$
b. $\sqrt[4]{4} \cdot \sqrt[4]{1,024}$
c. $-\sqrt[3]{\frac{8x^6}{125}}$
d. $8\sqrt[4]{3} - \sqrt[4]{48}$

Solution:

a. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$
b. First, express the product as a single radical expression. $\sqrt[4]{4,096} = 8$ because $8^4 = 4,096$
c. $\frac{-\sqrt[3]{8x^6}}{\sqrt[3]{125}}$ Write as quotient of two radical expressions.
$\frac{-2x^2}{5}$ Simplify.
d. $8\sqrt[4]{3} - 2\sqrt[4]{3}$ Simplify to get equal radicands.
$6\sqrt[4]{3}$ Add.

Note:
Exercise:

Problem: Simplify.

a. $\sqrt[3]{-216}$
b. $\frac{3\sqrt[4]{80}}{\sqrt[4]{5}}$
c. $6\sqrt[3]{9,000}+7\sqrt[3]{576}$

Solution:

a. -6
b. 6
c. $88\sqrt[3]{9}$

Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index n is even, then a cannot be negative.
Equation:

$$a^{\frac{1}{n}}=\sqrt[n]{a}$$

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an *n*th root. The numerator tells us the power and the denominator tells us the root.
Equation:

$$a^{\frac{m}{n}}=\left(\sqrt[n]{a}\right)^{m}=\sqrt[n]{a^{m}}$$

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

Note:
Rational Exponents
Rational exponents are another way to express principal *n*th roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is
Equation:

$$a^{\frac{m}{n}}=\left(\sqrt[n]{a}\right)^{m}=\sqrt[n]{a^{m}}$$

Note:
Given an expression with a rational exponent, write the expression as a radical.

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Example:
Exercise:

Problem:
Writing Rational Exponents as Radicals

Write $343^{\frac{2}{3}}$ as a radical. Simplify.

Solution:

The 2 tells us the power and the 3 tells us the root.

$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = \sqrt[3]{343^2}$

We know that $\sqrt[3]{343} = 7$ because $7^3 = 343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = 7^2 = 49$$

Note:
Exercise:

Problem: Write $9^{\frac{5}{2}}$ as a radical. Simplify.

Solution:

$\left(\sqrt{9}\right)^5 = 3^5 = 243$

Example:
Exercise:

Problem:
Writing Radicals as Rational Exponents

Write $\frac{4}{\sqrt[7]{a^2}}$ using a rational exponent.

Solution:

The power is 2 and the root is 7, so the rational exponent will be $\frac{2}{7}$. We get $\frac{4}{a^{\frac{2}{7}}}$. Using properties of exponents, we get $\frac{4}{\sqrt[7]{a^2}} = 4a^{\frac{-2}{7}}$.

Note:
Exercise:

Problem: Write $x\sqrt{(5y)^9}$ using a rational exponent.

Solution:

$x(5y)^{\frac{9}{2}}$

Example:
Exercise:

Problem:
Simplifying Rational Exponents

Simplify:

a. $5\left(2x^{\frac{3}{4}}\right)\left(3x^{\frac{1}{5}}\right)$
b. $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

Solution:

a.

$30x^{\frac{3}{4}}x^{\frac{1}{5}}$ Multiply the coefficients.

$30x^{\frac{3}{4}+\frac{1}{5}}$ Use properties of exponents.

$30x^{\frac{19}{20}}$ Simplify.

b.

$\left(\frac{9}{16}\right)^{\frac{1}{2}}$ Use definition of negative exponents.

$\sqrt{\frac{9}{16}}$ Rewrite as a radical.

$\frac{\sqrt{9}}{\sqrt{16}}$ Use the quotient rule.

$\frac{3}{4}$ Simplify.

Note:
Exercise:

Problem: Simplify $(8x)^{\frac{1}{3}}\left(14x^{\frac{6}{5}}\right)$.

Solution:

$28x^{\frac{23}{15}}$

Note:
Access these online resources for additional instruction and practice with radicals and rational exponents.

- Radicals
- Rational Exponents
- Simplify Radicals
- Rationalize Denominator

Key Concepts

- The principal square root of a number a is the nonnegative number that when multiplied by itself equals a. See [link].
- If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b See [link] and [link].

- If a and b are nonnegative, the square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b See [link] and [link].
- We can add and subtract radical expressions if they have the same radicand and the same index. See [link] and [link].
- Radical expressions written in simplest form do not contain a radical in the denominator. To eliminate the square root radical from the denominator, multiply both the numerator and the denominator by the conjugate of the denominator. See [link] and [link].
- The principal *n*th root of a is the number with the same sign as a that when raised to the *n*th power equals a. These roots have the same properties as square roots. See [link].
- Radicals can be rewritten as rational exponents and rational exponents can be rewritten as radicals. See [link] and [link].
- The properties of exponents apply to rational exponents. See [link].

Section Exercises

Verbal

Exercise:

Problem:

What does it mean when a radical does not have an index? Is the expression equal to the radicand? Explain.

Solution:

When there is no index, it is assumed to be 2 or the square root. The expression would only be equal to the radicand if the index were 1.

Exercise:

Problem: Where would radicals come in the order of operations? Explain why.

Exercise:

Problem: Every number will have two square roots. What is the principal square root?

Solution:

The principal square root is the nonnegative root of the number.

Exercise:

Problem: Can a radical with a negative radicand have a real square root? Why or why not?

Numeric

For the following exercises, simplify each expression.
Exercise:

Problem: $\sqrt{256}$

Solution:

16

Exercise:

Problem: $\sqrt{\sqrt{256}}$

Exercise:

Problem: $\sqrt{4\left(9+16\right)}$

Solution:

10

Exercise:

Problem: $\sqrt{289}-\sqrt{121}$

Exercise:

Problem: $\sqrt{196}$

Solution:

14

Exercise:

Problem: $\sqrt{1}$

Exercise:

Problem: $\sqrt{98}$

Solution:

$7\sqrt{2}$

Exercise:

Problem: $\sqrt{\frac{27}{64}}$

Exercise:

Problem: $\sqrt{\frac{81}{5}}$

Solution:

$\frac{9\sqrt{5}}{5}$

Exercise:

Problem: $\sqrt{800}$

Exercise:

Problem: $\sqrt{169}+\sqrt{144}$

Solution:

25

Exercise:

Problem: $\sqrt{\frac{8}{50}}$

Exercise:

Problem: $\frac{18}{\sqrt{162}}$

Solution:

$\sqrt{2}$

Exercise:

Problem: $\sqrt{192}$

Exercise:

Problem: $14\sqrt{6}-6\sqrt{24}$

Solution:

$2\sqrt{6}$

Exercise:

Problem: $15\sqrt{5}+7\sqrt{45}$

Exercise:

Problem: $\sqrt{150}$

Solution:

$5\sqrt{6}$

Exercise:

Problem: $\sqrt{\frac{96}{100}}$

Exercise:

Problem: $\left(\sqrt{42}\right)\left(\sqrt{30}\right)$

Solution:

$6\sqrt{35}$

Exercise:

Problem: $12\sqrt{3}-4\sqrt{75}$

Exercise:

Problem: $\sqrt{\frac{4}{225}}$

Solution:

$\frac{2}{15}$

Exercise:

Problem: $\sqrt{\frac{405}{324}}$

Exercise:

Problem: $\sqrt{\frac{360}{361}}$

Solution:

$\frac{6\sqrt{10}}{19}$

Exercise:

Problem: $\frac{5}{1+\sqrt{3}}$

Exercise:

Problem: $\frac{8}{1-\sqrt{17}}$

Solution:

$-\frac{1+\sqrt{17}}{2}$

Exercise:

Problem: $\sqrt[4]{16}$

Exercise:

Problem: $\sqrt[3]{128}+3\sqrt[3]{2}$

Solution:

$7\sqrt[3]{2}$

Exercise:

Problem: $\sqrt[5]{\frac{-32}{243}}$

Exercise:

Problem: $\frac{15\sqrt[4]{125}}{\sqrt[4]{5}}$

Solution:

$15\sqrt{5}$

Exercise:

Problem: $3\sqrt[3]{-432}+\sqrt[3]{16}$

Algebraic

For the following exercises, simplify each expression.

Exercise:

Problem: $\sqrt{400x^4}$

Solution:

$20x^2$

Exercise:

Problem: $\sqrt{4y^2}$

Exercise:

Problem: $\sqrt{49p}$

Solution:

$7\sqrt{p}$

Exercise:

Problem: $\left(144p^2q^6\right)^{\frac{1}{2}}$

Exercise:

Problem: $m^{\frac{5}{2}}\sqrt{289}$

Solution:

$17m^2\sqrt{m}$

Exercise:

Problem: $9\sqrt{3m^2}+\sqrt{27}$

Exercise:

Problem: $3\sqrt{ab^2}-b\sqrt{a}$

Solution:

$2b\sqrt{a}$

Exercise:

Problem: $\frac{4\sqrt{2n}}{\sqrt{16n^4}}$

Exercise:

Problem: $\sqrt{\frac{225x^3}{49x}}$

Solution:

$\frac{15x}{7}$

Exercise:

Problem: $3\sqrt{44z} + \sqrt{99z}$

Exercise:

Problem: $\sqrt{50y^8}$

Solution:

$5y^4\sqrt{2}$

Exercise:

Problem: $\sqrt{490bc^2}$

Exercise:

Problem: $\sqrt{\frac{32}{14d}}$

Solution:

$\frac{4\sqrt{7d}}{7d}$

Exercise:

Problem: $q^{\frac{3}{2}}\sqrt{63p}$

Exercise:

Problem: $\frac{\sqrt{8}}{1-\sqrt{3x}}$

Solution:

$\frac{2\sqrt{2}+2\sqrt{6x}}{1-3x}$

Exercise:

Problem: $\sqrt{\frac{20}{121d^4}}$

Exercise:

Problem: $w^{\frac{3}{2}}\sqrt{32} - w^{\frac{3}{2}}\sqrt{50}$

Solution:

$-w\sqrt{2w}$

Exercise:

Problem: $\sqrt{108x^4} + \sqrt{27x^4}$

Exercise:

Problem: $\frac{\sqrt{12x}}{2+2\sqrt{3}}$

Solution:

$\frac{3\sqrt{x}-\sqrt{3x}}{2}$

Exercise:

Problem: $\sqrt{147k^3}$

Exercise:

Problem: $\sqrt{125n^{10}}$

Solution:

$5n^5\sqrt{5}$

Exercise:

Problem: $\sqrt{\frac{42q}{36q^3}}$

Exercise:

Problem: $\sqrt{\frac{81m}{361m^2}}$

Solution:

$\frac{9\sqrt{m}}{19m}$

Exercise:

Problem: $\sqrt{72c}-2\sqrt{2c}$

Exercise:

Problem: $\sqrt{\frac{144}{324d^2}}$

Solution:

$\frac{2}{3d}$

Exercise:

Problem: $\sqrt[3]{24x^6}+\sqrt[3]{81x^6}$

Exercise:

Problem: $\sqrt[4]{\frac{162x^6}{16x^4}}$

Solution:

$\frac{3\sqrt[4]{2x^2}}{2}$

Exercise:

Problem: $\sqrt[3]{64y}$

Exercise:

Problem: $\sqrt[3]{128z^3}-\sqrt[3]{-16z^3}$

Solution:

$6z\sqrt[3]{2}$

Exercise:

Problem: $\sqrt[5]{1{,}024c^{10}}$

Real-World Applications

Exercise:

Problem:

A guy wire for a suspension bridge runs from the ground diagonally to the top of the closest pylon to make a triangle. We can use the Pythagorean Theorem to find the length of guy wire needed. The square of the distance between the wire on the ground and the pylon on the ground is 90,000 feet. The square of the height of the pylon is 160,000 feet. So the length of the guy wire can be found by evaluating $\sqrt{90{,}000+160{,}000}.$ What is the length of the guy wire?

Solution:

500 feet

Exercise:

Problem:

A car accelerates at a rate of $6-\frac{\sqrt{4}}{\sqrt{t}}\ \mathrm{m/s}^2$ where *t* is the time in seconds after the car moves from rest. Simplify the expression.

Extensions

For the following exercises, simplify each expression.

Exercise:

Problem: $\frac{\sqrt{8}-\sqrt{16}}{4-\sqrt{2}}-2^{\frac{1}{2}}$

Solution:

$\frac{-5\sqrt{2}-6}{7}$

Exercise:

Problem: $\frac{4^{\frac{3}{2}}-16^{\frac{3}{2}}}{8^{\frac{1}{3}}}$

Exercise:

Problem: $\frac{\sqrt{mn^3}}{a^2\sqrt{c^{-3}}}\cdot\frac{a^{-7}n^{-2}}{\sqrt{m^2c^4}}$

Solution:

$\frac{\sqrt{mnc}}{a^9 cmn}$

Exercise:

Problem: $\frac{a}{a-\sqrt{c}}$

Exercise:

Problem: $\frac{x\sqrt{64y}+4\sqrt{y}}{\sqrt{128y}}$

Solution:

$\frac{2\sqrt{2}x+\sqrt{2}}{4}$

Exercise:

Problem: $\left(\frac{\sqrt{250x^2}}{\sqrt{100b^3}}\right)\left(\frac{7\sqrt{b}}{\sqrt{125x}}\right)$

Exercise:

Problem: $\sqrt{\frac{\sqrt[3]{64}+\sqrt[4]{256}}{\sqrt{64}+\sqrt{256}}}$

Solution:

$\frac{\sqrt{3}}{3}$

Glossary

index
: the number above the radical sign indicating the *n*th root

principal *n*th root
: the number with the same sign as a that when raised to the *n*th power equals a

principal square root
: the nonnegative square root of a number a that, when multiplied by itself, equals a

radical
: the symbol used to indicate a root

radical expression
: an expression containing a radical symbol

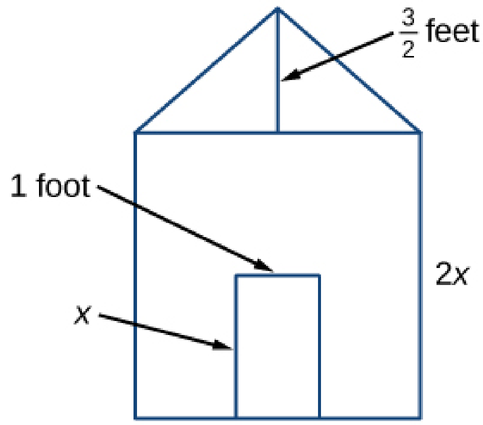
radicand

the number under the radical symbol

1.4 Polynomials
In this section students will:

- Identify the degree and leading coefficient of polynomials.
- Add and subtract polynomials.
- Multiply polynomials.
- Use FOIL to multiply binomials.
- Perform operations with polynomials of several variables.

Earl is building a doghouse, whose front is in the shape of a square topped with a triangle. There will be a rectangular door through which the dog can enter and exit the house. Earl wants to find the area of the front of the doghouse so that he can purchase the correct amount of paint. Using the measurements of the front of the house, shown in [link], we can create an expression that combines several variable terms, allowing us to solve this problem and others like it.



First find the area of the square in square feet.
Equation:

$$\begin{aligned} A &= s^2 \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

Then find the area of the triangle in square feet.
Equation:

$$\begin{aligned} A &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}(2x)\left(\tfrac{3}{2}\right) \\ &= \tfrac{3}{2}x \end{aligned}$$

Next find the area of the rectangular door in square feet.
Equation:

$$\begin{aligned} A &= lw \\ &= x \cdot 1 \\ &= x \end{aligned}$$

The area of the front of the doghouse can be found by adding the areas of the square and the triangle, and then subtracting the area of the rectangle. When we do this, we get $4x^2 + \frac{3}{2}x - x \text{ ft}^2$,or $4x^2 + \frac{1}{2}x \text{ ft}^2$.

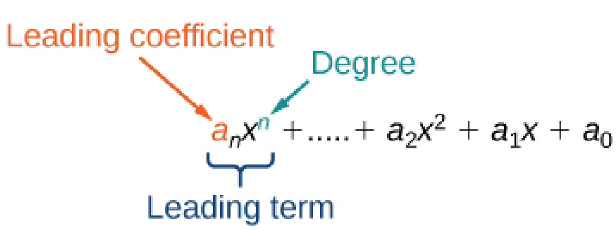
In this section, we will examine expressions such as this one, which combine several variable terms.

Identifying the Degree and Leading Coefficient of Polynomials

The formula just found is an example of a **polynomial**, which is a sum of or difference of terms, each consisting of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as 384π,is known as a **coefficient**. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product $a_i x^i$,such as $384\pi w$,is a **term of a polynomial**. If a term does not contain a variable, it is called a *constant*.

A polynomial containing only one term, such as $5x^4$,is called a **monomial**. A polynomial containing two terms, such as $2x - 9$,is called a **binomial**. A polynomial containing three terms, such as $-3x^2 + 8x - 7$,is called a **trinomial**.

We can find the **degree** of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers are descending, we say that it is in standard form.



Note:

Polynomials

A **polynomial** is an expression that can be written in the form

Equation:

$$a_n x^n + ... + a_2 x^2 + a_1 x + a_0$$

Each real number a_i is called a **coefficient**. The number a_0 that is not multiplied by a variable is called a *constant*. Each product $a_i x^i$ is a **term of a polynomial**. The highest power of the variable that occurs in the polynomial is called the **degree** of a polynomial. The **leading term** is the term with the highest power, and its coefficient is called the **leading coefficient**.

Note:

Given a polynomial expression, identify the degree and leading coefficient.

1. Find the highest power of *x* to determine the degree.
2. Identify the term containing the highest power of *x* to find the leading term.
3. Identify the coefficient of the leading term.

Example:

Exercise:

Problem:

Identifying the Degree and Leading Coefficient of a Polynomial

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

a. $3 + 2x^2 - 4x^3$
b. $5t^5 - 2t^3 + 7t$
c. $6p - p^3 - 2$

Solution:

a. The highest power of *x* is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.
b. The highest power of *t* is 5,so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that

term, 5.

c. The highest power of p is 3,so the degree is 3. The leading term is the term containing that degree, $-p^3$,The leading coefficient is the coefficient of that term, -1.

Note:
Exercise:

Problem:

Identify the degree, leading term, and leading coefficient of the polynomial $4x^2-x^6+2x-6$.

Solution:

The degree is 6, the leading term is $-x^6$,and the leading coefficient is -1.

Adding and Subtracting Polynomials

We can add and subtract polynomials by combining like terms, which are terms that contain the same variables raised to the same exponents. For example, $5x^2$ and $-2x^2$ are like terms, and can be added to get $3x^2$,but $3x$ and $3x^2$ are not like terms, and therefore cannot be added.

Note:
Given multiple polynomials, add or subtract them to simplify the expressions.

1. Combine like terms.
2. Simplify and write in standard form.

Example:
Exercise:

Problem:
Adding Polynomials

Find the sum.

$$\left(12x^2+9x-21\right)+\left(4x^3+8x^2-5x+20\right)$$

Solution:

$$4x^3+\left(12x^2+8x^2\right)+(9x-5x)+(-21+20) \qquad \text{Combine like terms.}$$
$$4x^3+20x^2+4x-1 \qquad \text{Simplify.}$$

Analysis

We can check our answers to these types of problems using a graphing calculator. To check, graph the problem as given along with the simplified answer. The two graphs should be equivalent. Be sure to use the same window to compare the graphs. Using different windows can make the expressions seem equivalent when they are not.

Note:
Exercise:

Problem: Find the sum.

$$\left(2x^3+5x^2-x+1\right)+\left(2x^2-3x-4\right)$$

Solution:

$$2x^3+7x^2-4x-3$$

Example:
Exercise:

Problem:
Subtracting Polynomials

Find the difference.

$$\left(7x^4-x^2+6x+1\right)-\left(5x^3-2x^2+3x+2\right)$$

Solution:

$7x^4 - 5x^3 + \left(-x^2 + 2x^2\right) + (6x - 3x) + (1 - 2)$	Combine like terms.
$7x^4 - 5x^3 + x^2 + 3x - 1$	Simplify.

Analysis

Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.

Note:
Exercise:

Problem: Find the difference.

$$\left(-7x^3 - 7x^2 + 6x - 2\right) - \left(4x^3 - 6x^2 - x + 7\right)$$

Solution:

$$-11x^3 - x^2 + 7x - 9$$

Multiplying Polynomials

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms. We can also use a shortcut called the FOIL method when multiplying binomials. Certain special products follow patterns that we can memorize and use instead of multiplying the polynomials by hand each time. We will look at a variety of ways to multiply polynomials.

Multiplying Polynomials Using the Distributive Property

To multiply a number by a polynomial, we use the distributive property. The number must be distributed to each term of the polynomial. We can distribute the 2 in $2(x + 7)$ to obtain the equivalent expression $2x + 14$. When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second. We then add the products together and combine like terms to simplify.

Note:
Given the multiplication of two polynomials, use the distributive property to simplify the expression.

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

Example:
Exercise:

Problem:
Multiplying Polynomials Using the Distributive Property

Find the product.

$(2x+1)\left(3x^2-x+4\right)$

Solution:

$2x\left(3x^2-x+4\right)+1\left(3x^2-x+4\right)$	Use the distributive property.
$\left(6x^3-2x^2+8x\right)+\left(3x^2-x+4\right)$	Multiply.
$6x^3+\left(-2x^2+3x^2\right)+(8x-x)+4$	Combine like terms.
$6x^3+x^2+7x+4$	Simplify.

Analysis

We can use a table to keep track of our work, as shown in [link]. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.

	$3x^2$	$-x$	$+4$
$2x$	$6x^3$	$-2x^2$	$8x$

$+1$	$3x^2$	$-x$	4

Note:
Exercise:

Problem: Find the product.

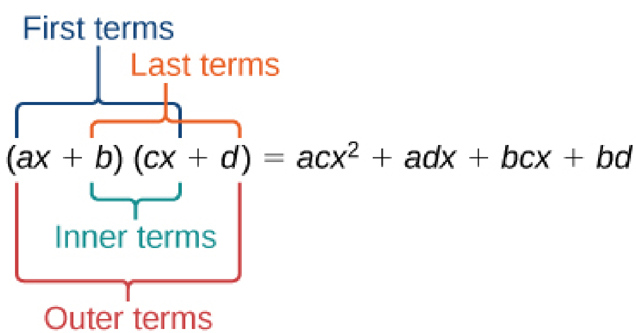
$$(3x+2)\left(x^3-4x^2+7\right)$$

Solution:

$$3x^4-10x^3-8x^2+21x+14$$

Using FOIL to Multiply Binomials

A shortcut called FOIL is sometimes used to find the product of two binomials. It is called FOIL because we multiply the **f**irst terms, the **o**uter terms, the **i**nner terms, and then the **l**ast terms of each binomial.



The FOIL method arises out of the distributive property. We are simply multiplying each term of the first binomial by each term of the second binomial, and then combining like terms.

Note:
Given two binomials, use FOIL to simplify the expression.

1. Multiply the first terms of each binomial.
2. Multiply the outer terms of the binomials.
3. Multiply the inner terms of the binomials.
4. Multiply the last terms of each binomial.
5. Add the products.
6. Combine like terms and simplify.

Example:
Exercise:

Problem:
Using FOIL to Multiply Binomials

Use FOIL to find the product.

$(2x - 18)(3x + 3)$

Solution:

Find the product of the first terms.

$$2x - 18 \qquad 3x + 3 \qquad\qquad 2x \cdot 3x = 6x^2$$

Find the product of the outer terms.

$$2x - 18 \qquad 3x + 3 \qquad\qquad 2x \cdot 3 = 6x$$

Find the product of the inner terms.

$$2x - 18 \qquad 3x + 3 \qquad\qquad -18 \cdot 3x = -54x$$

Find the product of the last terms.

$$2x - 18 \qquad 3x + 3 \qquad\qquad -18 \cdot 3 = -54$$

$6x^2+6x-54x-54$	Add the products.
$6x^2+(6x-54x)-54$	Combine like terms.
$6x^2-48x-54$	Simplify.

Note:
Exercise:

Problem: Use FOIL to find the product.

$(x+7)(3x-5)$

Solution:

$3x^2+16x{-}35$

Perfect Square Trinomials

Certain binomial products have special forms. When a binomial is squared, the result is called a **perfect square trinomial**. We can find the square by multiplying the binomial by itself. However, there is a special form that each of these perfect square trinomials takes, and memorizing the form makes squaring binomials much easier and faster. Let's look at a few perfect square trinomials to familiarize ourselves with the form.
Equation:

$$\begin{aligned}(x+5)^2 &= x^2+10x+25\\ (x-3)^2 &= x^2-6x+9\\ (4x-1)^2 &= 16x^2-8x+1\end{aligned}$$

Notice that the first term of each trinomial is the square of the first term of the binomial and, similarly, the last term of each trinomial is the square of the last term of the binomial. The middle term is double the product of the two terms. Lastly, we see that the first sign of the trinomial is the same as the sign of the binomial.

Note:

Perfect Square Trinomials

When a binomial is squared, the result is the first term squared added to double the product of both terms and the last term squared.

Equation:

$$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2$$

Note:

Given a binomial, square it using the formula for perfect square trinomials.

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

Example:

Exercise:

Problem:

Expanding Perfect Squares

Expand $(3x-8)^2$.

Solution:

Begin by squaring the first term and the last term. For the middle term of the trinomial, double the product of the two terms.

Equation:

$$(3x)^2 - 2(3x)(8) + (-8)^2$$

Simplify

Equation:

$$9x^2 - 48x + 64.$$

Note:
Exercise:

Problem: Expand $(4x-1)^2$.

Solution:

$16x^2-8x+1$

Difference of Squares

Another special product is called the **difference of squares**, which occurs when we multiply a binomial by another binomial with the same terms but the opposite sign. Let's see what happens when we multiply $(x+1)(x-1)$ using the FOIL method.

Equation:

$$\begin{aligned}(x+1)(x-1) &= x^2-x+x-1\\ &= x^2-1\end{aligned}$$

The middle term drops out, resulting in a difference of squares. Just as we did with the perfect squares, let's look at a few examples.

Equation:

$$\begin{aligned}(x+5)(x-5) &= x^2-25\\ (x+11)(x-11) &= x^2-121\\ (2x+3)(2x-3) &= 4x^2-9\end{aligned}$$

Because the sign changes in the second binomial, the outer and inner terms cancel each other out, and we are left only with the square of the first term minus the square of the last term.

Note:
Is there a special form for the sum of squares?
No. The difference of squares occurs because the opposite signs of the binomials cause the middle terms to disappear. There are no two binomials that multiply to equal a sum of squares.

Note:
Difference of Squares
When a binomial is multiplied by a binomial with the same terms separated by the opposite sign, the result is the square of the first term minus the square of the last term.
Equation:

$$(a+b)(a-b)=a^2-b^2$$

Note:
Given a binomial multiplied by a binomial with the same terms but the opposite sign, find the difference of squares.

1. Square the first term of the binomials.
2. Square the last term of the binomials.
3. Subtract the square of the last term from the square of the first term.

Example:
Exercise:

Problem:
Multiplying Binomials Resulting in a Difference of Squares

Multiply $(9x+4)(9x-4)$.

Solution:

Square the first term to get $(9x)^2=81x^2$. Square the last term to get $4^2=16$. Subtract the square of the last term from the square of the first term to find the product of $81x^2-16$.

Note:
Exercise:

Problem: Multiply $(2x+7)(2x-7)$.

Solution:

$4x^2 - 49$

Performing Operations with Polynomials of Several Variables

We have looked at polynomials containing only one variable. However, a polynomial can contain several variables. All of the same rules apply when working with polynomials containing several variables. Consider an example:

Equation:

$$\begin{array}{ll} (a+2b)(4a-b-c) & \\ a(4a-b-c)+2b(4a-b-c) & \text{Use the distributive property.} \\ 4a^2-ab-ac+8ab-2b^2-2bc & \text{Multiply.} \\ 4a^2+(-ab+8ab)-ac-2b^2-2bc & \text{Combine like terms.} \\ 4a^2+7ab-ac-2bc-2b^2 & \text{Simplify.} \end{array}$$

Example:

Exercise:

Problem:

Multiplying Polynomials Containing Several Variables

Multiply $(x+4)(3x-2y+5)$.

Solution:

Follow the same steps that we used to multiply polynomials containing only one variable.

Equation:

$$\begin{array}{ll} x(3x-2y+5)+4(3x-2y+5) & \text{Use the distributive property.} \\ 3x^2-2xy+5x+12x-8y+20 & \text{Multiply.} \\ 3x^2-2xy+(5x+12x)-8y+20 & \text{Combine like terms.} \\ 3x^2-2xy+17x-8y+20 & \text{Simplify.} \end{array}$$

Note:
Exercise:

Problem: Multiply $(3x-1)(2x+7y-9)$.

Solution:

$6x^2+21xy–29x–7y+9$

Note:
Access these online resources for additional instruction and practice with polynomials.

- Adding and Subtracting Polynomials
- Multiplying Polynomials
- Special Products of Polynomials

Key Equations

perfect square trinomial	$(x+a)^2=(x+a)(x+a)=x^2+2ax+a^2$
difference of squares	$(a+b)(a-b)=a^2-b^2$

Key Concepts

- A polynomial is a sum of terms each consisting of a variable raised to a non-negative integer power. The degree is the highest power of the variable that occurs in the polynomial. The leading term is the term containing the highest degree, and the leading coefficient is the coefficient of that term. See [link].
- We can add and subtract polynomials by combining like terms. See [link] and [link].
- To multiply polynomials, use the distributive property to multiply each term in the first polynomial by each term in the second. Then add the products. See [link].

- FOIL (First, Outer, Inner, Last) is a shortcut that can be used to multiply binomials. See [link].
- Perfect square trinomials and difference of squares are special products. See [link] and [link].
- Follow the same rules to work with polynomials containing several variables. See [link].

Section Exercises

Verbal

Exercise:

Problem:

Evaluate the following statement: The degree of a polynomial in standard form is the exponent of the leading term. Explain why the statement is true or false.

Solution:

The statement is true. In standard form, the polynomial with the highest value exponent is placed first and is the leading term. The degree of a polynomial is the value of the highest exponent, which in standard form is also the exponent of the leading term.

Exercise:

Problem:

Many times, multiplying two binomials with two variables results in a trinomial. This is not the case when there is a difference of two squares. Explain why the product in this case is also a binomial.

Exercise:

Problem:

You can multiply polynomials with any number of terms and any number of variables using four basic steps over and over until you reach the expanded polynomial. What are the four steps?

Solution:

Use the distributive property, multiply, combine like terms, and simplify.

Exercise:

Problem:

State whether the following statement is true and explain why or why not: A trinomial is always a higher degree than a monomial.

Algebraic

For the following exercises, identify the degree of the polynomial.

Exercise:

Problem: $7x - 2x^2 + 13$

Solution:

2

Exercise:

Problem: $14m^3 + m^2 - 16m + 8$

Exercise:

Problem: $-625a^8 + 16b^4$

Solution:

8

Exercise:

Problem: $200p - 30p^2m + 40m^3$

Exercise:

Problem: $x^2 + 4x + 4$

Solution:

2

Exercise:

Problem: $6y^4 - y^5 + 3y - 4$

For the following exercises, find the sum or difference.

Exercise:

Problem: $\left(12x^2 + 3x\right) - \left(8x^2 - 19\right)$

Solution:

$4x^2 + 3x + 19$

Exercise:

Problem: $\left(4z^3 + 8z^2 - z\right) + \left(-2z^2 + z + 6\right)$

Exercise:

Problem: $\left(6w^2 + 24w + 24\right) - \left(3w^2 - 6w + 3\right)$

Solution:

$3w^2 + 30w + 21$

Exercise:

Problem: $\left(7a^3 + 6a^2 - 4a - 13\right) + \left(-3a^3 - 4a^2 + 6a + 17\right)$

Exercise:

Problem: $\left(11b^4 - 6b^3 + 18b^2 - 4b + 8\right) - \left(3b^3 + 6b^2 + 3b\right)$

Solution:

$11b^4 - 9b^3 + 12b^2 - 7b + 8$

Exercise:

Problem: $\left(49p^2 - 25\right) + \left(16p^4 - 32p^2 + 16\right)$

For the following exercises, find the product.

Exercise:

Problem: $(4x+2)(6x-4)$

Solution:

$24x^2 - 4x - 8$

Exercise:

Problem: $\left(14c^2+4c\right)\left(2c^2-3c\right)$

Exercise:

Problem: $\left(6b^2-6\right)\left(4b^2-4\right)$

Solution:

$24b^4 - 48b^2 + 24$

Exercise:

Problem: $(3d-5)(2d+9)$

Exercise:

Problem: $(9v-11)(11v-9)$

Solution:

$99v^2 - 202v + 99$

Exercise:

Problem: $\left(4t^2+7t\right)\left(-3t^2+4\right)$

Exercise:

Problem: $(8n-4)\left(n^2+9\right)$

Solution:

$8n^3 - 4n^2 + 72n - 36$

For the following exercises, expand the binomial.

Exercise:

Problem: $(4x+5)^2$

Exercise:

Problem: $(3y-7)^2$

Solution:

$9y^2-42y+49$

Exercise:

Problem: $(12-4x)^2$

Exercise:

Problem: $(4p+9)^2$

Solution:

$16p^2+72p+81$

Exercise:

Problem: $(2m-3)^2$

Exercise:

Problem: $(3y-6)^2$

Solution:

$9y^2-36y+36$

Exercise:

Problem: $(9b+1)^2$

For the following exercises, multiply the binomials.

Exercise:

Problem: $(4c+1)(4c-1)$

Solution:

$16c^2 - 1$

Exercise:

Problem: $(9a-4)(9a+4)$

Exercise:

Problem: $(15n-6)(15n+6)$

Solution:

$225n^2 - 36$

Exercise:

Problem: $(25b+2)(25b-2)$

Exercise:

Problem: $(4+4m)(4-4m)$

Solution:

$-16m^2 + 16$

Exercise:

Problem: $(14p+7)(14p-7)$

Exercise:

Problem: $(11q-10)(11q+10)$

Solution:

$121q^2 - 100$

For the following exercises, multiply the polynomials.

Exercise:

Problem: $\left(2x^2+2x+1\right)(4x-1)$

Exercise:

Problem: $\left(4t^2+t-7\right)\left(4t^2-1\right)$

Solution:

$16t^4+4t^3-32t^2-t+7$

Exercise:

Problem: $(x-1)\left(x^2-2x+1\right)$

Exercise:

Problem: $(y-2)\left(y^2-4y-9\right)$

Solution:

y^3-6y^2-y+18

Exercise:

Problem: $(6k-5)\left(6k^2+5k-1\right)$

Exercise:

Problem: $\left(3p^2+2p-10\right)(p-1)$

Solution:

$3p^3-p^2-12p+10$

Exercise:

Problem: $(4m-13)\left(2m^2-7m+9\right)$

Exercise:

Problem: $(a+b)(a-b)$

Solution:

$a^2 - b^2$

Exercise:

Problem: $(4x - 6y)(6x - 4y)$

Exercise:

Problem: $(4t - 5u)^2$

Solution:

$16t^2 - 40tu + 25u^2$

Exercise:

Problem: $(9m + 4n - 1)(2m + 8)$

Exercise:

Problem: $(4t - x)(t - x + 1)$

Solution:

$4t^2 + x^2 + 4t - 5tx - x$

Exercise:

Problem: $\left(b^2 - 1\right)\left(a^2 + 2ab + b^2\right)$

Exercise:

Problem: $(4r - d)(6r + 7d)$

Solution:

$24r^2 + 22rd - 7d^2$

Exercise:

Problem: $(x + y)\left(x^2 - xy + y^2\right)$

Real-World Applications

Exercise:

Problem:

A developer wants to purchase a plot of land to build a house. The area of the plot can be described by the following expression: $(4x+1)(8x-3)$ where *x* is measured in meters. Multiply the binomials to find the area of the plot in standard form.

Solution:

$32x^2 - 4x - 3 \text{ m}^2$

Exercise:

Problem:

A prospective buyer wants to know how much grain a specific silo can hold. The area of the floor of the silo is $(2x+9)^2$. The height of the silo is $10x+10$,where *x* is measured in feet. Expand the square and multiply by the height to find the expression that shows how much grain the silo can hold.

Extensions

For the following exercises, perform the given operations.

Exercise:

Problem: $(4t-7)^2(2t+1)-\left(4t^2+2t+11\right)$

Solution:

$32t^3-100t^2+40t+38$

Exercise:

Problem: $(3b+6)(3b-6)\left(9b^2-36\right)$

Exercise:

Problem: $\left(a^2+4ac+4c^2\right)\left(a^2-4c^2\right)$

Solution:

$a^4 + 4a^3c - 16ac^3 - 16c^4$

Glossary

binomial
: a polynomial containing two terms

coefficient
: any real number a_i in a polynomial in the form $a_nx^n + ... + a_2x^2 + a_1x + a_0$

degree
: the highest power of the variable that occurs in a polynomial

difference of squares
: the binomial that results when a binomial is multiplied by a binomial with the same terms, but the opposite sign

leading coefficient
: the coefficient of the leading term

leading term
: the term containing the highest degree

monomial
: a polynomial containing one term

perfect square trinomial
: the trinomial that results when a binomial is squared

polynomial
: a sum of terms each consisting of a variable raised to a nonnegative integer power

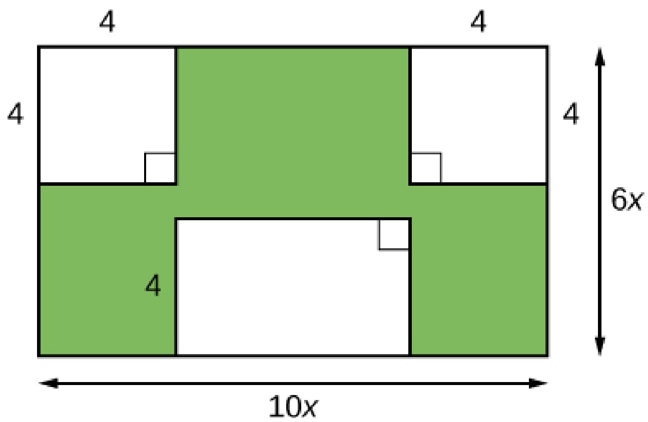
term of a polynomial
: any a_ix^i of a polynomial in the form $a_nx^n + ... + a_2x^2 + a_1x + a_0$

trinomial
: a polynomial containing three terms

1.5 Factoring Polynomials
In this section students will:

- Factor the greatest common factor of a polynomial.
- Factor a trinomial.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor the sum and difference of cubes.
- Factor expressions using fractional or negative exponents.

Imagine that we are trying to find the area of a lawn so that we can determine how much grass seed to purchase. The lawn is the green portion in [link].



The area of the entire region can be found using the formula for the area of a rectangle.
Equation:

$$\begin{aligned} A &= lw \\ &= 10x \cdot 6x \\ &= 60x^2 \text{ units}^2 \end{aligned}$$

The areas of the portions that do not require grass seed need to be subtracted from the area of the entire region. The two square regions each have an area of $A = s^2 = 4^2 = 16$ units^2. The other rectangular region has one side of length $10x - 8$ and one side of length 4,giving an area of $A = lw = 4(10x - 8) = 40x - 32$ units^2. So the region that must be subtracted has an area of $2(16) + 40x - 32 = 40x$ units^2.

The area of the region that requires grass seed is found by subtracting $60x^2 - 40x$ units^2. This area can also be expressed in factored form as $20x(3x - 2)$ units^2. We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

Factoring the Greatest Common Factor of a Polynomial

When we study fractions, we learn that the **greatest common factor** (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Note:
Greatest Common Factor
The **greatest common factor** (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.

Note:
Given a polynomial expression, factor out the greatest common factor.

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

Example:
Exercise:

Problem:
Factoring the Greatest Common Factor

Factor $6x^3y^3 + 45x^2y^2 + 21xy$.

Solution:

First, find the GCF of the expression. The GCF of $6, 45,$and 21 is 3. The GCF of x^3, x^2, and x is x. (Note that the GCF of a set of expressions in the form x^n will always be the

exponent of lowest degree.) And the GCF of y^3, y^2, and y is y. Combine these to find the GCF of the polynomial, $3xy$.

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that $3xy\left(2x^2y^2\right) = 6x^3y^3$, $3xy(15xy) = 45x^2y^2$, and $3xy(7) = 21xy$.

Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.
Equation:

$$(3xy)\left(2x^2y^2 + 15xy + 7\right)$$

Analysis

After factoring, we can check our work by multiplying. Use the distributive property to confirm that $(3xy)\left(2x^2y^2 + 15xy + 7\right) = 6x^3y^3 + 45x^2y^2 + 21xy$.

Note:
Exercise:

Problem: Factor $x\left(b^2 - a\right) + 6\left(b^2 - a\right)$ by pulling out the GCF.

Solution:

$(b^2 - a)(x + 6)$

Factoring a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial $x^2 + 5x + 6$ has a GCF of 1, but it can be written as the product of the factors $(x + 2)$ and $(x + 3)$.

Trinomials of the form $x^2 + bx + c$ can be factored by finding two numbers with a product of c and a sum of b. The trinomial $x^2 + 10x + 16$, for example, can be factored using the numbers 2 and 8 because the product of those numbers is 16 and their sum is 10. The trinomial can be rewritten as the product of $(x + 2)$ and $(x + 8)$.

Note:
Factoring a Trinomial with Leading Coefficient 1
A trinomial of the form $x^2 + bx + c$ can be written in factored form as $(x + p)(x + q)$ where $pq = c$ and $p + q = b$.

Note:
Can every trinomial be factored as a product of binomials?
No. Some polynomials cannot be factored. These polynomials are said to be prime.

Note:
Given a trinomial in the form $x^2 + bx + c$, factor it.

1. List factors of c.
2. Find p and q, a pair of factors of c with a sum of b.
3. Write the factored expression $(x + p)(x + q)$.

Example:
Exercise:

Problem:
Factoring a Trinomial with Leading Coefficient 1

Factor $x^2 + 2x - 15$.

Solution:

We have a trinomial with leading coefficient 1, $b = 2$, and $c = -15$. We need to find two numbers with a product of -15 and a sum of 2. In [link], we list factors until we find a pair with the desired sum.

Factors of -15	Sum of Factors
$1, -15$	-14

Factors of -15	Sum of Factors
$-1, 15$	14
$3, -5$	-2
$-3, 5$	2

Now that we have identified p and q as -3 and 5, write the factored form as $(x-3)(x+5)$.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(x-3)(x+5)=x^2+2x-15$.

Note:
Does the order of the factors matter?
No. Multiplication is commutative, so the order of the factors does not matter.

Note:
Exercise:

Problem: Factor x^2-7x+6.

Solution:

$(x–6)(x–1)$

Factoring by Grouping

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can **factor by grouping** by dividing the *x* term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial $2x^2+5x+3$ can be rewritten as $(2x+3)(x+1)$ using this process. We begin by rewriting the original expression as $2x^2+2x+3x+3$ and then factor each portion of the expression to obtain $2x(x+1)+3(x+1)$. We then pull out the GCF of $(x+1)$ to find the factored expression.

Note:
Factor by Grouping
To factor a trinomial in the form $ax^2 + bx + c$ by grouping, we find two numbers with a product of ac and a sum of b. We use these numbers to divide the x term into the sum of two terms and factor each portion of the expression separately, then factor out the GCF of the entire expression.

Note: Given a trinomial in the form $ax^2 + bx + c$, factor by grouping.

1. List factors of ac.
2. Find p and q, a pair of factors of ac with a sum of b.
3. Rewrite the original expression as $ax^2 + px + qx + c$.
4. Pull out the GCF of $ax^2 + px$.
5. Pull out the GCF of $qx + c$.
6. Factor out the GCF of the expression.

Example:
Exercise:

Problem:
Factoring a Trinomial by Grouping

Factor $5x^2 + 7x - 6$ by grouping.

Solution:

We have a trinomial with $a = 5, b = 7$, and $c = -6$. First, determine $ac = -30$. We need to find two numbers with a product of -30 and a sum of 7. In [link], we list factors until we find a pair with the desired sum.

Factors of -30	**Sum of Factors**
$1, -30$	-29
$-1, 30$	29

Factors of -30	Sum of Factors
$2, -15$	-13
$-2, 15$	13
$3, -10$	-7
$-3, 10$	7

So $p = -3$ and $q = 10$.
Equation:

$$\begin{array}{ll} 5x^2 - 3x + 10x - 6 & \text{Rewrite the original expression as } ax^2 + px + qx + c. \\ x(5x - 3) + 2(5x - 3) & \text{Factor out the GCF of each part.} \\ (5x - 3)(x + 2) & \text{Factor out the GCF of the expression.} \end{array}$$

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(5x - 3)(x + 2) = 5x^2 + 7x - 6$.

Note:
Exercise:

Problem: Factor a. $2x^2 + 9x + 9$ b. $6x^2 + x - 1$

Solution:

a. $(2x + 3)(x + 3)$ b. $(3x - 1)(2x + 1)$

Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.
Equation:

$$
\begin{array}{rcl}
a^2 + 2ab + b^2 & = & (a+b)^2 \\
& \text{and} & \\
a^2 - 2ab + b^2 & = & (a-b)^2
\end{array}
$$

We can use this equation to factor any perfect square trinomial.

Note:
Perfect Square Trinomials
A perfect square trinomial can be written as the square of a binomial:
Equation:

$$a^2 + 2ab + b^2 = (a+b)^2$$

Note:
Given a perfect square trinomial, factor it into the square of a binomial.

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of ab.
3. Write the factored form as $(a+b)^2$.

Example:
Exercise:

Problem:
Factoring a Perfect Square Trinomial

Factor $25x^2 + 20x + 4$.

Solution:

Notice that $25x^2$ and 4 are perfect squares because $25x^2 = (5x)^2$ and $4 = 2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2) = 20x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5x+2)^2$.

Note:
Exercise:

Problem: Factor $49x^2 - 14x + 1$.

Solution:

$(7x-1)^2$

Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.
Equation:

$$a^2 - b^2 = (a+b)(a-b)$$

We can use this equation to factor any differences of squares.

Note:
Differences of Squares
A difference of squares can be rewritten as two factors containing the same terms but opposite signs.
Equation:

$$a^2 - b^2 = (a+b)(a-b)$$

Note:
Given a difference of squares, factor it into binomials.

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a+b)(a-b)$.

Example:

Exercise:

Problem:
Factoring a Difference of Squares

Factor $9x^2 - 25.$

Solution:

Notice that $9x^2$ and 25 are perfect squares because $9x^2 = (3x)^2$ and $25 = 5^2.$ The polynomial represents a difference of squares and can be rewritten as $(3x + 5)(3x - 5).$

Note:
Exercise:

Problem: Factor $81y^2 - 100.$

Solution:

$(9y + 10)(9y - 10)$

Note:
Is there a formula to factor the sum of squares?
No. A sum of squares cannot be factored.

Factoring the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.
Equation:

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$

Similarly, the sum of cubes can be factored into a binomial and a trinomial, but with different signs.
Equation:

$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: **S**ame **O**pposite **A**lways **P**ositive. For example, consider the following example.

Equation:

$$x^3 - 2^3 = (x - 2)\left(x^2 + 2x + 4\right)$$

The sign of the first 2 is the *same* as the sign between $x^3 - 2^3.$ The sign of the $2x$ term is *opposite* the sign between $x^3 - 2^3.$ And the sign of the last term, 4, is *always positive*.

Note:
Sum and Difference of Cubes
We can factor the sum of two cubes as
Equation:

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$

We can factor the difference of two cubes as
Equation:

$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

Note:
Given a sum of cubes or difference of cubes, factor it.

1. Confirm that the first and last term are cubes, $a^3 + b^3$ or $a^3 - b^3$.
2. For a sum of cubes, write the factored form as $(a + b)\left(a^2 - ab + b^2\right).$ For a difference of cubes, write the factored form as $(a - b)\left(a^2 + ab + b^2\right).$

Example:
Exercise:

Problem:
Factoring a Sum of Cubes

Factor $x^3 + 512$.

Solution:

Notice that x^3 and 512 are cubes because $8^3 = 512$. Rewrite the sum of cubes as $(x+8)\left(x^2 - 8x + 64\right)$.

Analysis

After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.

Note:
Exercise:

Problem: Factor the sum of cubes: $216a^3 + b^3$.

Solution:

$(6a + b)(36a^2{-}6ab + b^2)$

Example:
Exercise:

Problem:
Factoring a Difference of Cubes

Factor $8x^3 - 125$.

Solution:

Notice that $8x^3$ and 125 are cubes because $8x^3 = (2x)^3$ and $125 = 5^3$. Write the difference of cubes as $(2x - 5)\left(4x^2 + 10x + 25\right)$.

Analysis

Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

Note:

Exercise:

Problem: Factor the difference of cubes: $1{,}000x^3 - 1$.

Solution:

$(10x - 1)\left(100x^2 + 10x + 1\right)$

Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2x^{\frac{1}{4}} + 5x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and being rewritten as $x^{\frac{1}{4}}\left(2 + 5x^{\frac{1}{2}}\right)$.

Example:
Exercise:

Problem:
Factoring an Expression with Fractional or Negative Exponents

Factor $3x(x+2)^{\frac{-1}{3}} + 4(x+2)^{\frac{2}{3}}$.

Solution:

Factor out the term with the lowest value of the exponent. In this case, that would be $(x+2)^{-\frac{1}{3}}$.
Equation:

$$\begin{array}{ll} (x+2)^{-\frac{1}{3}}(3x + 4(x+2)) & \text{Factor out the GCF.} \\ (x+2)^{-\frac{1}{3}}(3x + 4x + 8) & \text{Simplify.} \\ (x+2)^{-\frac{1}{3}}(7x + 8) & \end{array}$$

Note:
Exercise:

Problem: Factor $2(5a-1)^{\frac{3}{4}}+7a(5a-1)^{-\frac{1}{4}}$.

Solution:

$(5a–1)^{-\frac{1}{4}}(17a–2)$

Note:
Access these online resources for additional instruction and practice with factoring polynomials.

- Identify GCF
- Factor Trinomials when a Equals 1
- Factor Trinomials when a is not equal to 1
- Factor Sum or Difference of Cubes

Key Equations

difference of squares	$a^2-b^2=(a+b)(a-b)$
perfect square trinomial	$a^2+2ab+b^2=(a+b)^2$
sum of cubes	$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$
difference of cubes	$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$

- The greatest common factor, or GCF, can be factored out of a polynomial. Checking for a GCF should be the first step in any factoring problem. See [link].
- Trinomials with leading coefficient 1 can be factored by finding numbers that have a product of the third term and a sum of the second term. See [link].
- Trinomials can be factored using a process called factoring by grouping. See [link].
- Perfect square trinomials and the difference of squares are special products and can be factored using equations. See [link] and [link].

- The sum of cubes and the difference of cubes can be factored using equations. See [link] and [link].
- Polynomials containing fractional and negative exponents can be factored by pulling out a GCF. See [link].

Verbal

Exercise:

Problem:

If the terms of a polynomial do not have a GCF, does that mean it is not factorable? Explain.

Solution:

The terms of a polynomial do not have to have a common factor for the entire polynomial to be factorable. For example, $4x^2$ and $-9y^2$ don't have a common factor, but the whole polynomial is still factorable: $4x^2 - 9y^2 = (2x + 3y)(2x - 3y)$.

Exercise:

Problem:

A polynomial is factorable, but it is not a perfect square trinomial or a difference of two squares. Can you factor the polynomial without finding the GCF?

Exercise:

Problem: How do you factor by grouping?

Solution:

Divide the x term into the sum of two terms, factor each portion of the expression separately, and then factor out the GCF of the entire expression.

Algebraic

For the following exercises, find the greatest common factor.

Exercise:

Problem: $14x + 4xy - 18xy^2$

Exercise:

Problem: $49mb^2 - 35m^2ba + 77ma^2$

Solution:

$7m$

Exercise:

Problem: $30x^3y - 45x^2y^2 + 135xy^3$

Exercise:

Problem: $200p^3m^3 - 30p^2m^3 + 40m^3$

Solution:

$10m^3$

Exercise:

Problem: $36j^4k^2 - 18j^3k^3 + 54j^2k^4$

Exercise:

Problem: $6y^4 - 2y^3 + 3y^2 - y$

Solution:

y

For the following exercises, factor by grouping.

Exercise:

Problem: $6x^2 + 5x - 4$

Exercise:

Problem: $2a^2 + 9a - 18$

Solution:

$(2a-3)(a+6)$

Exercise:

Problem: $6c^2 + 41c + 63$

Exercise:

Problem: $6n^2 - 19n - 11$

Solution:

$(3n-11)(2n+1)$

Exercise:

Problem: $20w^2 - 47w + 24$

Exercise:

Problem: $2p^2 - 5p - 7$

Solution:

$(p+1)(2p-7)$

For the following exercises, factor the polynomial.

Exercise:

Problem: $7x^2 + 48x - 7$

Exercise:

Problem: $10h^2 - 9h - 9$

Solution:

$(5h+3)(2h-3)$

Exercise:

Problem: $2b^2 - 25b - 247$

Exercise:

Problem: $9d^2 - 73d + 8$

Solution:

$(9d-1)(d-8)$

Exercise:

Problem:$90v^2-181v+90$

Exercise:

Problem:$12t^2+t-13$

Solution:

$(12t+13)(t-1)$

Exercise:

Problem:$2n^2-n-15$

Exercise:

Problem:$16x^2-100$

Solution:

$(4x+10)(4x-10)$

Exercise:

Problem:$25y^2-196$

Exercise:

Problem:$121p^2-169$

Solution:

$(11p+13)(11p-13)$

Exercise:

Problem:$4m^2-9$

Exercise:

Problem:$361d^2-81$

Solution:

$(19d+9)(19d-9)$

Exercise:

Problem:$324x^2-121$

Exercise:

Problem:$144b^2-25c^2$

Solution:

$(12b+5c)(12b-5c)$

Exercise:

Problem:$16a^2-8a+1$

Exercise:

Problem:$49n^2+168n+144$

Solution:

$(7n+12)^2$

Exercise:

Problem:$121x^2-88x+16$

Exercise:

Problem:$225y^2+120y+16$

Solution:

$(15y+4)^2$

Exercise:

Problem:$m^2-20m+100$

Exercise:

Problem:$25p^2-120m+144$

Solution:

$(5p-12)^2$

Exercise:

Problem:$36q^2+60q+25$

For the following exercises, factor the polynomials.

Exercise:

Problem:x^3+216

Solution:

$(x+6)\left(x^2-6x+36\right)$

Exercise:

Problem:$27y^3-8$

Exercise:

Problem:$125a^3+343$

Solution:

$(5a+7)\left(25a^2-35a+49\right)$

Exercise:

Problem:b^3-8d^3

Exercise:

Problem:$64x^3{-}125$

Solution:

$(4x-5)\left(16x^2+20x+25\right)$

Exercise:

Problem:$729q^3+1331$

Exercise:

Problem:$125r^3+1{,}728s^3$

Solution:

$\left(5r+12s\right)\left(25r^{2}-60rs+144s^{2}\right)$

Exercise:

Problem: $4x(x-1)^{-\frac{2}{3}}+3(x-1)^{\frac{1}{3}}$

Exercise:

Problem: $3c(2c+3)^{-\frac{1}{4}}-5(2c+3)^{\frac{3}{4}}$

Solution:

$\left(2c+3\right)^{-\frac{1}{4}}\left(-7c-15\right)$

Exercise:

Problem: $3t(10t+3)^{\frac{1}{3}}+7(10t+3)^{\frac{4}{3}}$

Exercise:

Problem: $14x(x+2)^{-\frac{2}{5}}+5(x+2)^{\frac{3}{5}}$

Solution:

$\left(x+2\right)^{-\frac{2}{5}}\left(19x+10\right)$

Exercise:

Problem: $9y(3y-13)^{\frac{1}{5}}-2(3y-13)^{\frac{6}{5}}$

Exercise:

Problem: $5z(2z-9)^{-\frac{3}{2}}+11(2z-9)^{-\frac{1}{2}}$

Solution:

$\left(2z-9\right)^{-\frac{3}{2}}\left(27z-99\right)$

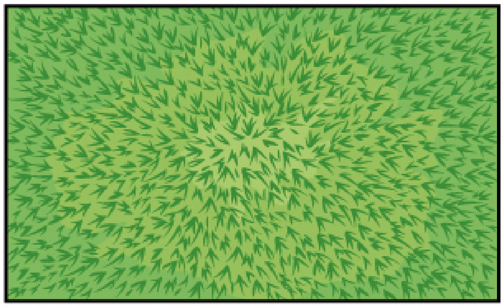
Exercise:

Problem: $6d(2d+3)^{-\frac{1}{6}}+5(2d+3)^{\frac{5}{6}}$

Real-World Applications

For the following exercises, consider this scenario:

Charlotte has appointed a chairperson to lead a city beautification project. The first act is to install statues and fountains in one of the city's parks. The park is a rectangle with an area of $98x^2 + 105x - 27\,\text{m}^2$, as shown in the figure below. The length and width of the park are perfect factors of the area.



$l \times w = 98x^2 + 105x - 27$

Exercise:

Problem: Factor by grouping to find the length and width of the park.

Solution:

$(14x-3)\,(7x+9)$

Exercise:

Problem:

A statue is to be placed in the center of the park. The area of the base of the statue is $4x^2 + 12x + 9\text{m}^2.$ Factor the area to find the lengths of the sides of the statue.

Exercise:

Problem:

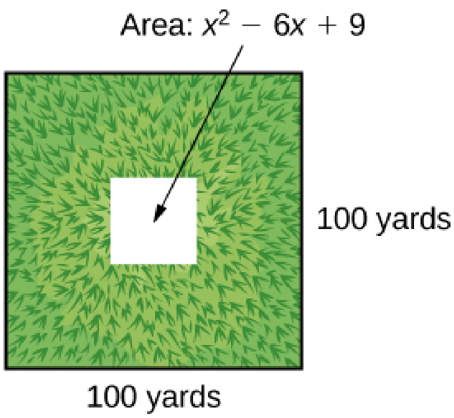
At the northwest corner of the park, the city is going to install a fountain. The area of the base of the fountain is $9x^2 - 25\text{m}^2.$ Factor the area to find the lengths of the sides of the fountain.

Solution:

$(3x+5)\,(3x-5)$

For the following exercise, consider the following scenario:

A school is installing a flagpole in the central plaza. The plaza is a square with side length 100 yd. as shown in the figure below. The flagpole will take up a square plot with area $x^2 - 6x + 9\text{yd}^2$.



Exercise:

Problem: Find the length of the base of the flagpole by factoring.

Extensions

For the following exercises, factor the polynomials completely.

Exercise:

Problem: $16x^4 - 200x^2 + 625$

Solution:

$(2x + 5)^2(2x - 5)^2$

Exercise:

Problem: $81y^4 - 256$

Exercise:

Problem: $16z^4 - 2{,}401a^4$

Solution:

$\left(4z^2 + 49a^2\right)(2z + 7a)(2z - 7a)$

Exercise:

Problem: $5x(3x+2)^{-\frac{2}{4}}+(12x+8)^{\frac{3}{2}}$

Exercise:

Problem: $\left(32x^3+48x^2-162x-243\right)^{-1}$

Solution:

$\frac{1}{(4x+9)(4x-9)(2x+3)}$

Glossary

factor by grouping
: a method for factoring a trinomial in the form ax^2+bx+c by dividing the *x* term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression

greatest common factor
: the largest polynomial that divides evenly into each polynomial

1.6 Rational Expressions
In this section students will:

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

A pastry shop has fixed costs of $\$280$ per week and variable costs of $\$9$ per box of pastries. The shop's costs per week in terms of x,the number of boxes made, is $280 + 9x.$ We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.
Equation:

$$\frac{280 + 9x}{x}$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a **rational expression**. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.
Equation:

$$\frac{x^2 + 8x + 16}{x^2 + 11x + 28}$$

We can factor the numerator and denominator to rewrite the expression.
Equation:

$$\frac{(x + 4)^2}{(x + 4)(x + 7)}$$

Then we can simplify that expression by canceling the common factor $(x + 4)$.
Equation:

$$\frac{x + 4}{x + 7}$$

Note:

Given a rational expression, simplify it.

1. Factor the numerator and denominator.
2. Cancel any common factors.

Example:
Exercise:

Problem:
Simplifying Rational Expressions

Simplify $\frac{x^2-9}{x^2+4x+3}$.

Solution:
Equation:

$$\frac{(x+3)(x-3)}{(x+3)(x+1)} \quad \text{Factor the numerator and the denominator.}$$
$$\frac{x-3}{x+1} \quad \text{Cancel common factor } (x+3).$$

Analysis

We can cancel the common factor because any expression divided by itself is equal to 1.

Note:
Can the x^2 term be cancelled in [link]?
No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.

Note:
Exercise:

Problem: Simplify $\frac{x-6}{x^2-36}$.

Solution:

$\frac{1}{x+6}$

Multiplying Rational Expressions

Multiplication of rational expressions works the same way as multiplication of any other fractions. We multiply the numerators to find the numerator of the product, and then multiply the denominators to find the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators just as we did when simplifying rational expressions. We are often able to simplify the product of rational expressions.

Note:
Given two rational expressions, multiply them.

1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

Example:
Exercise:

Problem:
Multiplying Rational Expressions

Multiply the rational expressions and show the product in simplest form:
Equation:

$$\frac{(x+5)(x-1)}{3(x+6)}\cdot\frac{(2x-1)}{(x+5)}$$

Solution:
Equation:

$$\frac{(x+5)(x-1)}{3(x+6)}\cdot\frac{(2x-1)}{(x+5)} \quad \text{Factor the numerator and denominator.}$$

$$\frac{(x+5)(x-1)(2x-1)}{3(x+6)(x+5)} \quad \text{Multiply numerators and denominators.}$$

$$\frac{\cancel{(x+5)}\,(x-1)(2x-1)}{3(x+6)\,\cancel{(x+5)}} \quad \text{Cancel common factors to simplify.}$$

$$\frac{(x-1)(2x-1)}{3(x+6)}$$

Note:
Exercise:

Problem: Multiply the rational expressions and show the product in simplest form:
Equation:

$$\frac{x^2+11x+30}{x^2+5x+6}\cdot\frac{x^2+7x+12}{x^2+8x+16}$$

Solution:

$\frac{(x+5)(x+6)}{(x+2)(x+4)}$

Dividing Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite $\frac{1}{x}\div\frac{x^2}{3}$ as the product $\frac{1}{x}\cdot\frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.
Equation:

$$\frac{1}{x}\cdot\frac{3}{x^2}=\frac{3}{x^3}$$

Note:
Given two rational expressions, divide them.

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.
4. Multiply the denominators.
5. Simplify.

Example:
Exercise:

Problem:
Dividing Rational Expressions

Divide the rational expressions and express the quotient in simplest form:
Equation:

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

Solution:
Equation:

$$\frac{9x^2-16}{3x^2+17x-28} \div \frac{3x^2-2x-8}{x^2+5x-14}$$

Note:
Exercise:

Problem: Divide the rational expressions and express the quotient in simplest form:
Equation:

$$\frac{9x^2-16}{3x^2+17x-28} \div \frac{3x^2-2x-8}{x^2+5x-14}$$

Solution:

1

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.
Equation:

$$\begin{aligned} \tfrac{5}{24}+\tfrac{1}{40} &= \tfrac{25}{120}+\tfrac{3}{120} \\ &= \tfrac{28}{120} \\ &= \tfrac{7}{30} \end{aligned}$$

We have to rewrite the fractions so they share a common denominator before we are able to add. We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance,

if the factored denominators were $(x+3)(x+4)$ and $(x+4)(x+5)$,then the LCD would be $(x+3)(x+4)(x+5)$.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of $(x+3)(x+4)$ by $\frac{x+5}{x+5}$ and the expression with a denominator of $(x+4)(x+5)$ by $\frac{x+3}{x+3}$.

Note:
Given two rational expressions, add or subtract them.

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

Example:
Exercise:

Problem:
Adding Rational Expressions

Add the rational expressions:
Equation:

$$\frac{5}{x}+\frac{6}{y}$$

Solution:

First, we have to find the LCD. In this case, the LCD will be $xy.$ We then multiply each expression by the appropriate form of 1 to obtain xy as the denominator for each fraction.
Equation:

$$\begin{array}{c}\frac{5}{x}\cdot\frac{y}{y}+\frac{6}{y}\cdot\frac{x}{x}\\ \frac{5y}{xy}+\frac{6x}{xy}\end{array}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.
Equation:

$$\frac{6x+5y}{xy}$$

Analysis

Multiplying by $\frac{y}{y}$ or $\frac{x}{x}$ does not change the value of the original expression because any number divided by itself is 1, and multiplying an expression by 1 gives the original expression.

Example:
Exercise:

Problem:
Subtracting Rational Expressions

Subtract the rational expressions:
Equation:

$$\frac{6}{x^2+4x+4}-\frac{2}{x^2-4}$$

Solution:
Equation:

$\frac{6}{(x+2)^2}-\frac{2}{(x+2)(x-2)}$	Factor.
$\frac{6}{(x+2)^2}\cdot\frac{x-2}{x-2}-\frac{2}{(x+2)(x-2)}\cdot\frac{x+2}{x+2}$	Multiply each fraction to get LCD as denominator.
$\frac{6(x-2)}{(x+2)^2(x-2)}-\frac{2(x+2)}{(x+2)^2(x-2)}$	Multiply.
$\frac{6x-12-(2x+4)}{(x+2)^2(x-2)}$	Apply distributive property.
$\frac{4x-16}{(x+2)^2(x-2)}$	Subtract.
$\frac{4(x-4)}{(x+2)^2(x-2)}$	Simplify.

Note:
Do we have to use the LCD to add or subtract rational expressions?
No. Any common denominator will work, but it is easiest to use the LCD.

Note:
Exercise:

Problem: Subtract the rational expressions: $\frac{3}{x+5} - \frac{1}{x-3}$.

Solution:

$\frac{2(x-7)}{(x+5)(x-3)}$

Simplifying Complex Rational Expressions

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression $\frac{a}{\frac{1}{b}+c}$ can be simplified by rewriting the numerator as the fraction $\frac{a}{1}$ and combining the expressions in the denominator as $\frac{1+bc}{b}$. We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get $\frac{a}{1} \cdot \frac{b}{1+bc}$, which is equal to $\frac{ab}{1+bc}$.

Note:
Given a complex rational expression, simplify it.

1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.
5. Multiply.
6. Simplify.

Example:
Exercise:

Problem:
Simplifying Complex Rational Expressions

Simplify: $\frac{y+\frac{1}{x}}{\frac{x}{y}}$.

Solution:

Begin by combining the expressions in the numerator into one expression.

Equation:

$$y \cdot \frac{x}{x} + \frac{1}{x} \qquad \text{Multiply by } \frac{x}{x} \text{ to get LCD as denominator.}$$
$$\frac{xy}{x} + \frac{1}{x}$$
$$\frac{xy+1}{x} \qquad \text{Add numerators.}$$

Now the numerator is a single rational expression and the denominator is a single rational expression.

Equation:

$$\frac{\frac{xy+1}{x}}{\frac{x}{y}}$$

We can rewrite this as division, and then multiplication.

Equation:

$$\frac{xy+1}{x} \div \frac{x}{y}$$
$$\frac{xy+1}{x} \cdot \frac{y}{x} \qquad \text{Rewrite as multiplication.}$$
$$\frac{y(xy+1)}{x^2} \qquad \text{Multiply.}$$

Note:

Exercise:

Problem: Simplify: $\frac{\frac{x}{y} - \frac{y}{x}}{y}$

Solution:

$\frac{x^2-y^2}{xy^2}$

Note:

Can a complex rational expression always be simplified?

Yes. We can always rewrite a complex rational expression as a simplified rational expression.

Note:

Access these online resources for additional instruction and practice with rational expressions.

Key Concepts

- Rational expressions can be simplified by cancelling common factors in the numerator and denominator. See [link].
- We can multiply rational expressions by multiplying the numerators and multiplying the denominators. See [link].
- To divide rational expressions, multiply by the reciprocal of the second expression. See [link].
- Adding or subtracting rational expressions requires finding a common denominator. See [link] and [link].
- Complex rational expressions have fractions in the numerator or the denominator. These expressions can be simplified. See [link].

Section Exercises

Verbal

Exercise:

Problem: How can you use factoring to simplify rational expressions?

Solution:

You can factor the numerator and denominator to see if any of the terms can cancel one another out.

Exercise:

Problem: How do you use the LCD to combine two rational expressions?

Exercise:

Problem:

Tell whether the following statement is true or false and explain why: You only need to find the LCD when adding or subtracting rational expressions.

Solution:

True. Multiplication and division do not require finding the LCD because the denominators can be combined through those operations, whereas addition and subtraction require like terms.

Algebraic

For the following exercises, simplify the rational expressions.

Exercise:

Problem: $\frac{x^2-16}{x^2-5x+4}$

Exercise:

Problem: $\frac{y^2+10y+25}{y^2+11y+30}$

Solution:

$\frac{y+5}{y+6}$

Exercise:

Problem: $\frac{6a^2-24a+24}{6a^2-24}$

Exercise:

Problem: $\frac{9b^2+18b+9}{3b+3}$

Solution:

$3b+3$

Exercise:

Problem: $\frac{m-12}{m^2-144}$

Exercise:

Problem: $\frac{2x^2+7x-4}{4x^2+2x-2}$

Solution:

$\frac{x+4}{2x+2}$

Exercise:

Problem: $\frac{6x^2+5x-4}{3x^2+19x+20}$

Exercise:

Problem: $\frac{a^2+9a+18}{a^2+3a-18}$

Solution:

$\frac{a+3}{a-3}$

Exercise:

Problem: $\frac{3c^2+25c-18}{3c^2-23c+14}$

Exercise:

Problem: $\frac{12n^2-29n-8}{28n^2-5n-3}$

Solution:

$\frac{3n-8}{7n-3}$

For the following exercises, multiply the rational expressions and express the product in simplest form.

Exercise:

Problem: $\frac{x^2-x-6}{2x^2+x-6} \cdot \frac{2x^2+7x-15}{x^2-9}$

Exercise:

Problem: $\frac{c^2+2c-24}{c^2+12c+36} \cdot \frac{c^2-10c+24}{c^2-8c+16}$

Solution:

$\frac{c-6}{c+6}$

Exercise:

Problem: $\frac{2d^2+9d-35}{d^2+10d+21} \cdot \frac{3d^2+2d-21}{3d^2+14d-49}$

Exercise:

Problem: $\frac{10h^2-9h-9}{2h^2-19h+24} \cdot \frac{h^2-16h+64}{5h^2-37h-24}$

Solution:

1

Exercise:

Problem: $\frac{6b^2+13b+6}{4b^2-9} \cdot \frac{6b^2+31b-30}{18b^2-3b-10}$

Exercise:

Problem: $\frac{2d^2+15d+25}{4d^2-25} \cdot \frac{2d^2-15d+25}{25d^2-1}$

Solution:

$\frac{d^2-25}{25d^2-1}$

Exercise:

Problem: $\frac{6x^2-5x-50}{15x^2-44x-20} \cdot \frac{20x^2-7x-6}{2x^2+9x+10}$

Exercise:

Problem: $\frac{t^2-1}{t^2+4t+3} \cdot \frac{t^2+2t-15}{t^2-4t+3}$

Solution:

$\frac{t+5}{t+3}$

Exercise:

Problem: $\frac{2n^2-n-15}{6n^2+13n-5} \cdot \frac{12n^2-13n+3}{4n^2-15n+9}$

Exercise:

Problem: $\frac{36x^2-25}{6x^2+65x+50} \cdot \frac{3x^2+32x+20}{18x^2+27x+10}$

Solution:

$\frac{6x-5}{6x+5}$

For the following exercises, divide the rational expressions.

Exercise:

Problem: $\frac{3y^2-7y-6}{2y^2-3y-9} \div \frac{y^2+y-2}{2y^2+y-3}$

Exercise:

Problem: $\frac{6p^2+p-12}{8p^2+18p+9} \div \frac{6p^2-11p+4}{2p^2+11p-6}$

Solution:

$\frac{p+6}{4p+3}$

Exercise:

Problem: $\frac{q^2-9}{q^2+6q+9} \div \frac{q^2-2q-3}{q^2+2q-3}$

Exercise:

Problem: $\frac{18d^2+77d-18}{27d^2-15d+2} \div \frac{3d^2+29d-44}{9d^2-15d+4}$

Solution:

$\frac{2d+9}{d+11}$

Exercise:

Problem: $\frac{16x^2+18x-55}{32x^2-36x-11} \div \frac{2x^2+17x+30}{4x^2+25x+6}$

Exercise:

Problem: $\frac{144b^2-25}{72b^2-6b-10} \div \frac{18b^2-21b+5}{36b^2-18b-10}$

Solution:

$\frac{12b+5}{3b-1}$

Exercise:

Problem: $\frac{16a^2-24a+9}{4a^2+17a-15} \div \frac{16a^2-9}{4a^2+11a+6}$

Exercise:

Problem: $\frac{22y^2+59y+10}{12y^2+28y-5} \div \frac{11y^2+46y+8}{24y^2-10y+1}$

Solution:

$\frac{4y-1}{y+4}$

Exercise:

Problem: $\frac{9x^2+3x-20}{3x^2-7x+4} \div \frac{6x^2+4x-10}{x^2-2x+1}$

For the following exercises, add and subtract the rational expressions, and then simplify.

Exercise:

Problem: $\frac{4}{x} + \frac{10}{y}$

Solution:

$\frac{10x+4y}{xy}$

Exercise:

Problem: $\frac{12}{2q} - \frac{6}{3p}$

Exercise:

Problem: $\frac{4}{a+1} + \frac{5}{a-3}$

Solution:

$\frac{9a-7}{a^2-2a-3}$

Exercise:

Problem: $\frac{c+2}{3} - \frac{c-4}{4}$

Exercise:

Problem: $\frac{y+3}{y-2} + \frac{y-3}{y+1}$

Solution:

$\frac{2y^2-y+9}{y^2-y-2}$

Exercise:

Problem: $\frac{x-1}{x+1} - \frac{2x+3}{2x+1}$

Exercise:

Problem: $\frac{3z}{z+1} + \frac{2z+5}{z-2}$

Solution:

$\frac{5z^2+z+5}{z^2-z-2}$

Exercise:

Problem: $\frac{4p}{p+1} - \frac{p+1}{4p}$

Exercise:

Problem: $\frac{x}{x+1} + \frac{y}{y+1}$

Solution:

$\frac{x+2xy+y}{x+xy+y+1}$

For the following exercises, simplify the rational expression.

Exercise:

Problem: $\frac{\frac{6}{y}-\frac{4}{x}}{y}$

Exercise:

Problem: $\frac{\frac{2}{a}+\frac{7}{b}}{b}$

Solution:

$\frac{2b+7a}{ab^2}$

Exercise:

Problem: $\frac{\frac{x}{4}-\frac{p}{8}}{p}$

Exercise:

Problem: $\frac{\frac{3}{a}+\frac{b}{6}}{\frac{2b}{3a}}$

Solution:

$\frac{18+ab}{4b}$

Exercise:

Problem: $\frac{\frac{3}{x+1}+\frac{2}{x-1}}{\frac{x-1}{x+1}}$

Exercise:

Problem: $\frac{\frac{a}{b}-\frac{b}{a}}{\frac{a+b}{ab}}$

Solution:

$a-b$

Exercise:

Problem: $\frac{\frac{2x}{3}+\frac{4x}{7}}{\frac{x}{2}}$

Exercise:

Problem: $\dfrac{\frac{2c}{c+2}+\frac{c-1}{c+1}}{\frac{2c+1}{c+1}}$

Solution:

$\dfrac{3c^2+3c-2}{2c^2+5c+2}$

Exercise:

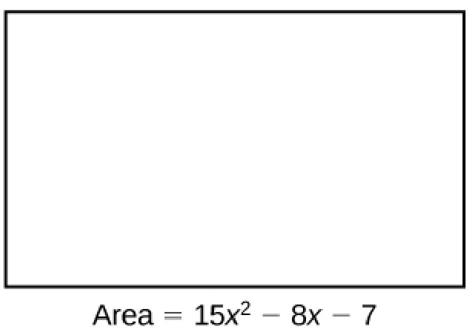
Problem: $\dfrac{\frac{x}{y}-\frac{y}{x}}{\frac{x}{y}+\frac{y}{x}}$

Real-World Applications

Exercise:

Problem:

Brenda is placing tile on her bathroom floor. The area of the floor is $15x^2 - 8x - 7$ ft^2. The area of one tile is $x^2 - 2x + 1\text{ft}^2$. To find the number of tiles needed, simplify the rational expression: $\frac{15x^2-8x-7}{x^2-2x+1}$.



Solution:

$\dfrac{15x+7}{x-1}$

Exercise:

Problem:

The area of Sandy's yard is $25x^2 - 625$ ft^2. A patch of sod has an area of $x^2 - 10x + 25$ ft^2. Divide the two areas and simplify to find how many pieces of sod Sandy needs to cover her yard.

Exercise:

Problem:

Aaron wants to mulch his garden. His garden is $x^2 + 18x + 81 \text{ ft}^2$. One bag of mulch covers $x^2 - 81 \text{ ft}^2$. Divide the expressions and simplify to find how many bags of mulch Aaron needs to mulch his garden.

Solution:

$\frac{x+9}{x-9}$

Extensions

For the following exercises, perform the given operations and simplify.

Exercise:

Problem: $\frac{x^2+x-6}{x^2-2x-3} \cdot \frac{2x^2-3x-9}{x^2-x-2} \div \frac{10x^2+27x+18}{x^2+2x+1}$

Exercise:

Problem: $\frac{\frac{3y^2-10y+3}{3y^2+5y-2} \cdot \frac{2y^2-3y-20}{2y^2-y-15}}{y-4}$

Solution:

$\frac{1}{y+2}$

Exercise:

Problem: $\frac{\frac{4a+1}{2a-3} + \frac{2a-3}{2a+3}}{\frac{4a^2+9}{a}}$

Exercise:

Problem: $\frac{x^2+7x+12}{x^2+x-6} \div \frac{3x^2+19x+28}{8x^2-4x-24} \div \frac{2x^2+x-3}{3x^2+4x-7}$

Solution:

4

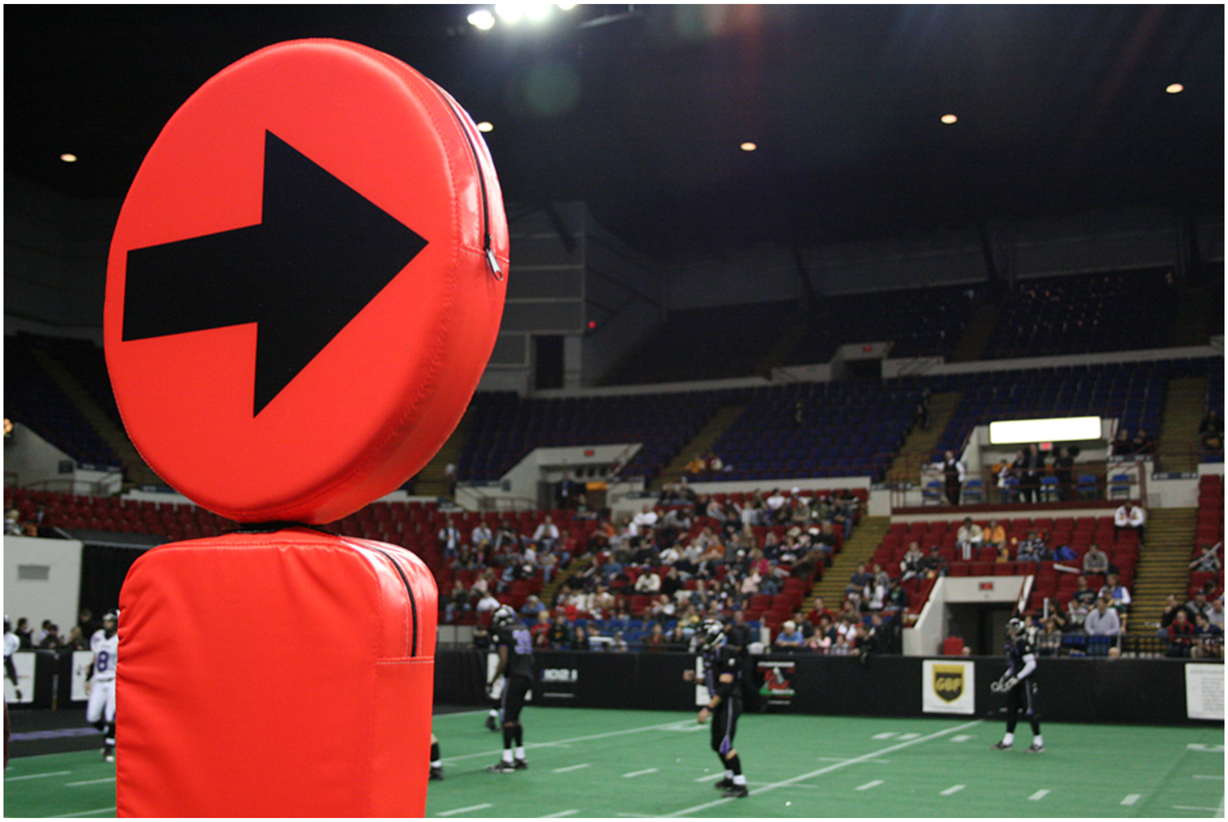
Glossary

least common denominator
: the smallest multiple that two denominators have in common

rational expression

the quotient of two polynomial expressions

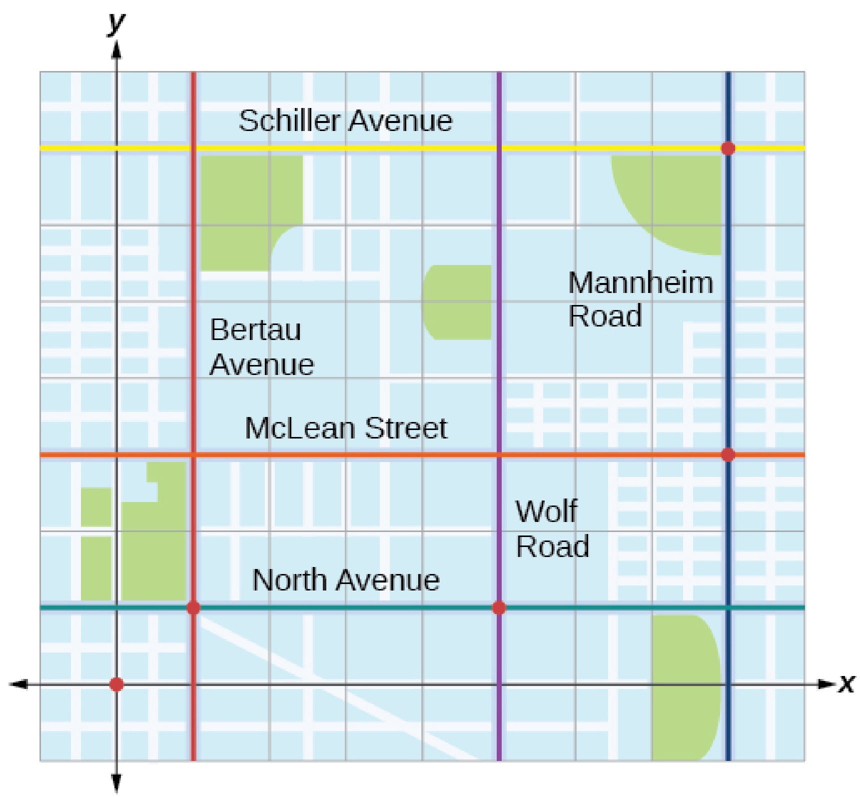
Introduction to Equations and Inequalities
class="introduction"



For most people, the term territorial possession indicates restrictions, usually dealing with trespassing or rite of passage and takes place in some foreign location. What most Americans do not realize is that from September through December, territorial possession dominates our lifestyles while watching the NFL. In this area, territorial possession is governed by the referees who make their decisions based on what the chains reveal. If the ball is at point $A\ (x_1, y_1)$,then it is up to the quarterback to decide which route to point $B\ (x_2, y_2)$,the end zone, is most feasible.

The Rectangular Coordinate Systems and Graphs
In this section you will:

- Plot ordered pairs in a Cartesian coordinate system.
- Graph equations by plotting points.
- Graph equations with a graphing utility.
- Find x-intercepts and y-intercepts.
- Use the distance formula.
- Use the midpoint formula.



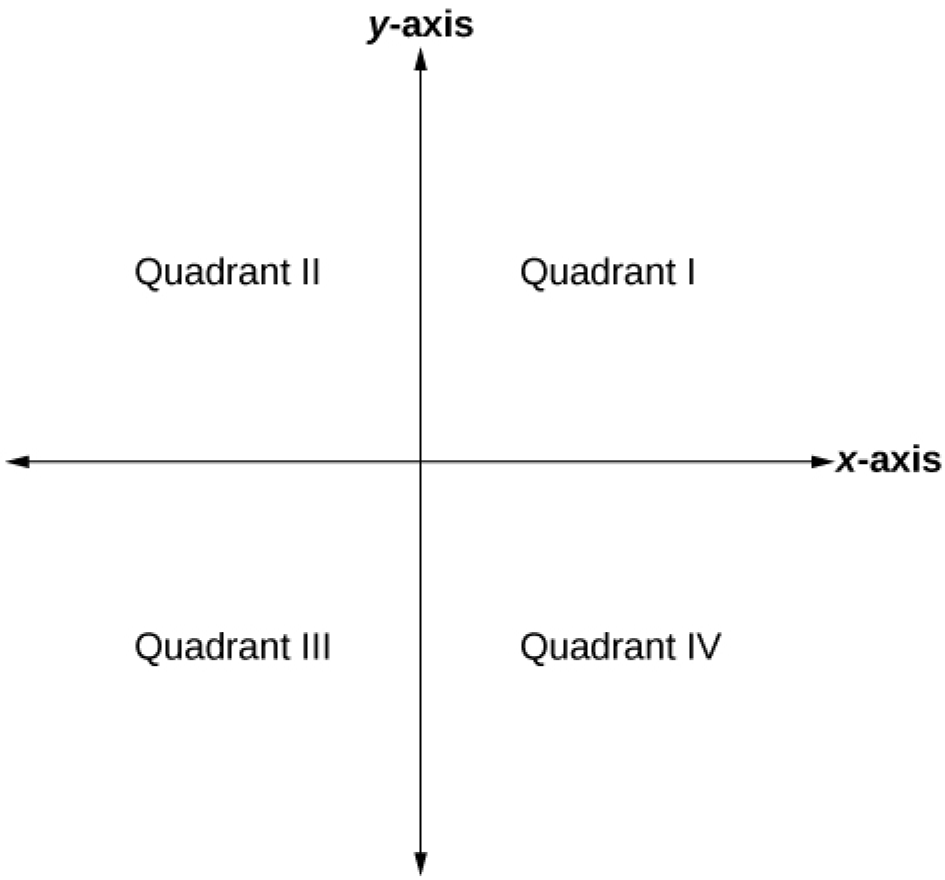
Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in [link]. Laying a rectangular coordinate grid over the map, we can see that each stop aligns with an intersection of grid lines. In this section, we will learn how to use grid lines to describe locations and changes in locations.

Plotting Ordered Pairs in the Cartesian Coordinate System

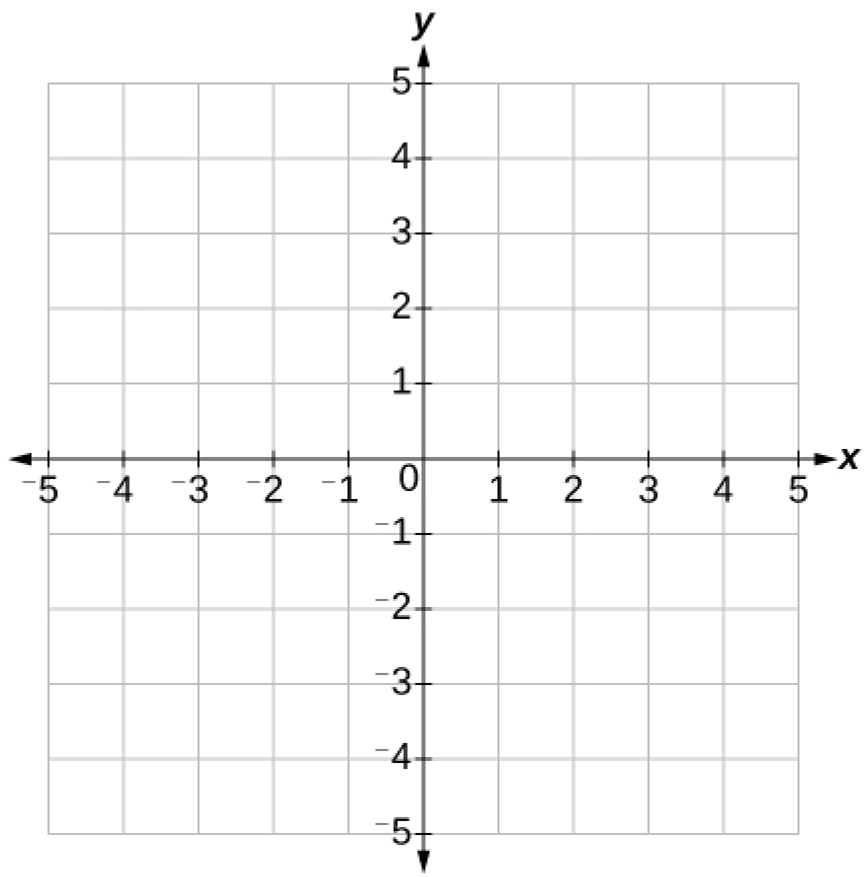
An old story describes how seventeenth-century philosopher/mathematician René Descartes invented the system that has become the foundation of algebra while sick in bed. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly’s location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers—the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes’ grid system existed centuries earlier, it was Descartes who introduced the components that comprise the **Cartesian coordinate system**, a grid system having perpendicular axes. Descartes named the horizontal axis the ***x*-axis** and the vertical axis the ***y*-axis**.

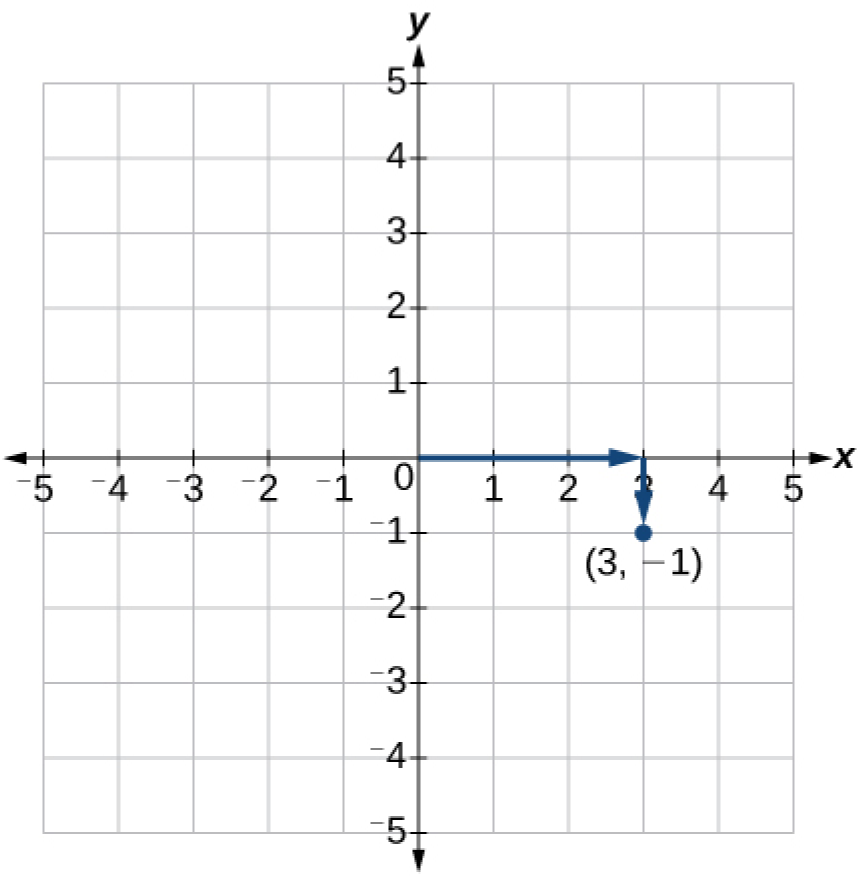
The Cartesian coordinate system, also called the rectangular coordinate system, is based on a two-dimensional plane consisting of the x-axis and the y-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a **quadrant**; the quadrants are numbered counterclockwise as shown in [link]



The center of the plane is the point at which the two axes cross. It is known as the **origin**, or point $(0, 0)$.From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the *x*-axis and up the *y*-axis; decreasing, negative numbers to the left on the *x*-axis and down the *y*-axis. The axes extend to positive and negative infinity as shown by the arrowheads in [link].



Each point in the plane is identified by its ***x*-coordinate**, or horizontal displacement from the origin, and its ***y*-coordinate**, or vertical displacement from the origin. Together, we write them as an **ordered pair** indicating the combined distance from the origin in the form (x, y). An ordered pair is also known as a coordinate pair because it consists of *x*- and *y*-coordinates. For example, we can represent the point $(3, -1)$ in the plane by moving three units to the right of the origin in the horizontal direction, and one unit down in the vertical direction. See [link].



When dividing the axes into equally spaced increments, note that the *x*-axis may be considered separately from the *y*-axis. In other words, while the *x*-axis may be divided and labeled according to consecutive integers, the *y*-axis may be divided and labeled by increments of 2, or 10, or 100. In fact, the axes may represent other units, such as years against the balance in a savings account, or quantity against cost, and so on. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

Note:
Cartesian Coordinate System
A two-dimensional plane where the

- *x*-axis is the horizontal axis
- *y*-axis is the vertical axis

A point in the plane is defined as an ordered pair, (x, y),such that *x* is determined by its horizontal distance from the origin and *y* is determined by its vertical distance from the origin.

Example:

Exercise:

Problem:

Plotting Points in a Rectangular Coordinate System

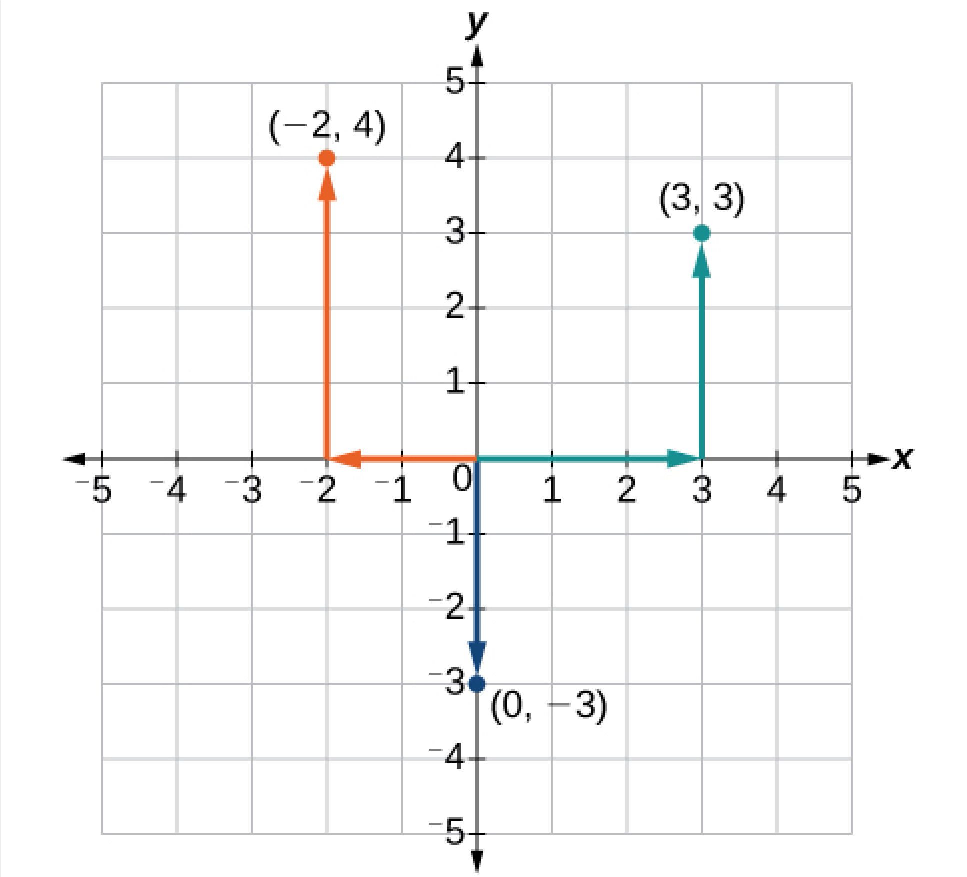
Plot the points $(-2, 4)$,$(3, 3)$,and $(0, -3)$ in the plane.

Solution:

To plot the point $(-2, 4)$,begin at the origin. The *x*-coordinate is –2, so move two units to the left. The *y*-coordinate is 4, so then move four units up in the positive *y* direction.

To plot the point $(3, 3)$,begin again at the origin. The *x*-coordinate is 3, so move three units to the right. The *y*-coordinate is also 3, so move three units up in the positive *y* direction.

To plot the point $(0, -3)$,begin again at the origin. The *x*-coordinate is 0. This tells us not to move in either direction along the *x*-axis. The *y*-coordinate is –3, so move three units down in the negative *y* direction. See the graph in [link].



Analysis

Note that when either coordinate is zero, the point must be on an axis. If the *x*-coordinate is zero, the point is on the *y*-axis. If the *y*-coordinate is zero, the point is on the *x*-axis.

Graphing Equations by Plotting Points

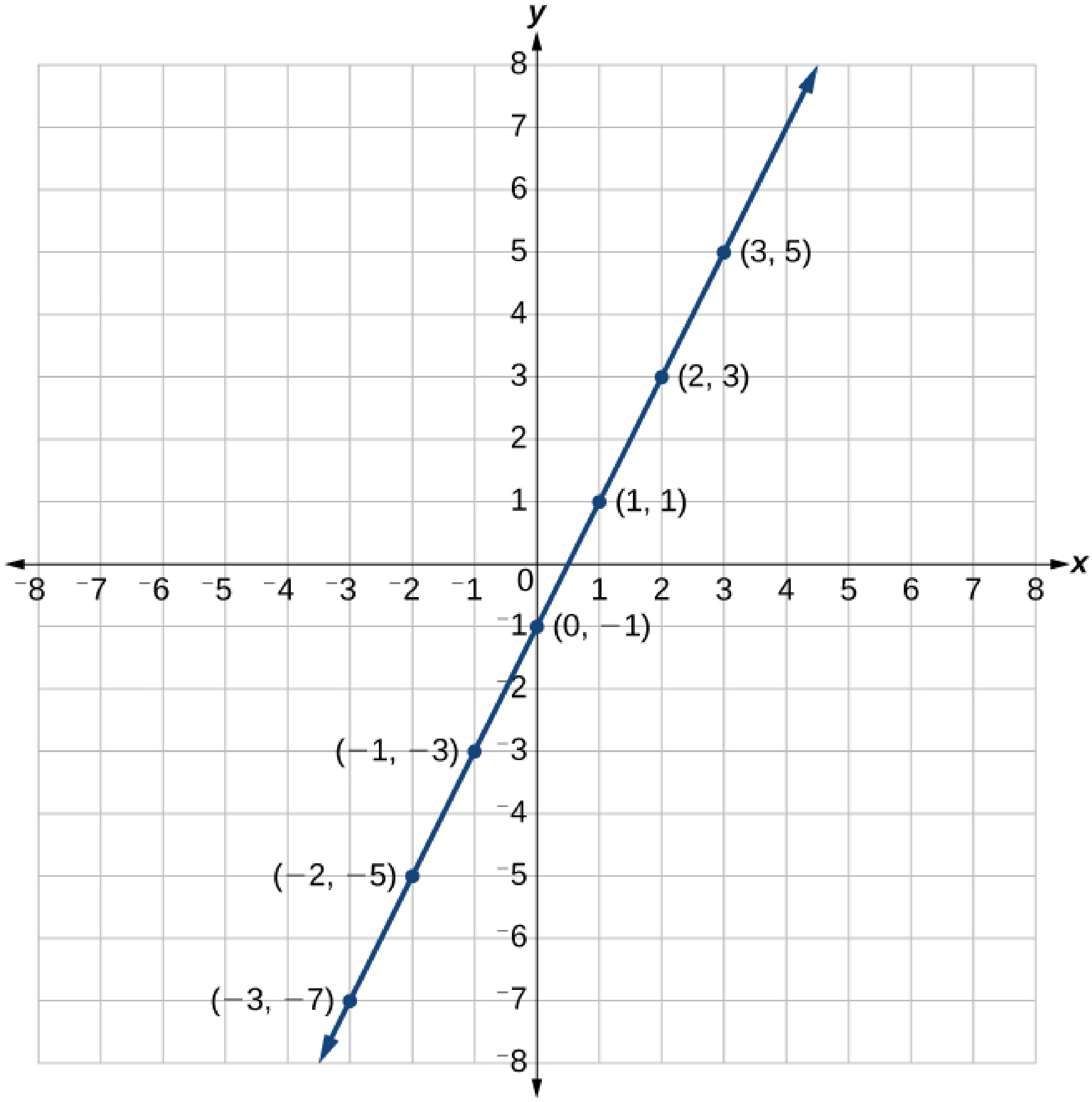
We can plot a set of points to represent an equation. When such an equation contains both an *x* variable and a *y* variable, it is called an **equation in two variables**. Its graph is called a **graph in two variables**. Any graph on a two-dimensional plane is a graph in two variables.

Suppose we want to graph the equation $y = 2x - 1$. We can begin by substituting a value for *x* into the equation and determining the resulting

value of *y*. Each pair of *x*- and *y*-values is an ordered pair that can be plotted. [link] lists values of *x* from –3 to 3 and the resulting values for *y*.

x	$y=2x-1$	(x,y)
-3	$y=2(-3)-1=-7$	$(-3,-7)$
-2	$y=2(-2)-1=-5$	$(-2,-5)$
-1	$y=2(-1)-1=-3$	$(-1,-3)$
0	$y=2(0)-1=-1$	$(0,-1)$
1	$y=2(1)-1=1$	$(1,1)$
2	$y=2(2)-1=3$	$(2,3)$
3	$y=2(3)-1=5$	$(3,5)$

We can plot the points in the table. The points for this particular equation form a line, so we can connect them. See [link]**.** This is not true for all equations.



Note that the *x*-values chosen are arbitrary, regardless of the type of equation we are graphing. Of course, some situations may require particular values of *x* to be plotted in order to see a particular result. Otherwise, it is logical to choose values that can be calculated easily, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two to graph a line. Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

Note:

Given an equation, graph by plotting points.

1. Make a table with one column labeled *x*, a second column labeled with the equation, and a third column listing the resulting ordered pairs.
2. Enter *x*-values down the first column using positive and negative values. Selecting the *x*-values in numerical order will make the graphing simpler.
3. Select *x*-values that will yield *y*-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

Example:
Exercise:

Problem:
Graphing an Equation in Two Variables by Plotting Points

Graph the equation $y=-x+2$ by plotting points.

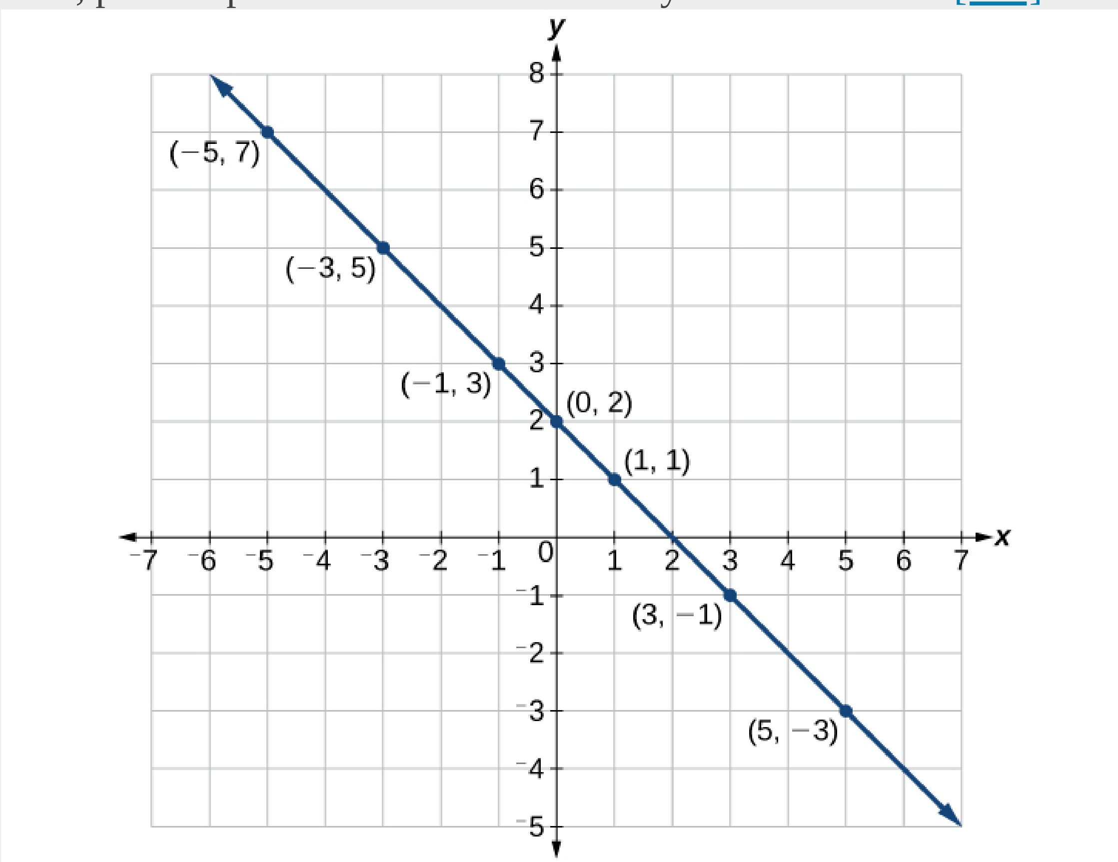
Solution:

First, we construct a table similar to [link]. Choose *x* values and calculate *y*.

x	$y=-x+2$	(x,y)
-5	$y=-(-5)+2=7$	$(-5,7)$

-3	$y=-(-3)+2=5$	$(-3,5)$
-1	$y=-(-1)+2=3$	$(-1,3)$
0	$y=-(0)+2=2$	$(0,2)$
1	$y=-(1)+2=1$	$(1,1)$
3	$y=-(3)+2=-1$	$(3,-1)$
5	$y=-(5)+2=-3$	$(5,-3)$

Now, plot the points. Connect them if they form a line. See [link]



Note:

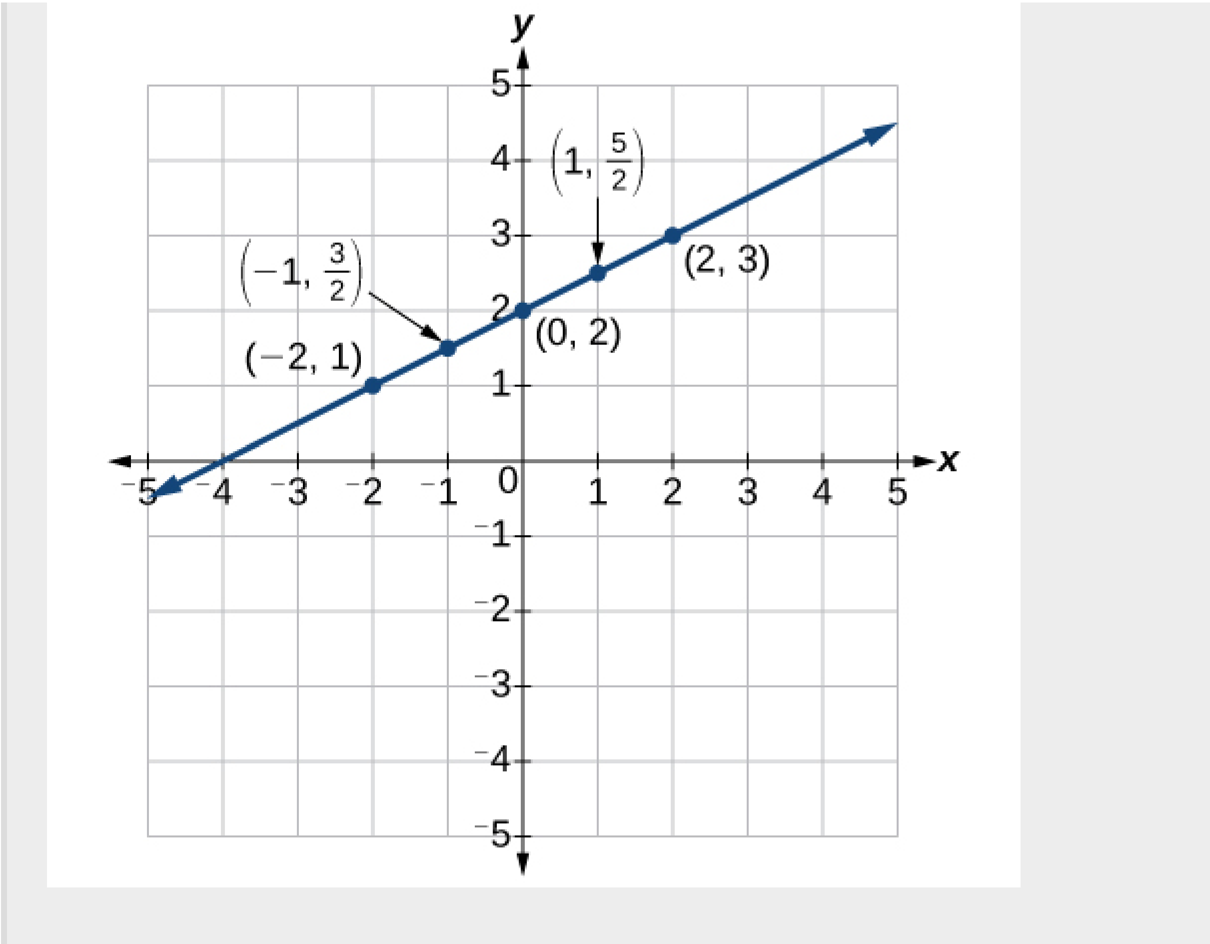
Exercise:

Problem:

Construct a table and graph the equation by plotting points: $y = \frac{1}{2}x + 2$.

Solution:

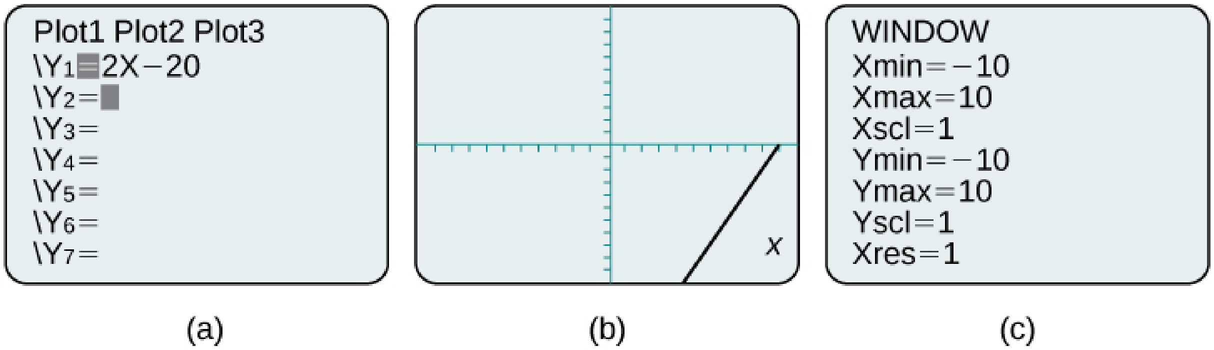
x	$y = \frac{1}{2}x + 2$	(x, y)
-2	$y = \frac{1}{2}(-2) + 2 = 1$	$(-2, 1)$
-1	$y = \frac{1}{2}(-1) + 2 = \frac{3}{2}$	$\left(-1, \frac{3}{2}\right)$
0	$y = \frac{1}{2}(0) + 2 = 2$	$(0, 2)$
1	$y = \frac{1}{2}(1) + 2 = \frac{5}{2}$	$\left(1, \frac{5}{2}\right)$
2	$y = \frac{1}{2}(2) + 2 = 3$	$(2, 3)$



Graphing Equations with a Graphing Utility

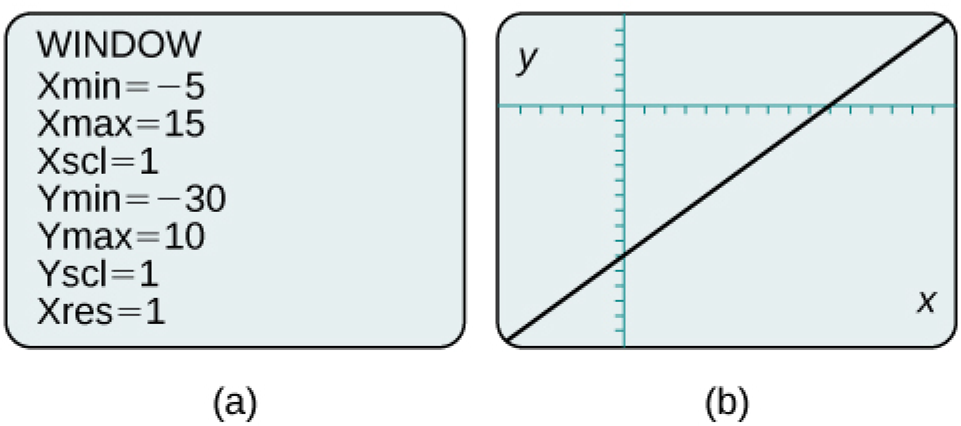
Most graphing calculators require similar techniques to graph an equation. The equations sometimes have to be manipulated so they are written in the style $y =$______. The TI-84 Plus, and many other calculator makes and models, have a mode function, which allows the window (the screen for viewing the graph) to be altered so the pertinent parts of a graph can be seen.

For example, the equation $y = 2x - 20$ has been entered in the TI-84 Plus shown in [link]**a.** In [link]**b,** the resulting graph is shown. Notice that we cannot see on the screen where the graph crosses the axes. The standard window screen on the TI-84 Plus shows $-10 \le x \le 10$,and $-10 \le y \le 10$. See [link]**c**.



a. Enter the equation. b. This is the graph in the original window. c. These are the original settings.

By changing the window to show more of the positive *x*-axis and more of the negative *y*-axis, we have a much better view of the graph and the *x*- and *y*-intercepts. See [link]**a** and [link]**b.**



a. This screen shows the new window settings. b. We can clearly view the intercepts in the new window.

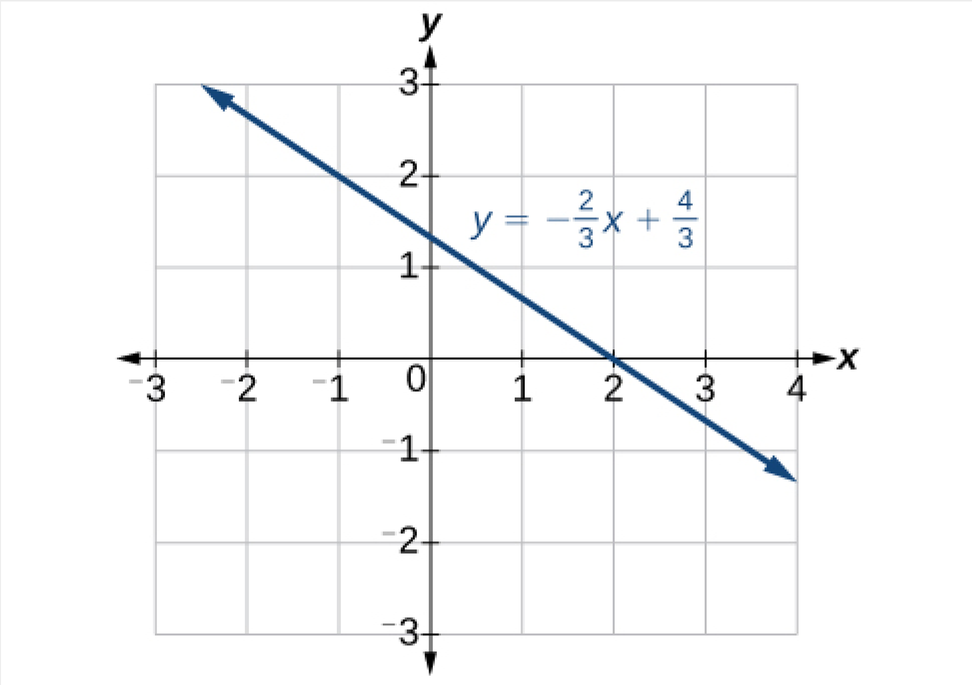
Example:
Exercise:

Problem:
Using a Graphing Utility to Graph an Equation

Use a graphing utility to graph the equation: $y = -\frac{2}{3}x - \frac{4}{3}$.

Solution:

Enter the equation in the *y*= function of the calculator. Set the window settings so that both the *x*- and *y*- intercepts are showing in the window. See [link].



Finding *x*-intercepts and *y*-intercepts

The **intercepts** of a graph are points at which the graph crosses the axes. The ***x*-intercept** is the point at which the graph crosses the *x*-axis. At this point, the *y*-coordinate is zero. The ***y*-intercept** is the point at which the graph crosses the *y*-axis. At this point, the *x*-coordinate is zero.

To determine the *x*-intercept, we set *y* equal to zero and solve for *x*. Similarly, to determine the *y*-intercept, we set *x* equal to zero and solve for *y*. For example, lets find the intercepts of the equation $y = 3x - 1$.

To find the *x*-intercept, set $y = 0$.

Equation:

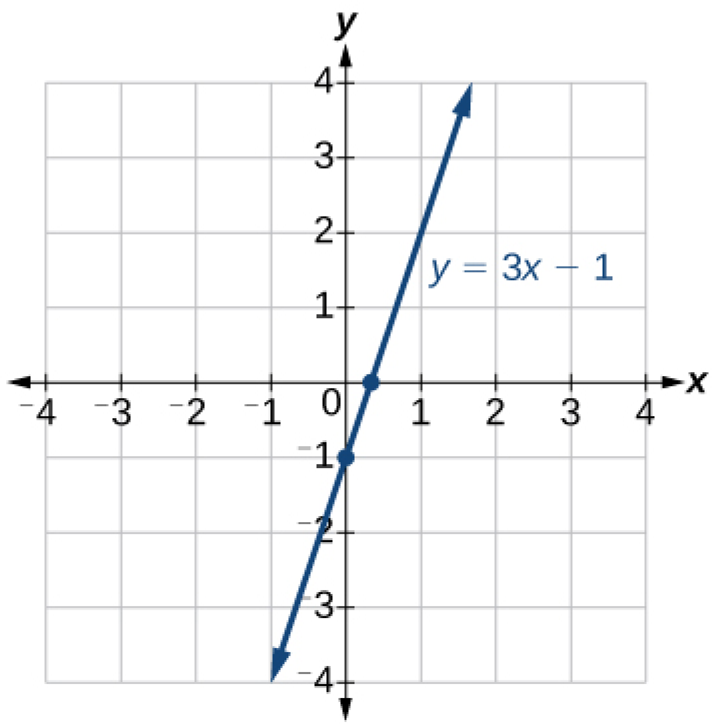
$$\begin{array}{ll} y = 3x - 1 & \\ 0 = 3x - 1 & \\ 1 = 3x & \\ \frac{1}{3} = x & \\ \left(\frac{1}{3}, 0\right) & \qquad x\text{−intercept} \end{array}$$

To find the *y*-intercept, set $x = 0$.

Equation:

$$\begin{array}{ll} y = 3x - 1 & \\ y = 3(0) - 1 & \\ y = -1 & \\ (0, -1) & \qquad y\text{−intercept} \end{array}$$

We can confirm that our results make sense by observing a graph of the equation as in [link]. Notice that the graph crosses the axes where we predicted it would.



Note:
Given an equation, find the intercepts.

- Find the *x*-intercept by setting $y = 0$ and solving for x.
- Find the *y*-intercept by setting $x = 0$ and solving for y.

Example:
Exercise:

Problem:
Finding the Intercepts of the Given Equation

Find the intercepts of the equation $y = -3x - 4$. Then sketch the graph using only the intercepts.

Solution:

Set $y=0$ to find the *x*-intercept.

Equation:

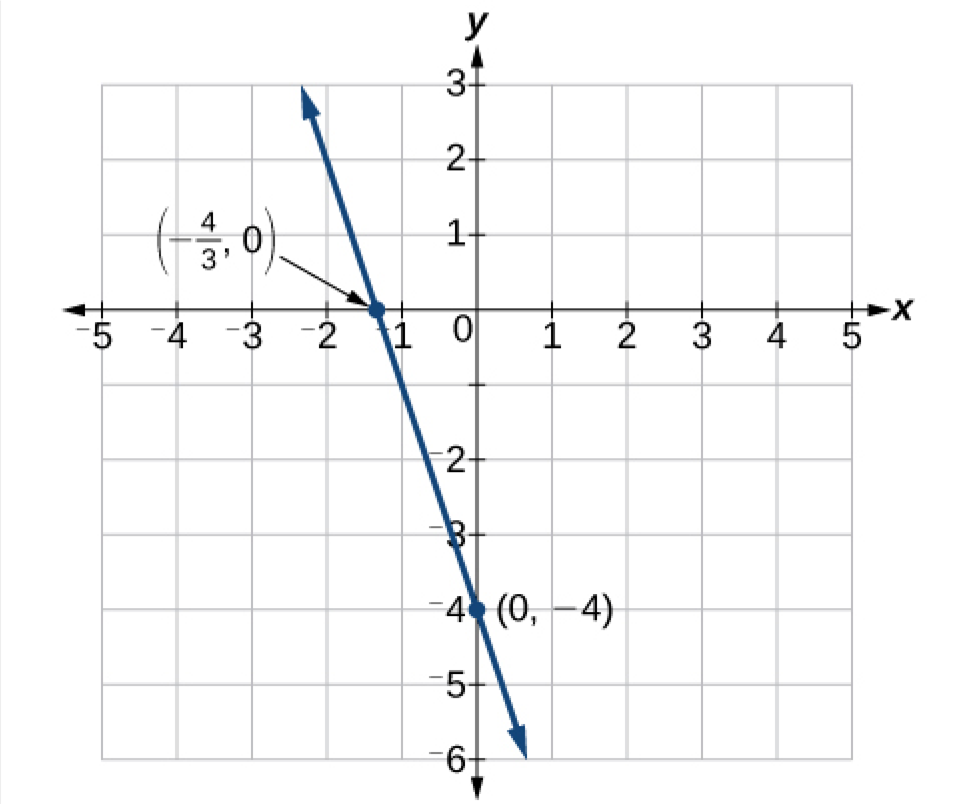
$$\begin{aligned} y &= -3x-4 \\ 0 &= -3x-4 \\ 4 &= -3x \\ -\frac{4}{3} &= x \\ \left(-\frac{4}{3},0\right) & \qquad\qquad x\text{−intercept} \end{aligned}$$

Set $x=0$ to find the *y*-intercept.

Equation:

$$\begin{aligned} &y = -3x-4 \\ &y = -3(0)-4 \\ &y = -4 \\ &(0,-4) \qquad\qquad y\text{−intercept} \end{aligned}$$

Plot both points, and draw a line passing through them as in [link].



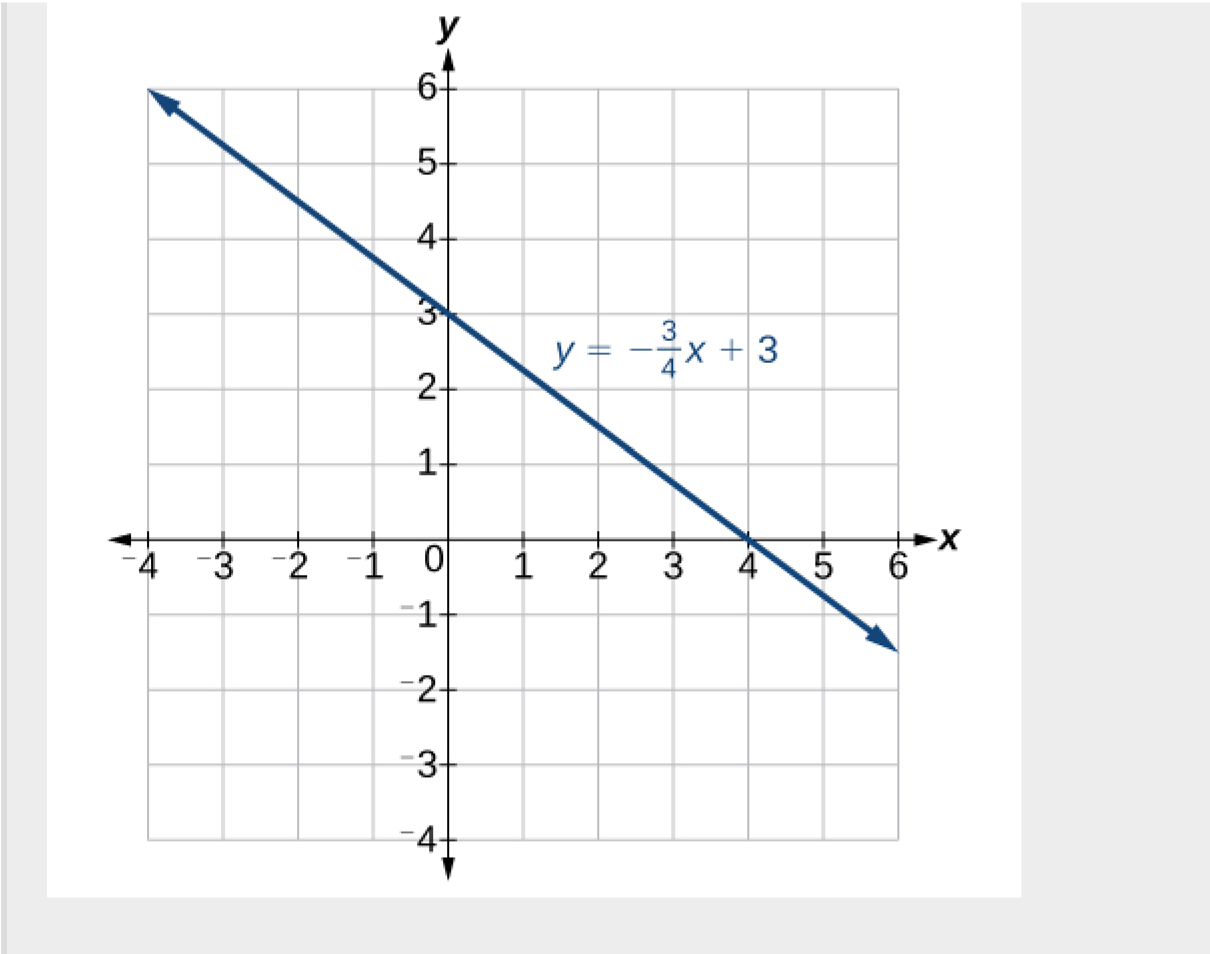
Note:
Exercise:

Problem:

Find the intercepts of the equation and sketch the graph: $y=-\frac{3}{4}x+3.$

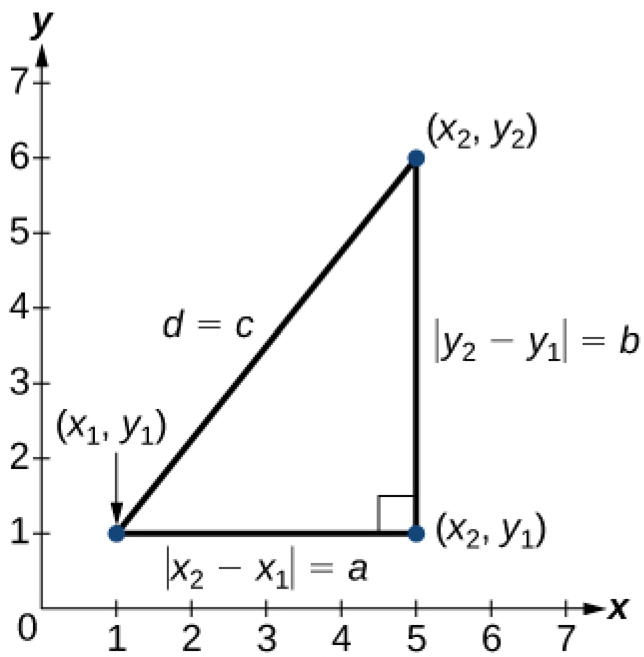
Solution:

x-intercept is $(4,0)$;*y*-intercept is $(0,3)$.



Using the Distance Formula

Derived from the Pythagorean Theorem, the **distance formula** is used to find the distance between two points in the plane. The Pythagorean Theorem, $a^2 + b^2 = c^2$,is based on a right triangle where a and b are the lengths of the legs adjacent to the right angle, and c is the length of the hypotenuse. See [link].



The relationship of sides $|x_2 - x_1|$ and $|y_2 - y_1|$ to side *d* is the same as that of sides *a* and *b* to side *c*. We use the absolute value symbol to indicate that the length is a positive number because the absolute value of any number is positive. (For example, $|-3| = 3.$) The symbols $|x_2 - x_1|$ and $|y_2 - y_1|$ indicate that the lengths of the sides of the triangle are positive. To find the length *c*, take the square root of both sides of the Pythagorean Theorem.

Equation:

$$c^2 = a^2 + b^2 \to c = \sqrt{a^2 + b^2}$$

It follows that the distance formula is given as

Equation:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \to d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We do not have to use the absolute value symbols in this definition because any number squared is positive.

Note:
The Distance Formula
Given endpoints (x_1, y_1) and (x_2, y_2),the distance between two points is given by
Equation:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

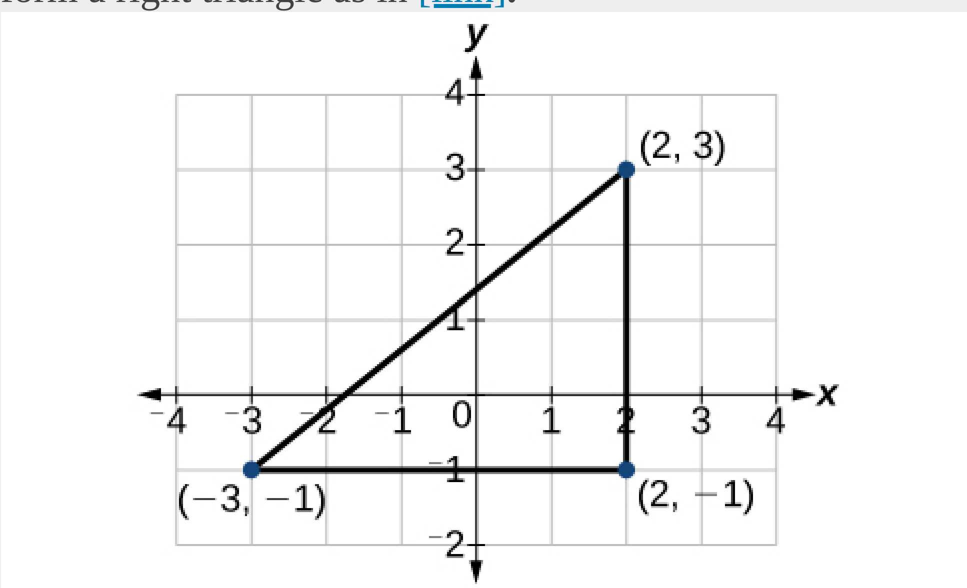
Example:
Exercise:

Problem:
Finding the Distance between Two Points

Find the distance between the points $(-3, -1)$ and $(2, 3)$.

Solution:

Let us first look at the graph of the two points. Connect the points to form a right triangle as in [link].



Then, calculate the length of *d* using the distance formula.

Equation:

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d &= \sqrt{(2 - (-3))^2 + (3 - (-1))^2} \\
&= \sqrt{(5)^2 + (4)^2} \\
&= \sqrt{25 + 16} \\
&= \sqrt{41}
\end{aligned}
$$

Note:
Exercise:

Problem: Find the distance between two points: $(1, 4)$ and $(11, 9)$.

Solution:

$\sqrt{125} = 5\sqrt{5}$

Example:
Exercise:

Problem:
Finding the Distance between Two Locations

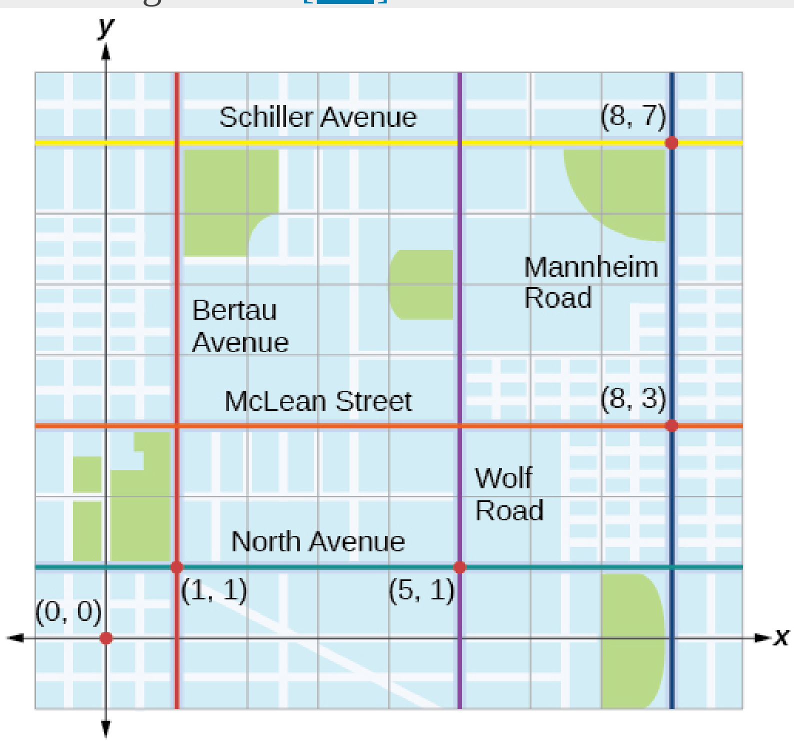
Let's return to the situation introduced at the beginning of this section.

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot

in [link]. Find the total distance that Tracie traveled. Compare this with the distance between her starting and final positions.

Solution:

The first thing we should do is identify ordered pairs to describe each position. If we set the starting position at the origin, we can identify each of the other points by counting units east (right) and north (up) on the grid. For example, the first stop is 1 block east and 1 block north, so it is at $(1, 1)$. The next stop is 5 blocks to the east, so it is at $(5, 1)$. After that, she traveled 3 blocks east and 2 blocks north to $(8, 3)$. Lastly, she traveled 4 blocks north to $(8, 7)$. We can label these points on the grid as in [link].



Next, we can calculate the distance. Note that each grid unit represents 1,000 feet.

- From her starting location to her first stop at $(1, 1)$,Tracie might have driven north 1,000 feet and then east 1,000 feet, or vice versa. Either way, she drove 2,000 feet to her first stop.

- Her second stop is at $(5, 1)$. So from $(1, 1)$ to $(5, 1)$,Tracie drove east 4,000 feet.
- Her third stop is at $(8, 3)$. There are a number of routes from $(5, 1)$ to $(8, 3)$. Whatever route Tracie decided to use, the distance is the same, as there are no angular streets between the two points. Let's say she drove east 3,000 feet and then north 2,000 feet for a total of 5,000 feet.
- Tracie's final stop is at $(8, 7)$. This is a straight drive north from $(8, 3)$ for a total of 4,000 feet.

Next, we will add the distances listed in [link].

From/To	Number of Feet Driven
$(0, 0)$ to $(1, 1)$	2,000
$(1, 1)$ to$(5, 1)$	4,000
$(5, 1)$ to $(8, 3)$	5,000
$(8, 3)$ to $(8, 7)$	4,000
Total	15,000

The total distance Tracie drove is 15,000 feet, or 2.84 miles. This is not, however, the actual distance between her starting and ending positions. To find this distance, we can use the distance formula between the points $(0, 0)$ and $(8, 7)$.

Equation:

$$\begin{aligned} d &= \sqrt{(8-0)^2 + (7-0)^2} \\ &= \sqrt{64+49} \\ &= \sqrt{113} \\ &\approx 10.63 \text{ units} \end{aligned}$$

At 1,000 feet per grid unit, the distance between Elmhurst, IL, to Franklin Park is 10,630.14 feet, or 2.01 miles. The distance formula results in a shorter calculation because it is based on the hypotenuse of a right triangle, a straight diagonal from the origin to the point $(8, 7)$. Perhaps you have heard the saying "as the crow flies," which means the shortest distance between two points because a crow can fly in a straight line even though a person on the ground has to travel a longer distance on existing roadways.

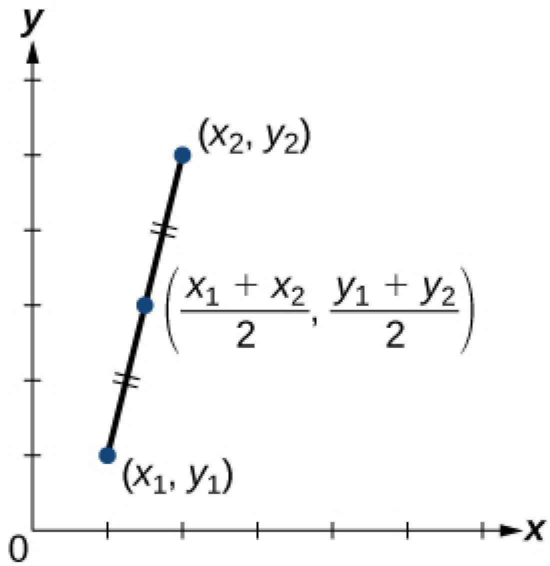
Using the Midpoint Formula

When the endpoints of a line segment are known, we can find the point midway between them. This point is known as the midpoint and the formula is known as the **midpoint formula**. Given the endpoints of a line segment, (x_1, y_1) and (x_2, y_2),the midpoint formula states how to find the coordinates of the midpoint M.

Equation:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

A graphical view of a midpoint is shown in [link]. Notice that the line segments on either side of the midpoint are congruent.



Example:
Exercise:

Problem:
Finding the Midpoint of the Line Segment

Find the midpoint of the line segment with the endpoints $(7, -2)$ and $(9, 5)$.

Solution:

Use the formula to find the midpoint of the line segment.
Equation:

$$\begin{aligned}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) &= \left(\frac{7+9}{2}, \frac{-2+5}{2}\right)\\ &= \left(8, \frac{3}{2}\right)\end{aligned}$$

Note:
Exercise:

Problem:

Find the midpoint of the line segment with endpoints $(-2,-1)$ and $(-8,6)$.

Solution:

$\left(-5,\frac{5}{2}\right)$

Example:
Exercise:

Problem:
Finding the Center of a Circle

The diameter of a circle has endpoints $(-1,-4)$ and $(5,-4)$. Find the center of the circle.

Solution:

The center of a circle is the center, or midpoint, of its diameter. Thus, the midpoint formula will yield the center point.
Equation:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$
$$\left(\frac{-1+5}{2},\frac{-4-4}{2}\right)=\left(\frac{4}{2},-\frac{8}{2}\right)=(2,-4)$$

Note:
Access these online resources for additional instruction and practice with the Cartesian coordinate system.

- Plotting points on the coordinate plane
- Find x and y intercepts based on the graph of a line

Key Concepts

- We can locate, or plot, points in the Cartesian coordinate system using ordered pairs, which are defined as displacement from the *x*-axis and displacement from the *y*-axis. See [link].
- An equation can be graphed in the plane by creating a table of values and plotting points. See [link].
- Using a graphing calculator or a computer program makes graphing equations faster and more accurate. Equations usually have to be entered in the form $y=$_____. See [link].
- Finding the *x*- and *y*-intercepts can define the graph of a line. These are the points where the graph crosses the axes. See [link].
- The distance formula is derived from the Pythagorean Theorem and is used to find the length of a line segment. See [link] and [link].
- The midpoint formula provides a method of finding the coordinates of the midpoint dividing the sum of the *x*-coordinates and the sum of the *y*-coordinates of the endpoints by 2. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Is it possible for a point plotted in the Cartesian coordinate system to not lie in one of the four quadrants? Explain.

Solution:

Answers may vary. Yes. It is possible for a point to be on the *x*-axis or on the *y*-axis and therefore is considered to NOT be in one of the quadrants.

Exercise:

Problem:

Describe the process for finding the *x*-intercept and the *y*-intercept of a graph algebraically.

Exercise:

Problem:

Describe in your own words what the *y*-intercept of a graph is.

Solution:

The *y*-intercept is the point where the graph crosses the *y*-axis.

Exercise:

Problem:

When using the distance formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, explain the correct order of operations that are to be performed to obtain the correct answer.

Algebraic

For each of the following exercises, find the *x*-intercept and the *y*-intercept without graphing. Write the coordinates of each intercept.

Exercise:

Problem:$y=-3x+6$

Solution:

The *x*-intercept is $(2, 0)$ and the *y*-intercept is $(0, 6)$.

Exercise:

Problem:$4y = 2x - 1$

Exercise:

Problem:$3x - 2y = 6$

Solution:

The *x*-intercept is $(2, 0)$ and the *y*-intercept is $(0, -3)$.

Exercise:

Problem:$4x - 3 = 2y$

Exercise:

Problem:$3x + 8y = 9$

Solution:

The *x*-intercept is $(3, 0)$ and the *y*-intercept is $\left(0, \frac{9}{8}\right)$.

Exercise:

Problem:$2x - \frac{2}{3} = \frac{3}{4}y + 3$

For each of the following exercises, solve the equation for *y* in terms of *x*.

Exercise:

Problem:$4x + 2y = 8$

Solution:

$y = 4 - 2x$

Exercise:

Problem:$3x - 2y = 6$

Exercise:

Problem:$2x = 5 - 3y$

Solution:

$y = \frac{5-2x}{3}$

Exercise:

Problem:$x - 2y = 7$

Exercise:

Problem:$5y + 4 = 10x$

Solution:

$y = 2x - \frac{4}{5}$

Exercise:

Problem:$5x + 2y = 0$

For each of the following exercises, find the distance between the two points. Simplify your answers, and write the exact answer in simplest radical form for irrational answers.

Exercise:

Problem:$(-4, 1)$ and $(3, -4)$

Solution:

$d = \sqrt{74}$

Exercise:

Problem: $(2, -5)$ and $(7, 4)$

Exercise:

Problem: $(5, 0)$ and $(5, 6)$

Solution:

$d = \sqrt{36} = 6$

Exercise:

Problem: $(-4, 3)$ and $(10, 3)$

Exercise:

Problem:

Find the distance between the two points given using your calculator, and round your answer to the nearest hundredth.

$(19, 12)$ and $(41, 71)$

Solution:

$d \approx 62.97$

For each of the following exercises, find the coordinates of the midpoint of the line segment that joins the two given points.

Exercise:

Problem: $(-5, -6)$ and $(4, 2)$

Exercise:

Problem:$(-1, 1)$ and $(7, -4)$

Solution:

$\left(3, \frac{-3}{2}\right)$

Exercise:

Problem:$(-5, -3)$ and $(-2, -8)$

Exercise:

Problem:$(0, 7)$ and $(4, -9)$

Solution:

$(2, -1)$

Exercise:

Problem:$(-43, 17)$ and $(23, -34)$

Graphical

For each of the following exercises, identify the information requested.

Exercise:

Problem: What are the coordinates of the origin?

Solution:

$(0, 0)$

Exercise:

Problem: If a point is located on the *y*-axis, what is the *x*-coordinate?

Exercise:

Problem: If a point is located on the *x*-axis, what is the *y*-coordinate?

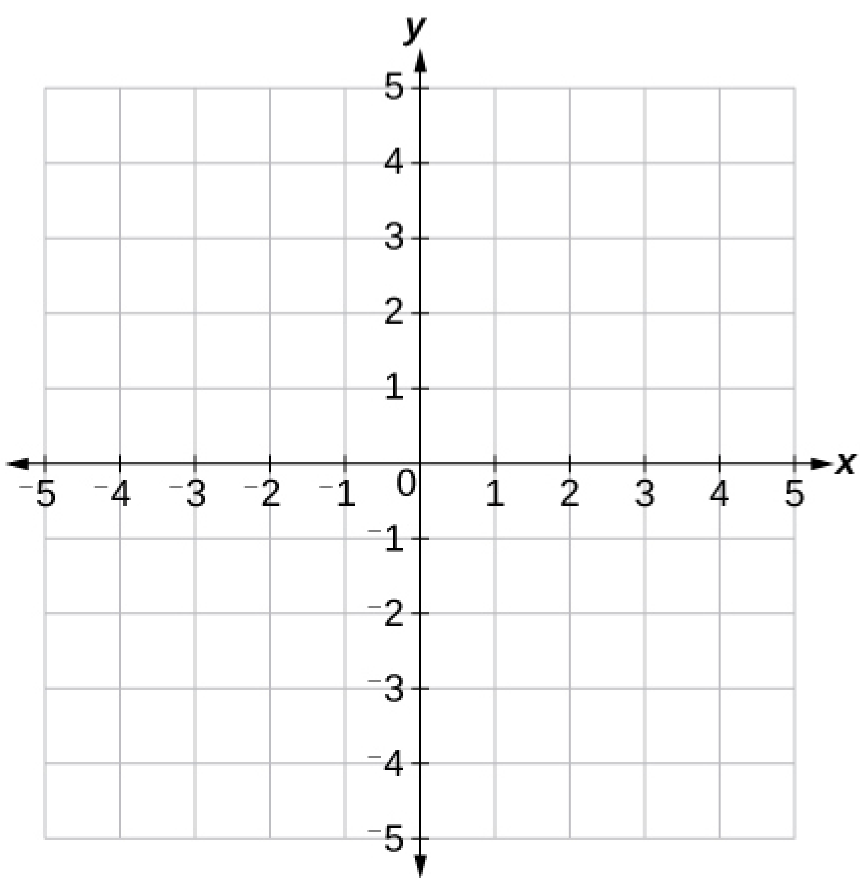
Solution:

$y=0$

For each of the following exercises, plot the three points on the given coordinate plane. State whether the three points you plotted appear to be collinear (on the same line).

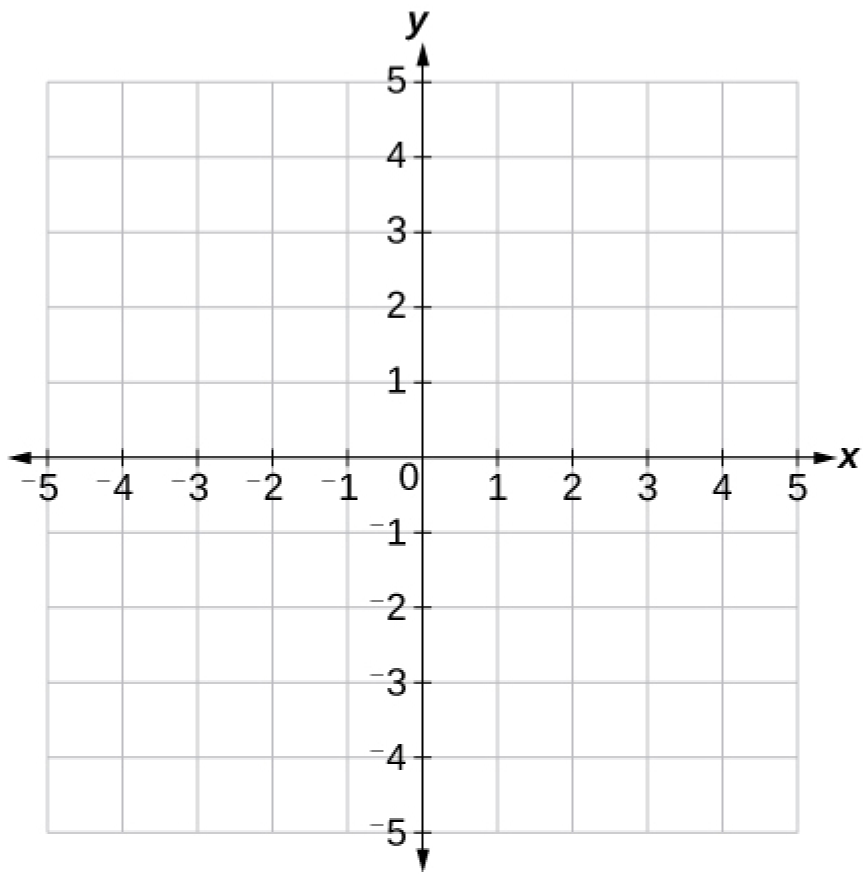
Exercise:

Problem:$(4,1)$ $(-2,-3)$ $(5,0)$

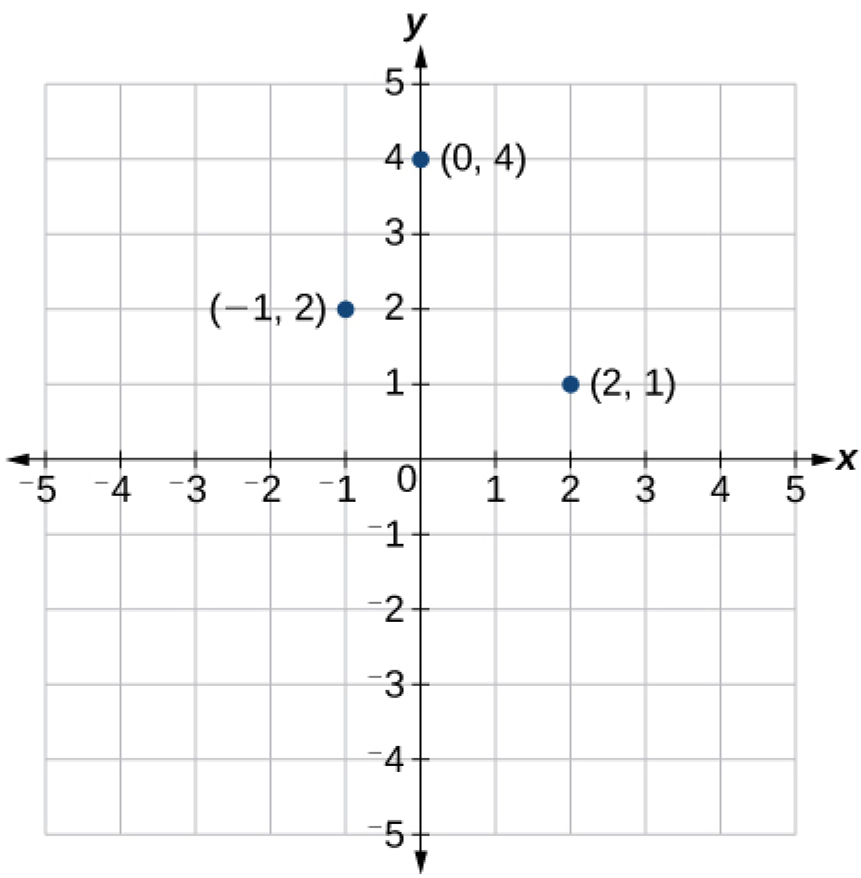


Exercise:

Problem: $(-1,2)(0,4)(2,1)$



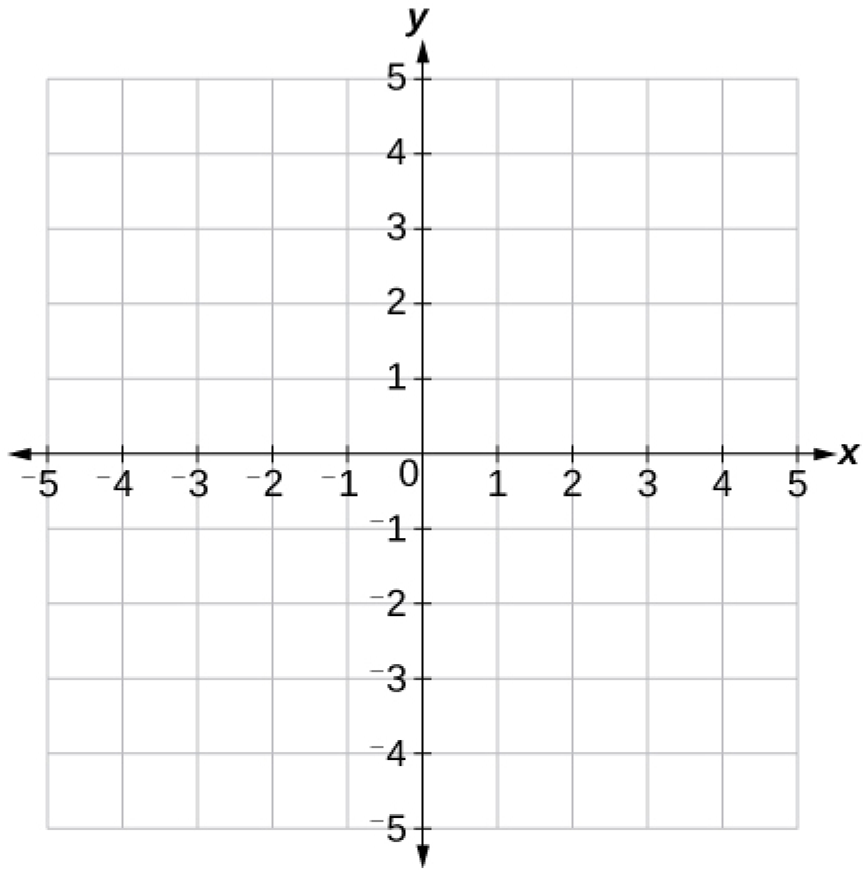
Solution:



not collinear

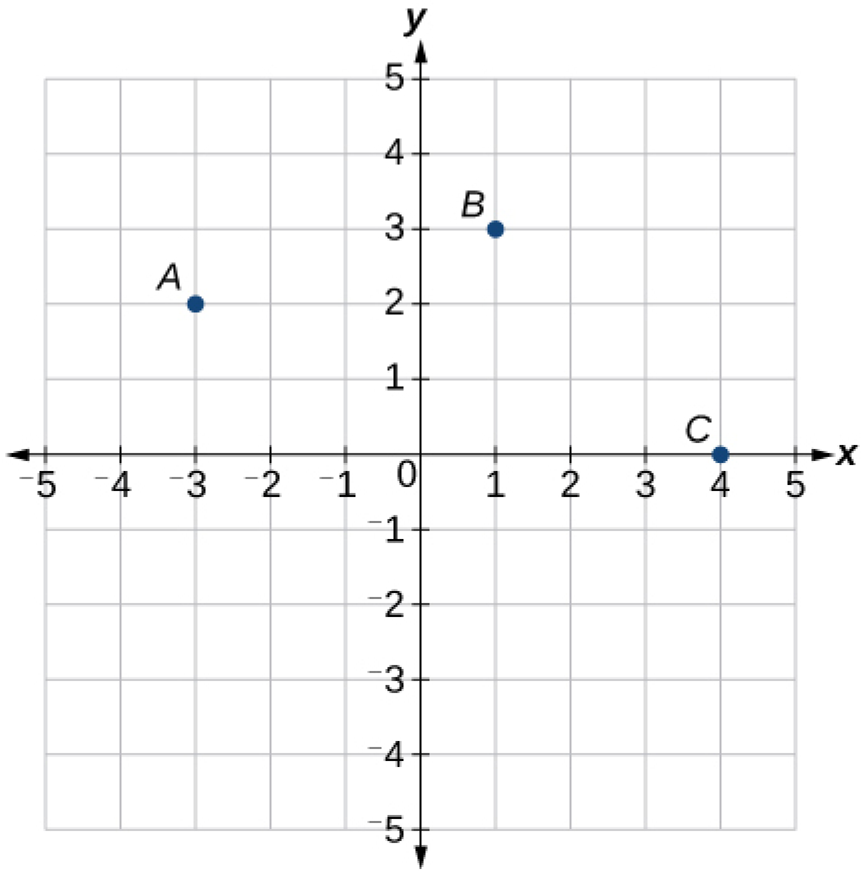
Exercise:

Problem:$(-3, 0)$ $(-3, 4)$ $(-3, -3)$



Exercise:

Problem: Name the coordinates of the points graphed.



Solution:

$(-3, 2), (1, 3), (4, 0)$

Exercise:

Problem:

Name the quadrant in which the following points would be located. If the point is on an axis, name the axis.

$a. (-3, -4)$
$b. (-5, 0)$
$c. (1, -4)$
$d. (-2, 7)$
$e. (0, -3)$

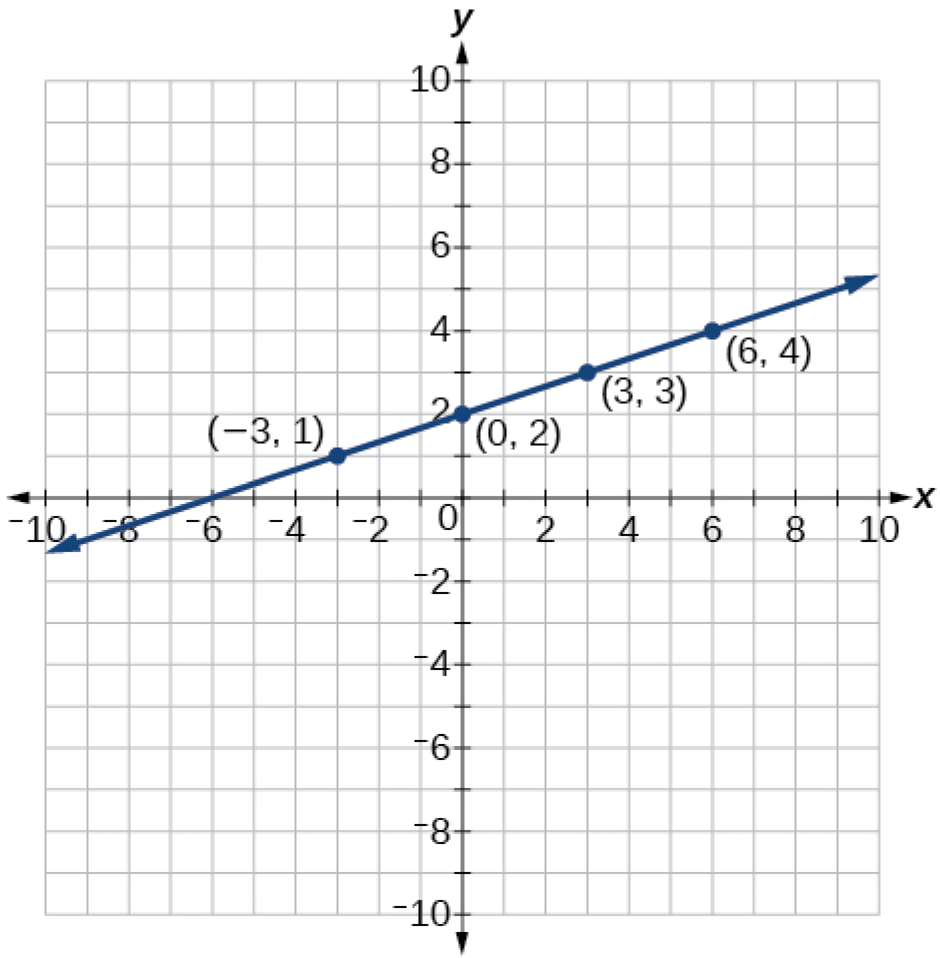
For each of the following exercises, construct a table and graph the equation by plotting at least three points.

Exercise:

Problem: $y=\frac{1}{3}x+2$

Solution:

x	y
-3	1
0	2
3	3
6	4



Exercise:

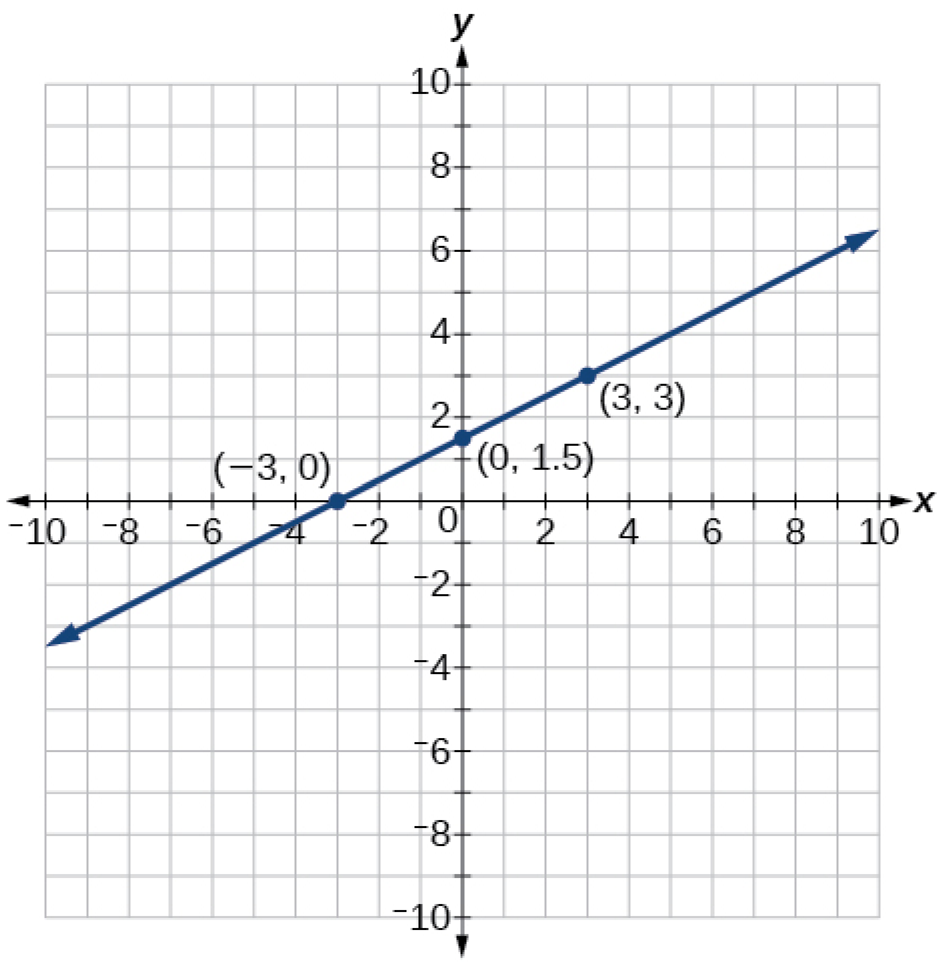
Problem: $y = -3x + 1$

Exercise:

Problem: $2y = x + 3$

Solution:

x	y
–3	0
0	1.5
3	3



Numeric

For each of the following exercises, find and plot the *x*- and *y*-intercepts, and graph the straight line based on those two points.

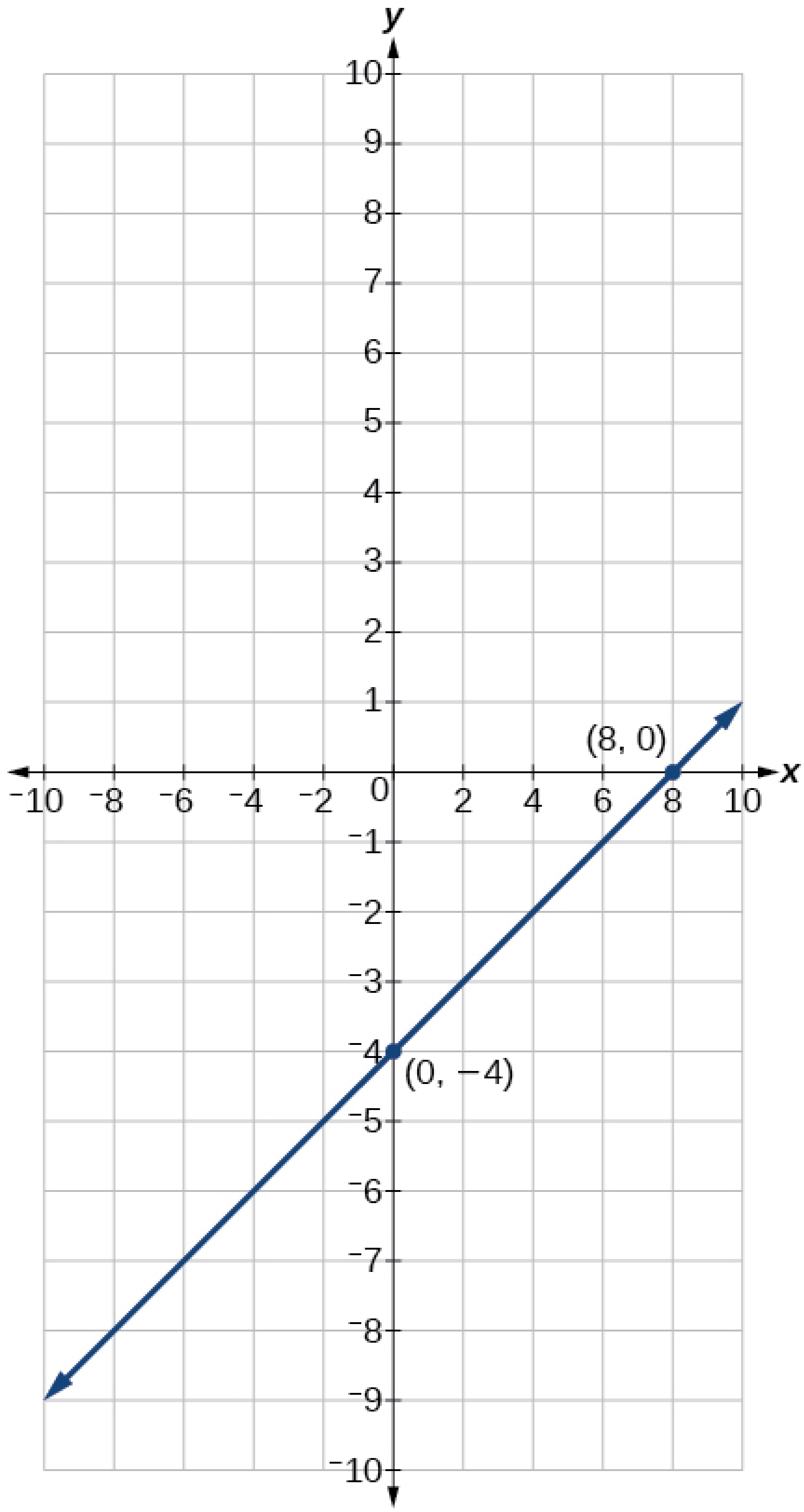
Exercise:

Problem: $4x - 3y = 12$

Exercise:

Problem: $x - 2y = 8$

Solution:



y
x
(8, 0)
(0, −4)

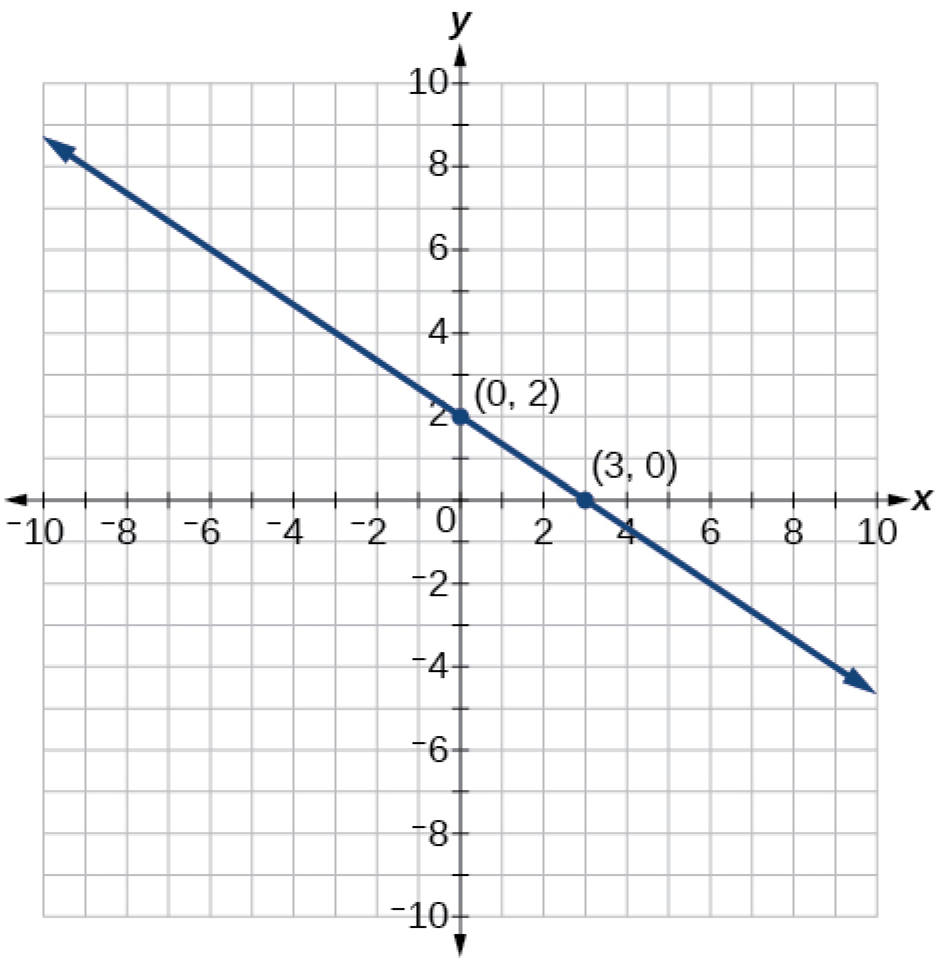
Exercise:

Problem: $y-5=5x$

Exercise:

Problem: $3y=-2x+6$

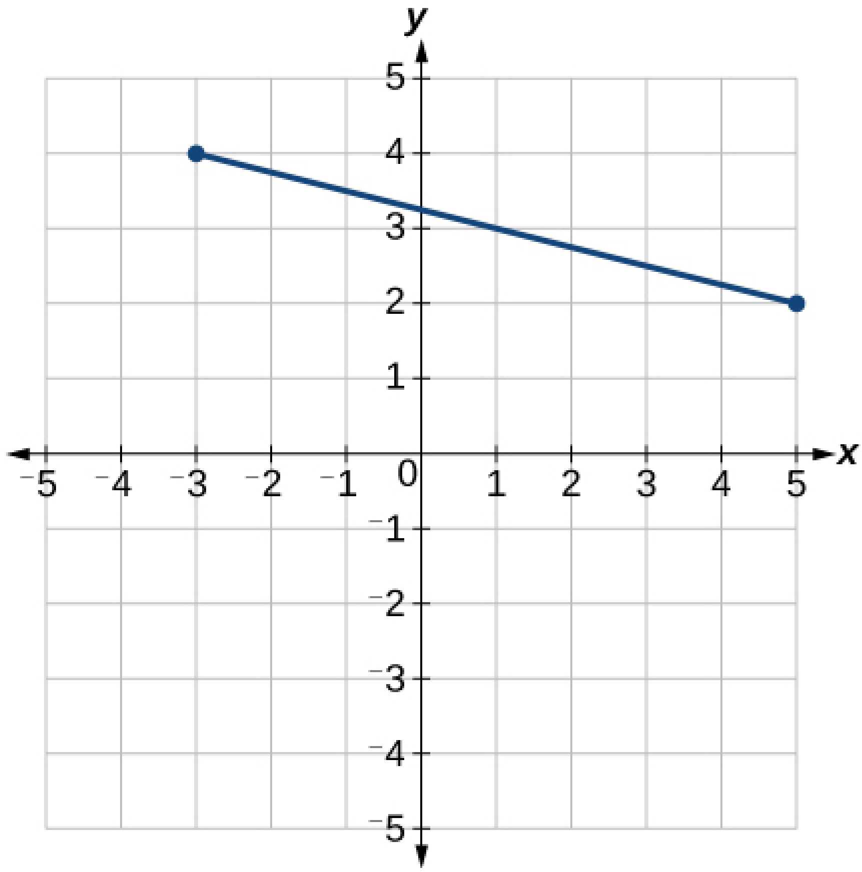
Solution:



Exercise:

Problem: $y=\frac{x-3}{2}$

For each of the following exercises, use the graph in the figure below.



Exercise:

Problem:

Find the distance between the two endpoints using the distance formula. Round to three decimal places.

Solution:

$d = 8.246$

Exercise:

Problem:

Find the coordinates of the midpoint of the line segment connecting the two points.

Exercise:

Problem: Find the distance that $(-3,4)$ is from the origin.

Solution:

$d=5$

Exercise:

Problem:

Find the distance that $(5,2)$ is from the origin. Round to three decimal places.

Exercise:

Problem: Which point is closer to the origin?

Solution:

$(-3,4)$

Technology

For the following exercises, use your graphing calculator to input the linear graphs in the Y= graph menu.

After graphing it, use the 2$^{\text{nd}}$ CALC button and 1:value button, hit enter. At the lower part of the screen you will see "x=" and a blinking cursor. You may enter any number for *x* and it will display the *y* value for any *x* value you input. Use this and plug in *x* = 0, thus finding the *y*-intercept, for each of the following graphs.

Exercise:

Problem:$\mathrm{Y}_1=-2x+5$

Exercise:

Problem: $\mathrm{Y}_1 = \frac{3x-8}{4}$

Solution:

$x = 0 \qquad y = -2$

Exercise:

Problem: $\mathrm{Y}_1 = \frac{x+5}{2}$

For the following exercises, use your graphing calculator to input the linear graphs in the Y= graph menu.

After graphing it, use the 2^{nd} CALC button and 2:zero button, hit enter. At the lower part of the screen you will see “left bound?” and a blinking cursor on the graph of the line. Move this cursor to the left of the *x*-intercept, hit ENTER. Now it says “right bound?” Move the cursor to the right of the *x*-intercept, hit enter. Now it says “guess?” Move your cursor to the left somewhere in between the left and right bound near the *x*-intercept. Hit enter. At the bottom of your screen it will display the coordinates of the *x*-intercept or the “zero” to the *y*-value. Use this to find the *x*-intercept.

Note: With linear/straight line functions the zero is not really a “guess,” but it is necessary to enter a “guess” so it will search and find the exact *x*-intercept between your right and left boundaries. With other types of functions (more than one *x*-intercept), they may be irrational numbers so “guess” is more appropriate to give it the correct limits to find a very close approximation between the left and right boundaries.

Exercise:

Problem: $\mathrm{Y}_1 = -8x + 6$

Solution:

$x = 0.75 \quad y = 0$

Exercise:

Problem: $\text{Y}_1 = 4x - 7$

Exercise:

Problem: $\text{Y}_1 = \frac{3x+5}{4}$ Round your answer to the nearest thousandth.

Solution:

$x = -1.667 \quad y = 0$

Extensions

Exercise:

Problem:

A man drove 10 mi directly east from his home, made a left turn at an intersection, and then traveled 5 mi north to his place of work. If a road was made directly from his home to his place of work, what would its distance be to the nearest tenth of a mile?

Exercise:

Problem:

If the road was made in the previous exercise, how much shorter would the man's one-way trip be every day?

Solution:

$15{-}11.2 \ = \ 3.8$ mi shorter

Exercise:

Problem:

Given these four points: $A\,(1,3), B\,(-3,5), C\,(4,7),$ and $D\,(5,-4)$,find the coordinates of the midpoint of line segments AB and CD.

Exercise:

Problem:

After finding the two midpoints in the previous exercise, find the distance between the two midpoints to the nearest thousandth.

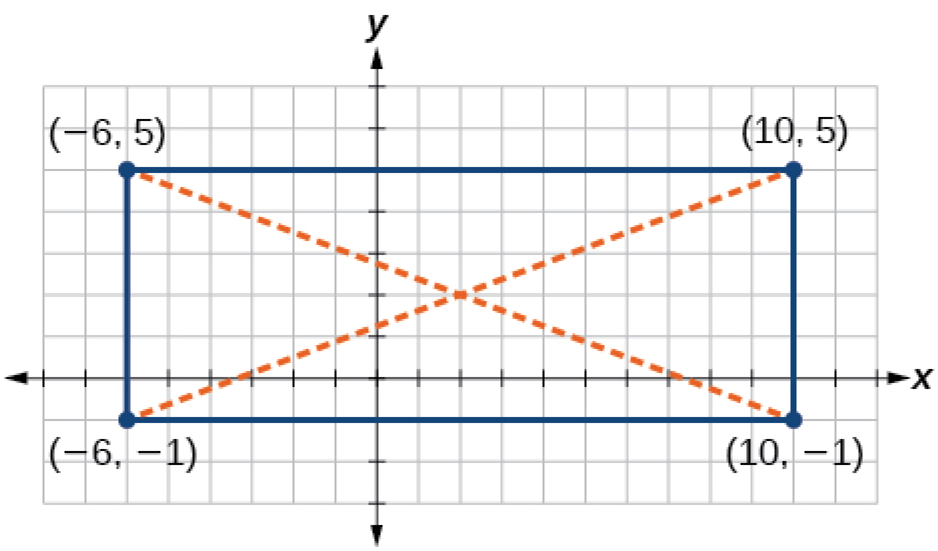
Solution:

6.042

Exercise:

Problem:

Given the graph of the rectangle shown and the coordinates of its vertices, prove that the diagonals of the rectangle are of equal length.



Exercise:

Problem:

In the previous exercise, find the coordinates of the midpoint for each diagonal.

Solution:

Midpoint of each diagonal is the same point $(2,2)$. Note this is a characteristic of rectangles, but not other quadrilaterals.

Real-World Applications

Exercise:

Problem:

The coordinates on a map for San Francisco are $(53,17)$ and those for Sacramento are $(123,78)$. Note that coordinates represent miles. Find the distance between the cities to the nearest mile.

Exercise:

Problem:

If San Jose's coordinates are $(76,-12)$,where the coordinates represent miles, find the distance between San Jose and San Francisco to the nearest mile.

Solution:

37 mi

Exercise:

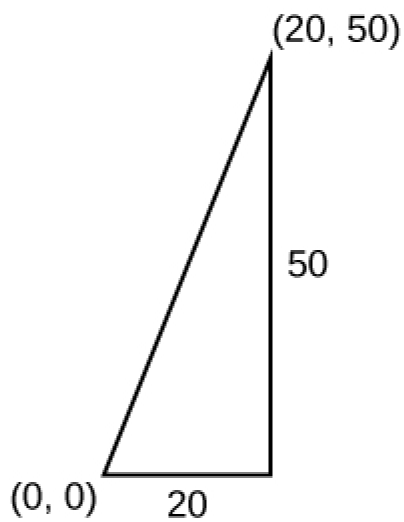
Problem:

A small craft in Lake Ontario sends out a distress signal. The coordinates of the boat in trouble were $(49, 64)$. One rescue boat is at the coordinates $(60, 82)$ and a second Coast Guard craft is at coordinates $(58, 47)$. Assuming both rescue craft travel at the same rate, which one would get to the distressed boat the fastest?

Exercise:

Problem:

A man on the top of a building wants to have a guy wire extend to a point on the ground 20 ft from the building. To the nearest foot, how long will the wire have to be if the building is 50 ft tall?



Solution:

54 ft

Exercise:

Problem:

If we rent a truck and pay a $75/day fee plus $.20 for every mile we travel, write a linear equation that would express the total cost y,using x to represent the number of miles we travel. Graph this function on your graphing calculator and find the total cost for one day if we travel 70 mi.

Glossary

Cartesian coordinate system
: a grid system designed with perpendicular axes invented by René Descartes

distance formula
: a formula that can be used to find the length of a line segment if the endpoints are known

equation in two variables
: a mathematical statement, typically written in *x* and *y*, in which two expressions are equal

graph in two variables
: the graph of an equation in two variables, which is always shown in two variables in the two-dimensional plane

intercepts
: the points at which the graph of an equation crosses the *x*-axis and the *y*-axis

midpoint formula
: a formula to find the point that divides a line segment into two parts of equal length

ordered pair
: a pair of numbers indicating horizontal displacement and vertical displacement from the origin; also known as a coordinate pair, (x, y)

origin
: the point where the two axes cross in the center of the plane, described by the ordered pair $(0, 0)$

quadrant
: one quarter of the coordinate plane, created when the axes divide the plane into four sections

x-axis
: the common name of the horizontal axis on a coordinate plane; a number line increasing from left to right

x-coordinate
: the first coordinate of an ordered pair, representing the horizontal displacement and direction from the origin

x-intercept
: the point where a graph intersects the *x*-axis; an ordered pair with a *y*-coordinate of zero

y-axis
: the common name of the vertical axis on a coordinate plane; a number line increasing from bottom to top

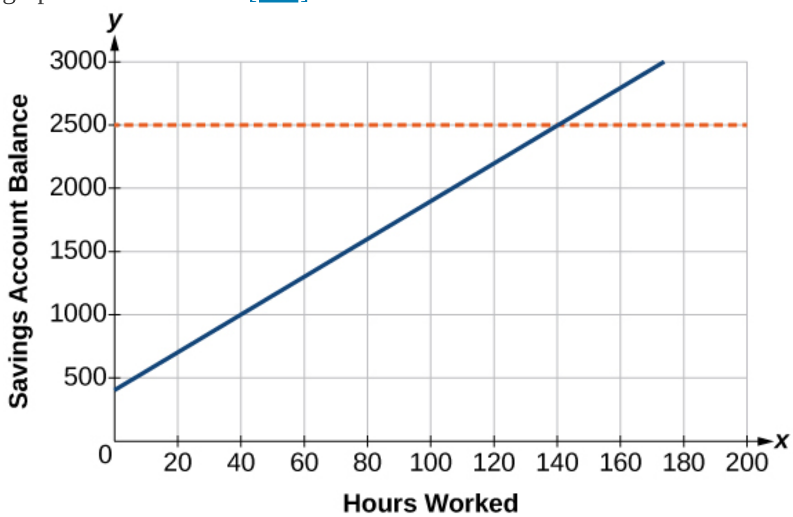
y-coordinate
: the second coordinate of an ordered pair, representing the vertical displacement and direction from the origin

y-intercept
: a point where a graph intercepts the *y*-axis; an ordered pair with an *x*-coordinate of zero

Linear Equations in One Variable
In this section you will:

- Solve equations in one variable algebraically.
- Solve a rational equation.
- Find a linear equation.
- Given the equations of two lines, determine whether their graphs are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Caroline is a full-time college student planning a spring break vacation. To earn enough money for the trip, she has taken a part-time job at the local bank that pays $15.00/hr, and she opened a savings account with an initial deposit of $400 on January 15. She arranged for direct deposit of her payroll checks. If spring break begins March 20 and the trip will cost approximately $2,500, how many hours will she have to work to earn enough to pay for her vacation? If she can only work 4 hours per day, how many days per week will she have to work? How many weeks will it take? In this section, we will investigate problems like this and others, which generate graphs like the line in [link].



Solving Linear Equations in One Variable

A **linear equation** is an equation of a straight line, written in one variable. The only power of the variable is 1. Linear equations in one variable may take the form $ax+b=0$ and are solved using basic algebraic operations.

We begin by classifying linear equations in one variable as one of three types: identity, conditional, or inconsistent. An **identity equation** is true for all values of the variable. Here is an example of an identity equation.
Equation:

$$3x=2x+x$$

The **solution set** consists of all values that make the equation true. For this equation, the solution set is all real numbers because any real number substituted for x will make the equation true.

A **conditional equation** is true for only some values of the variable. For example, if we are to solve the equation $5x+2=3x-6$,we have the following:
Equation:

$$\begin{aligned} 5x+2 &= 3x-6 \\ 2x &= -8 \\ x &= -4 \end{aligned}$$

The solution set consists of one number: $\{-4\}$. It is the only solution and, therefore, we have solved a conditional equation.

An **inconsistent equation** results in a false statement. For example, if we are to solve $5x-15=5\left(x-4\right)$,we have the following:

Equation:

$$\begin{aligned} 5x-15 &= 5x-20 & & \\ 5x-15-5x &= 5x-20-5x & & \text{Subtract } 5x \text{ from both sides.} \\ -15 &\neq -20 & & \text{False statement} \end{aligned}$$

Indeed, $-15 \neq -20$. There is no solution because this is an inconsistent equation.

Solving linear equations in one variable involves the fundamental properties of equality and basic algebraic operations. A brief review of those operations follows.

Note:
Linear Equation in One Variable
A linear equation in one variable can be written in the form

Equation:

$$ax+b=0$$

where *a* and *b* are real numbers, $a\neq 0$.

Note:
Given a linear equation in one variable, use algebra to solve it.
The following steps are used to manipulate an equation and isolate the unknown variable, so that the last line reads $x=$_________,if *x* is the unknown. There is no set order, as the steps used depend on what is given:

1. We may add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
2. Apply the distributive property as needed: $a\left(b+c\right)=ab+ac$.
3. Isolate the variable on one side of the equation.
4. When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

Example:
Exercise:

Problem:
Solving an Equation in One Variable

Solve the following equation: $2x + 7 = 19$.

Solution:

This equation can be written in the form $ax + b = 0$ by subtracting 19 from both sides. However, we may proceed to solve the equation in its original form by performing algebraic operations.

Equation:

$$\begin{aligned} 2x + 7 &= 19 & \\ 2x &= 12 & \text{Subtract 7 from both sides.} \\ x &= 6 & \text{Multiply both sides by } \tfrac{1}{2} \text{ or divide by 2.} \end{aligned}$$

The solution is 6

Note:
Exercise:

Problem: Solve the linear equation in one variable: $2x + 1 = -9$.

Solution:

$x = -5$

Example:
Exercise:

Problem:
Solving an Equation Algebraically When the Variable Appears on Both Sides

Solve the following equation: $4\left(x-3\right) + 12 = 15-5\left(x + 6\right)$.

Solution:

Apply standard algebraic properties.

Equation:

$$\begin{aligned} 4(x-3) + 12 &= 15 - 5(x+6) & \\ 4x - 12 + 12 &= 15 - 5x - 30 & \text{Apply the distributive property.} \\ 4x &= -15 - 5x & \text{Combine like terms.} \\ 9x &= -15 & \text{Place } x- \text{ terms on one side and simplify.} \\ x &= -\tfrac{15}{9} & \text{Multiply both sides by } \tfrac{1}{9}\text{, the reciprocal of 9.} \\ x &= -\tfrac{5}{3} & \end{aligned}$$

Analysis

This problem requires the distributive property to be applied twice, and then the properties of algebra are used to reach the final line, $x = -\frac{5}{3}$.

Note:
Exercise:

Problem: Solve the equation in one variable: $-2\left(3x - 1\right) + x = 14 - x$.

Solution:

$x = -3$

Solving a Rational Equation

In this section, we look at rational equations that, after some manipulation, result in a linear equation. If an equation contains at least one rational expression, it is a considered a **rational equation**.

Recall that a rational number is the ratio of two numbers, such as $\frac{2}{3}$ or $\frac{7}{2}$. A rational expression is the ratio, or quotient, of two polynomials. Here are three examples.
Equation:

$$\frac{x+1}{x^2-4}, \ \frac{1}{x-3}, \text{ or } \frac{4}{x^2+x-2}$$

Rational equations have a variable in the denominator in at least one of the terms. Our goal is to perform algebraic operations so that the variables appear in the numerator. In fact, we will eliminate all denominators by multiplying both sides of the equation by the least common denominator (LCD).

Finding the LCD is identifying an expression that contains the highest power of all of the factors in all of the denominators. We do this because when the equation is multiplied by the LCD, the common factors in the LCD and in each denominator will equal one and will cancel out.

Example:
Exercise:

Problem:
Solving a Rational Equation

Solve the rational equation: $\frac{7}{2x} - \frac{5}{3x} = \frac{22}{3}$.

Solution:

We have three denominators; $2x$, $3x$,and 3. The LCD must contain $2x$, $3x$,and 3. An LCD of $6x$ contains all three denominators. In other words, each denominator can be divided evenly into the LCD. Next, multiply both sides of the equation by the LCD $6x$.
Equation:

$$
\begin{array}{rcll}
(6x)\left[\frac{7}{2x}-\frac{5}{3x}\right] & = & \left[\frac{22}{3}\right](6x) & \\
(6x)\left(\frac{7}{2x}\right)-(6x)\left(\frac{5}{3x}\right) & = & \left(\frac{22}{3}\right)(6x) & \text{Use the distributive property.} \\
(\cancel{6x})\left(\frac{7}{\cancel{2x}}\right)-(\cancel{6x})\left(\frac{5}{\cancel{3x}}\right) & = & \left(\frac{22}{\cancel{3}}\right)(\cancel{6}x) & \text{Cancel out the common factors.} \\
3(7)-2(5) & = & 22(2x) & \text{Multiply remaining factors by each numerator.} \\
21-10 & = & 44x & \\
11 & = & 44x & \\
\frac{11}{44} & = & x & \\
\frac{1}{4} & = & x &
\end{array}
$$

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or subtracted—such as $(x+1)$. Always consider a binomial as an individual factor—the terms cannot be separated. For example, suppose a problem has three terms and the denominators are x,$x-1$,and $3x-3$. First, factor all denominators. We then have x,$(x-1)$,and $3(x-1)$ as the denominators. (Note the parentheses placed around the second denominator.) Only the last two denominators have a common factor of $(x-1)$. The x in the first denominator is separate from the x in the $(x-1)$ denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the x,one factor of $(x-1)$,and the 3. Thus, the LCD is the following:

Equation:

$$x\,(x-1)3=3x\,(x-1)$$

So, both sides of the equation would be multiplied by $3x\,(x-1)$. Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out.

Another example is a problem with two denominators, such as x and x^2+2x. Once the second denominator is factored as $x^2+2x=x\,(x+2)$,there is a common factor of x in both denominators and the LCD is $x\,(x+2)$.

Sometimes we have a rational equation in the form of a proportion; that is, when one fraction equals another fraction and there are no other terms in the equation.

Equation:

$$\frac{a}{b}=\frac{c}{d}$$

We can use another method of solving the equation without finding the LCD: cross-multiplication. We multiply terms by crossing over the equal sign.

If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}\times\frac{c}{d}$.

Multiply $a\,(d)$ and $b\,(c)$,which results in $ad=bc$.

Any solution that makes a denominator in the original expression equal zero must be excluded from the possibilities.

Note:
Rational Equations
A **rational equation** contains at least one rational expression where the variable appears in at least one of the denominators.

Note:
Given a rational equation, solve it.

1. Factor all denominators in the equation.
2. Find and exclude values that set each denominator equal to zero.
3. Find the LCD.
4. Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.
5. Solve the remaining equation.
6. Make sure to check solutions back in the original equations to avoid a solution producing zero in a denominator

Example:
Exercise:

Problem:
Solving a Rational Equation without Factoring

Solve the following rational equation:
Equation:

$$\frac{2}{x} - \frac{3}{2} = \frac{7}{2x}$$

Solution:

We have three denominators: x,2,and $2x$. No factoring is required. The product of the first two denominators is equal to the third denominator, so, the LCD is $2x$. Only one value is excluded from a solution set, 0. Next, multiply the whole equation (both sides of the equal sign) by $2x$.
Equation:

$$\begin{aligned}
2x\left[\tfrac{2}{x} - \tfrac{3}{2}\right] &= \left[\tfrac{7}{2x}\right]2x \\
2\cancel{x}\left(\frac{2}{\cancel{x}}\right) - \cancel{2}x\left(\frac{3}{\cancel{2}}\right) &= \left(\frac{7}{\cancel{2x}}\right)\cancel{2x} && \text{Distribute } 2x. \\
2(2) - 3x &= 7 && \text{Denominators cancel out.} \\
4 - 3x &= 7 \\
-3x &= 3 \\
x &= -1 \\
& \quad \text{or } \{-1\}
\end{aligned}$$

The proposed solution is −1, which is not an excluded value, so the solution set contains one number, $x = -1$,or $\{-1\}$ written in set notation.

Note:
Exercise:

Problem: Solve the rational equation: $\frac{2}{3x} = \frac{1}{4} - \frac{1}{6x}$.

Solution:

$x = \frac{10}{3}$

Example:
Exercise:

Problem:
Solving a Rational Equation by Factoring the Denominator

Solve the following rational equation: $\frac{1}{x} = \frac{1}{10} - \frac{3}{4x}$.

Solution:

First find the common denominator. The three denominators in factored form are $x, 10 = 2 \cdot 5$,and $4x = 2 \cdot 2 \cdot x.$ The smallest expression that is divisible by each one of the denominators is $20x.$ Only $x = 0$ is an excluded value. Multiply the whole equation by $20x.$
Equation:

$$\begin{array}{ccc} 20x\left(\frac{1}{x}\right) & = & \left(\frac{1}{10} - \frac{3}{4x}\right)20x \\ 20 & = & 2x - 15 \\ 35 & = & 2x \\ \frac{35}{2} & = & x \end{array}$$

The solution is $\frac{35}{2}$.

Note:
Exercise:

Problem: Solve the rational equation: $-\frac{5}{2x} + \frac{3}{4x} = -\frac{7}{4}$.

Solution:

$x = 1$

Example:
Exercise:

Problem:
Solving Rational Equations with a Binomial in the Denominator

Solve the following rational equations and state the excluded values:

a. $\frac{3}{x-6}=\frac{5}{x}$
b. $\frac{x}{x-3}=\frac{5}{x-3}-\frac{1}{2}$
c. $\frac{x}{x-2}=\frac{5}{x-2}-\frac{1}{2}$

Solution:

a. The denominators x and $x-6$ have nothing in common. Therefore, the LCD is the product $x\left(x-6\right)$. However, for this problem, we can cross-multiply.
Equation:

$$\begin{aligned} \frac{3}{x-6} &= \frac{5}{x} \\ 3x &= 5(x-6) && \text{Distribute.} \\ 3x &= 5x-30 \\ -2x &= -30 \\ x &= 15 \end{aligned}$$

The solution is 15. The excluded values are 6 and 0.

b. The LCD is $2\left(x-3\right)$. Multiply both sides of the equation by $2\left(x-3\right)$.
Equation:

$$\begin{aligned} 2(x-3)\left[\frac{x}{x-3}\right] &= \left[\frac{5}{x-3}-\frac{1}{2}\right]2(x-3) \\ \frac{2\,\cancel{(x-3)}\,x}{\cancel{x-3}} &= \frac{2\,\cancel{(x-3)}\,5}{\cancel{x-3}}-\frac{\cancel{2}\,(x-3)}{\cancel{2}} \\ 2x &= 10-(x-3) \\ 2x &= 10-x+3 \\ 2x &= 13-x \\ 3x &= 13 \\ x &= \frac{13}{3} \end{aligned}$$

The solution is $\frac{13}{3}$. The excluded value is 3.

c. The least common denominator is $2\left(x-2\right)$. Multiply both sides of the equation by $x\left(x-2\right)$.
Equation:

$$\begin{aligned} 2(x-2)\left[\frac{x}{x-2}\right] &= \left[\frac{5}{x-2}-\frac{1}{2}\right]2(x-2) \\ 2x &= 10-(x-2) \\ 2x &= 12-x \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

The solution is 4. The excluded value is 2.

Note:
Exercise:

Problem: Solve $\frac{-3}{2x+1} = \frac{4}{3x+1}$. State the excluded values.

Solution:

$x = -\frac{7}{17}$. Excluded values are $x = -\frac{1}{2}$ and $x = -\frac{1}{3}$.

Example:
Exercise:

Problem:
Solving a Rational Equation with Factored Denominators and Stating Excluded Values

Solve the rational equation after factoring the denominators: $\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{x^2-1}$. State the excluded values.

Solution:

We must factor the denominator x^2-1. We recognize this as the difference of squares, and factor it as $(x-1)(x+1)$. Thus, the LCD that contains each denominator is $(x-1)(x+1)$. Multiply the whole equation by the LCD, cancel out the denominators, and solve the remaining equation.
Equation:

$$\begin{array}{rcll} (x-1)(x+1)\left[\frac{2}{x+1} - \frac{1}{x-1}\right] & = & \left[\frac{2x}{(x-1)(x+1)}\right](x-1)(x+1) & \\ 2(x-1) - 1(x+1) & = & 2x & \\ 2x - 2 - x - 1 & = & 2x & \text{Distribute the negative sign.} \\ -3 - x & = & 0 & \\ -3 & = & x & \end{array}$$

The solution is -3. The excluded values are 1 and -1.

Note:
Exercise:

Problem: Solve the rational equation: $\frac{2}{x-2} + \frac{1}{x+1} = \frac{1}{x^2-x-2}$.

Solution:

$x = \frac{1}{3}$

Finding a Linear Equation

Perhaps the most familiar form of a linear equation is the slope-intercept form, written as $y = mx + b$,where $m = \text{slope}$ and $b = y\text{−intercept}$. Let us begin with the slope.

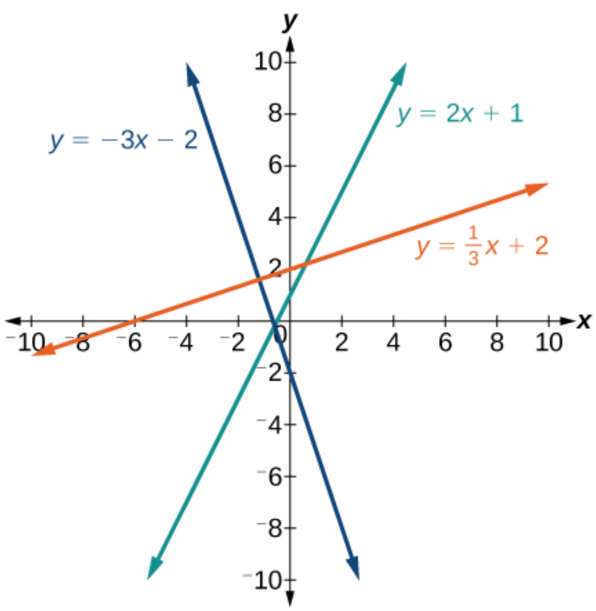
The Slope of a Line

The **slope** of a line refers to the ratio of the vertical change in *y* over the horizontal change in *x* between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If the slope is positive, the line slants to the right. If the slope is negative, the line slants to the left. As the slope increases, the line becomes steeper. Some examples are shown in [link]. The lines indicate the following slopes: $m = -3$,$m = 2$,and $m = \frac{1}{3}$.



Note:
The Slope of a Line
The slope of a line, *m*, represents the change in *y* over the change in *x*. Given two points, (x_1, y_1) and (x_2, y_2), the following formula determines the slope of a line containing these points:
Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example:
Exercise:

Problem:
Finding the Slope of a Line Given Two Points

Find the slope of a line that passes through the points $(2, -1)$ and $(-5, 3)$.

Solution:

We substitute the *y*-values and the *x*-values into the formula.
Equation:

$$\begin{aligned} m &= \frac{3-(-1)}{-5-2} \\ &= \frac{4}{-7} \\ &= -\frac{4}{7} \end{aligned}$$

The slope is $-\frac{4}{7}$.

Analysis

It does not matter which point is called (x_1, y_1) or (x_2, y_2). As long as we are consistent with the order of the *y* terms and the order of the *x* terms in the numerator and denominator, the calculation will yield the same result.

Note:
Exercise:

Problem: Find the slope of the line that passes through the points $(-2, 6)$ and $(1, 4)$.

Solution:

$m = -\frac{2}{3}$

Example:
Exercise:

Problem:
Identifying the Slope and *y*-intercept of a Line Given an Equation

Identify the slope and *y*-intercept, given the equation $y = -\frac{3}{4}x - 4$.

Solution:

As the line is in $y = mx + b$ form, the given line has a slope of $m = -\frac{3}{4}$. The *y*-intercept is $b = -4$.

Analysis

The *y*-intercept is the point at which the line crosses the *y*-axis. On the *y*-axis, $x = 0$. We can always identify the *y*-intercept when the line is in slope-intercept form, as it will always equal *b*. Or, just substitute $x = 0$ and

solve for *y*.

The Point-Slope Formula

Given the slope and one point on a line, we can find the equation of the line using the point-slope formula.
Equation:

$$y - y_1 = m\left(x - x_1\right)$$

This is an important formula, as it will be used in other areas of college algebra and often in calculus to find the equation of a tangent line. We need only one point and the slope of the line to use the formula. After substituting the slope and the coordinates of one point into the formula, we simplify it and write it in slope-intercept form.

Note:
The Point-Slope Formula
Given one point and the slope, the point-slope formula will lead to the equation of a line:
Equation:

$$y - y_1 = m\left(x - x_1\right)$$

Example:
Exercise:

Problem:
Finding the Equation of a Line Given the Slope and One Point

Write the equation of the line with slope $m = -3$ and passing through the point $(4, 8)$. Write the final equation in slope-intercept form.

Solution:

Using the point-slope formula, substitute -3 for *m* and the point $(4, 8)$ for (x_1, y_1).
Equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 8 &= -3(x - 4) \\ y - 8 &= -3x + 12 \\ y &= -3x + 20 \end{aligned}$$

Analysis

Note that any point on the line can be used to find the equation. If done correctly, the same final equation will be obtained.

Note:

Exercise:

Problem:

Given $m = 4$,find the equation of the line in slope-intercept form passing through the point $(2, 5)$.

Solution:

$y = 4x{-}3$

Example:
Exercise:

Problem:
Finding the Equation of a Line Passing Through Two Given Points

Find the equation of the line passing through the points $(3, 4)$ and $(0, -3)$. Write the final equation in slope-intercept form.

Solution:

First, we calculate the slope using the slope formula and two points.
Equation:

$$\begin{aligned} m &= \tfrac{-3-4}{0-3} \\ &= \tfrac{-7}{-3} \\ &= \tfrac{7}{3} \end{aligned}$$

Next, we use the point-slope formula with the slope of $\frac{7}{3}$,and either point. Let's pick the point $(3, 4)$ for (x_1, y_1).
Equation:

$$\begin{aligned} y - 4 &= \tfrac{7}{3}(x - 3) & \\ y - 4 &= \tfrac{7}{3}x - 7 & \text{Distribute the } \tfrac{7}{3}. \\ y &= \tfrac{7}{3}x - 3 & \end{aligned}$$

In slope-intercept form, the equation is written as $y = \frac{7}{3}x - 3$.

Analysis

To prove that either point can be used, let us use the second point $(0, -3)$ and see if we get the same equation.
Equation:

$$\begin{aligned} y - (-3) &= \tfrac{7}{3}(x - 0) \\ y + 3 &= \tfrac{7}{3}x \\ y &= \tfrac{7}{3}x - 3 \end{aligned}$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Standard Form of a Line

Another way that we can represent the equation of a line is in standard form. Standard form is given as
Equation:

$$Ax + By = C$$

where A,B,and $C$are integers. The *x*- and *y*-terms are on one side of the equal sign and the constant term is on the other side.

Example:
Exercise:

Problem:
Finding the Equation of a Line and Writing It in Standard Form

Find the equation of the line with $m = -6$ and passing through the point $\left(\frac{1}{4}, -2\right)$. Write the equation in standard form.

Solution:

We begin using the point-slope formula.
Equation:

$$\begin{array}{rcl} y - (-2) & = & -6\left(x - \frac{1}{4}\right) \\ y + 2 & = & -6x + \frac{3}{2} \end{array}$$

From here, we multiply through by 2, as no fractions are permitted in standard form, and then move both variables to the left aside of the equal sign and move the constants to the right.
Equation:

$$\begin{array}{rcl} 2(y + 2) & = & \left(-6x + \frac{3}{2}\right)2 \\ 2y + 4 & = & -12x + 3 \\ 12x + 2y & = & -1 \end{array}$$

This equation is now written in standard form.

Note:
Exercise:

Problem:

Find the equation of the line in standard form with slope $m = -\frac{1}{3}$ and passing through the point $\left(1, \frac{1}{3}\right)$.

Solution:

$x+3y=2$

Vertical and Horizontal Lines

The equations of vertical and horizontal lines do not require any of the preceding formulas, although we can use the formulas to prove that the equations are correct. The equation of a vertical line is given as

Equation:

$$x=c$$

where *c* is a constant. The slope of a vertical line is undefined, and regardless of the *y*-value of any point on the line, the *x*-coordinate of the point will be *c*.

Suppose that we want to find the equation of a line containing the following points: $(-3,-5),(-3,1),(-3,3)$,and $(-3,5).$ First, we will find the slope.

Equation:

$$m=\frac{5-3}{-3-(-3)}=\frac{2}{0}$$

Zero in the denominator means that the slope is undefined and, therefore, we cannot use the point-slope formula. However, we can plot the points. Notice that all of the *x*-coordinates are the same and we find a vertical line through $x=-3$. See [link].

The equation of a horizontal line is given as

Equation:

$$y=c$$

where *c* is a constant. The slope of a horizontal line is zero, and for any *x*-value of a point on the line, the *y*-coordinate will be *c*.

Suppose we want to find the equation of a line that contains the following set of points: $(-2,-2),(0,-2),(3,-2)$,and $(5,-2)$.We can use the point-slope formula. First, we find the slope using any two points on the line.

Equation:

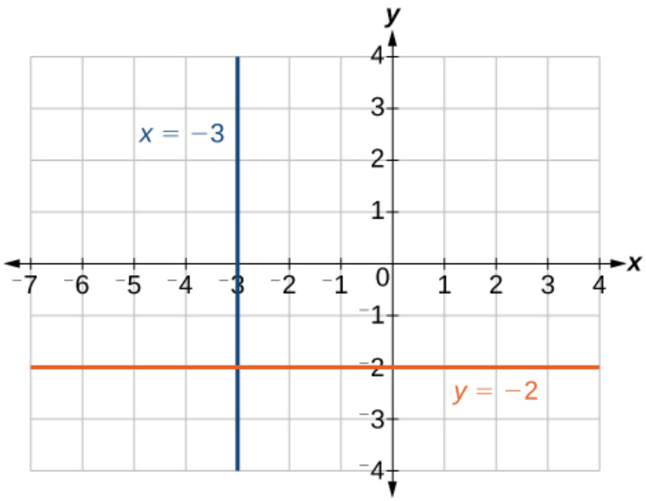
$$\begin{array}{rcl}m&=&\frac{-2-(-2)}{0-(-2)}\\&=&\frac{0}{2}\\&=&0\end{array}$$

Use any point for (x_1,y_1) in the formula, or use the *y*-intercept.

Equation:

$$\begin{array}{rcl}y-(-2)&=&0(x-3)\\y+2&=&0\\y&=&-2\end{array}$$

The graph is a horizontal line through $y = -2$. Notice that all of the *y*-coordinates are the same. See [link].



The line *x* = −3 is a vertical line. The line *y* = −2 is a horizontal line.

Example:
Exercise:

Problem:
Finding the Equation of a Line Passing Through the Given Points

Find the equation of the line passing through the given points: $(1, -3)$ and $(1, 4)$.

Solution:

The *x*-coordinate of both points is 1. Therefore, we have a vertical line, $x = 1$.

Note:
Exercise:

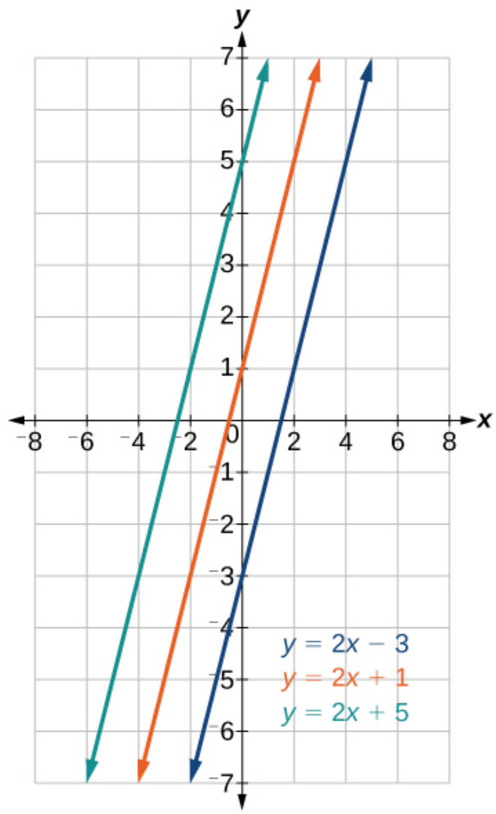
Problem: Find the equation of the line passing through $(-5, 2)$ and $(2, 2)$.

Solution:

Horizontal line: $y = 2$

Determining Whether Graphs of Lines are Parallel or Perpendicular

Parallel lines have the same slope and different y-intercepts. Lines that are parallel to each other will never intersect. For example, [link] shows the graphs of various lines with the same slope, $m = 2$.



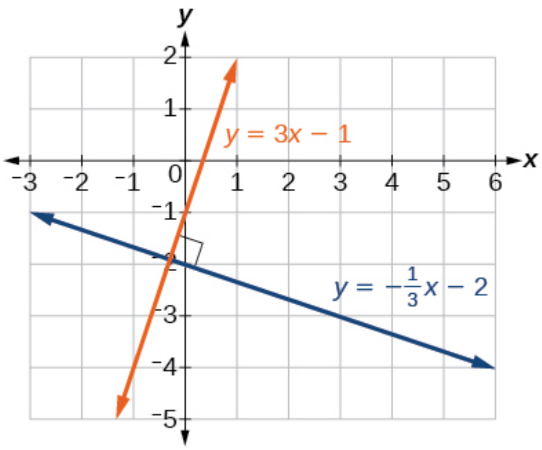
Parallel lines

All of the lines shown in the graph are parallel because they have the same slope and different y-intercepts.

Lines that are perpendicular intersect to form a 90°-angle. The slope of one line is the negative reciprocal of the other. We can show that two lines are perpendicular if the product of the two slopes is $-1 : m_1 \cdot m_2 = -1$. For example, [link] shows the graph of two perpendicular lines. One line has a slope of 3; the other line has a slope of $-\frac{1}{3}$.

Equation:

$$\begin{array}{rcl} m_1 \cdot m_2 & = & -1 \\ 3 \cdot \left(-\frac{1}{3}\right) & = & -1 \end{array}$$



Perpendicular lines

Example:
Exercise:

Problem:
Graphing Two Equations, and Determining Whether the Lines are Parallel, Perpendicular, or Neither

Graph the equations of the given lines, and state whether they are parallel, perpendicular, or neither: $3y=-4x+3$ and $3x-4y=8$.

Solution:

The first thing we want to do is rewrite the equations so that both equations are in slope-intercept form.

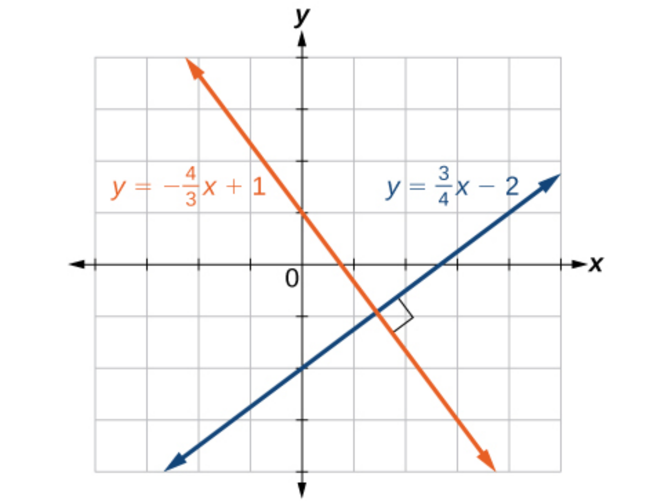
First equation:
Equation:

$$\begin{array}{rcl}3y & = & -4x+3\\ y & = & -\frac{4}{3}x+1\end{array}$$

Second equation:
Equation:

$$\begin{array}{rcl}3x-4y & = & 8\\ -4y & = & -3x+8\\ y & = & \frac{3}{4}x-2\end{array}$$

See the graph of both lines in [link]



From the graph, we can see that the lines appear perpendicular, but we must compare the slopes.

Equation:

$$\begin{aligned} m_1 &= -\tfrac{4}{3} \\ m_2 &= \tfrac{3}{4} \\ m_1 \cdot m_2 &= \left(-\tfrac{4}{3}\right)\left(\tfrac{3}{4}\right) = -1 \end{aligned}$$

The slopes are negative reciprocals of each other, confirming that the lines are perpendicular.

Note:

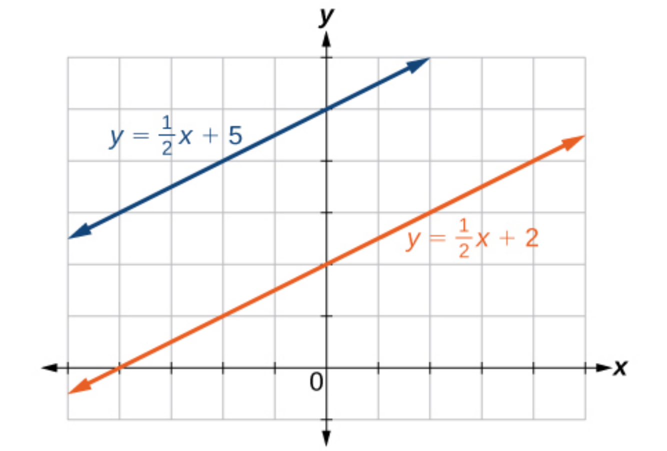
Exercise:

Problem:

Graph the two lines and determine whether they are parallel, perpendicular, or neither: $2y - x = 10$ and $2y = x + 4$.

Solution:

Parallel lines: equations are written in slope-intercept form.



Writing the Equations of Lines Parallel or Perpendicular to a Given Line

As we have learned, determining whether two lines are parallel or perpendicular is a matter of finding the slopes. To write the equation of a line parallel or perpendicular to another line, we follow the same principles as we do for finding the equation of any line. After finding the slope, use the point-slope formula to write the equation of the new line.

Note:
Given an equation for a line, write the equation of a line parallel or perpendicular to it.

1. Find the slope of the given line. The easiest way to do this is to write the equation in slope-intercept form.
2. Use the slope and the given point with the point-slope formula.
3. Simplify the line to slope-intercept form and compare the equation to the given line.

Example:
Exercise:

Problem:
Writing the Equation of a Line Parallel to a Given Line Passing Through a Given Point

Write the equation of line parallel to a $5x + 3y = 1$ and passing through the point $(3, 5)$.

Solution:

First, we will write the equation in slope-intercept form to find the slope.
Equation:

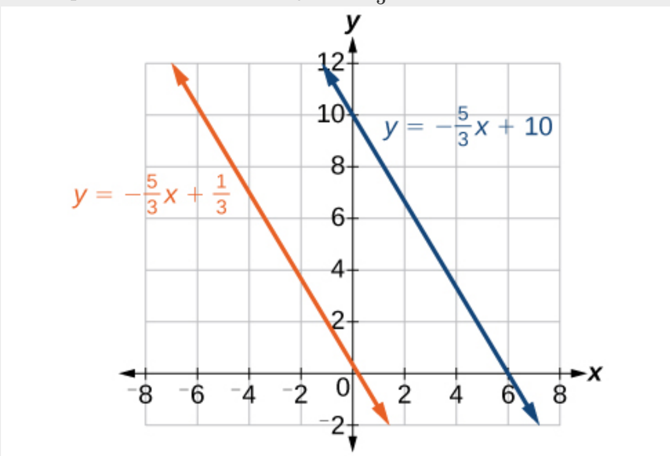
$$\begin{array}{rcl} 5x + 3y & = & 1 \\ 3y & = & 5x + 1 \\ y & = & -\frac{5}{3}x + \frac{1}{3} \end{array}$$

The slope is $m = -\frac{5}{3}$. The y-intercept is $\frac{1}{3}$,but that really does not enter into our problem, as the only thing we need for two lines to be parallel is the same slope. The one exception is that if the y-intercepts are the same, then the two lines are the same line. The next step is to use this slope and the given point with the point-slope formula.

Equation:

$$\begin{aligned} y - 5 &= -\frac{5}{3}(x - 3) \\ y - 5 &= -\frac{5}{3}x + 5 \\ y &= -\frac{5}{3}x + 10 \end{aligned}$$

The equation of the line is $y = -\frac{5}{3}x + 10$. See [link].



Note:

Exercise:

Problem: Find the equation of the line parallel to $5x = 7 + y$ and passing through the point $(-1, -2)$.

Solution:

$y = 5x + 3$

Example:

Exercise:

Problem:

Finding the Equation of a Line Perpendicular to a Given Line Passing Through a Given Point

Find the equation of the line perpendicular to $5x - 3y + 4 = 0$ and passing through the point $(-4, 1)$.

Solution:

The first step is to write the equation in slope-intercept form.

Equation:

$$\begin{array}{rcl} 5x-3y+4 & = & 0 \\ -3y & = & -5x-4 \\ y & = & \frac{5}{3}x+\frac{4}{3} \end{array}$$

We see that the slope is $m=\frac{5}{3}$. This means that the slope of the line perpendicular to the given line is the negative reciprocal, or $-\frac{3}{5}$. Next, we use the point-slope formula with this new slope and the given point.

Equation:

$$\begin{array}{rcl} y-1 & = & -\frac{3}{5}(x-(-4)) \\ y-1 & = & -\frac{3}{5}x-\frac{12}{5} \\ y & = & -\frac{3}{5}x-\frac{12}{5}+\frac{5}{5} \\ y & = & -\frac{3}{5}x-\frac{7}{5} \end{array}$$

Note:
Access these online resources for additional instruction and practice with linear equations.

- Solving rational equations
- Equation of a line given two points
- Finding the equation of a line perpendicular to another line through a given point
- Finding the equation of a line parallel to another line through a given point

Key Concepts

- We can solve linear equations in one variable in the form $ax+b=0$ using standard algebraic properties. See [link] and [link].
- A rational expression is a quotient of two polynomials. We use the LCD to clear the fractions from an equation. See [link] and [link].
- All solutions to a rational equation should be verified within the original equation to avoid an undefined term, or zero in the denominator. See [link] and [link].
- Given two points, we can find the slope of a line using the slope formula. See [link].
- We can identify the slope and *y*-intercept of an equation in slope-intercept form. See [link].
- We can find the equation of a line given the slope and a point. See [link].
- We can also find the equation of a line given two points. Find the slope and use the point-slope formula. See [link].
- The standard form of a line has no fractions. See [link].
- Horizontal lines have a slope of zero and are defined as $y=c$,where *c* is a constant.
- Vertical lines have an undefined slope (zero in the denominator), and are defined as $x=c$,where *c* is a constant. See [link].
- Parallel lines have the same slope and different *y*-intercepts. See [link].
- Perpendicular lines have slopes that are negative reciprocals of each other unless one is horizontal and the other is vertical. See [link].

Section Exercises

Verbal

Exercise:

Problem: What does it mean when we say that two lines are parallel?

Solution:

It means they have the same slope.

Exercise:

Problem:

What is the relationship between the slopes of perpendicular lines (assuming neither is horizontal nor vertical)?

Exercise:

Problem:

How do we recognize when an equation, for example $y = 4x + 3$,will be a straight line (linear) when graphed?

Solution:

The exponent of the x variable is 1. It is called a first-degree equation.

Exercise:

Problem: What does it mean when we say that a linear equation is inconsistent?

Exercise:

Problem: When solving the following equation:

$\frac{2}{x-5} = \frac{4}{x+1}$

explain why we must exclude $x = 5$ and $x = -1$ as possible solutions from the solution set.

Solution:

If we insert either value into the equation, they make an expression in the equation undefined (zero in the denominator).

Algebraic

For the following exercises, solve the equation for x.

Exercise:

Problem:$7x + 2 = 3x - 9$

Exercise:

Problem:$4x - 3 = 5$

Solution:

$x = 2$

Exercise:

Problem: $3(x+2) - 12 = 5(x+1)$

Exercise:

Problem: $12 - 5(x+3) = 2x - 5$

Solution:

$x = \frac{2}{7}$

Exercise:

Problem: $\frac{1}{2} - \frac{1}{3}x = \frac{4}{3}$

Exercise:

Problem: $\frac{x}{3} - \frac{3}{4} = \frac{2x+3}{12}$

Solution:

$x = 6$

Exercise:

Problem: $\frac{2}{3}x + \frac{1}{2} = \frac{31}{6}$

Exercise:

Problem: $3(2x-1) + x = 5x + 3$

Solution:

$x = 3$

Exercise:

Problem: $\frac{2x}{3} - \frac{3}{4} = \frac{x}{6} + \frac{21}{4}$

Exercise:

Problem: $\frac{x+2}{4} - \frac{x-1}{3} = 2$

Solution:

$x = -14$

For the following exercises, solve each rational equation for x. State all x-values that are excluded from the solution set.

Exercise:

Problem: $\frac{3}{x} - \frac{1}{3} = \frac{1}{6}$

Exercise:

Problem: $2 - \frac{3}{x+4} = \frac{x+2}{x+4}$

Solution:

$x \neq -4$; $x = -3$

Exercise:

Problem: $\frac{3}{x-2} = \frac{1}{x-1} + \frac{7}{(x-1)(x-2)}$

Exercise:

Problem: $\frac{3x}{x-1} + 2 = \frac{3}{x-1}$

Solution:

$x \neq 1$; when we solve this we get $x = 1$, which is excluded, therefore NO solution

Exercise:

Problem: $\frac{5}{x+1} + \frac{1}{x-3} = \frac{-6}{x^2-2x-3}$

Exercise:

Problem: $\frac{1}{x} = \frac{1}{5} + \frac{3}{2x}$

Solution:

$x \neq 0$; $x = -\frac{5}{2}$

For the following exercises, find the equation of the line using the point-slope formula.

Write all the final equations using the slope-intercept form.

Exercise:

Problem: $(0, 3)$ with a slope of $\frac{2}{3}$

Exercise:

Problem: $(1, 2)$ with a slope of $-\frac{4}{5}$

Solution:

$y = -\frac{4}{5}x + \frac{14}{5}$

Exercise:

Problem: *x*-intercept is 1, and $(-2, 6)$

Exercise:

Problem: y-intercept is 2, and $(4, -1)$

Solution:

$y = -\frac{3}{4}x + 2$

Exercise:

Problem: $(-3, 10)$ and $(5, -6)$

Exercise:

Problem: $(1, 3)\ \text{ and }\ (5, 5)$

Solution:

$y = \frac{1}{2}x + \frac{5}{2}$

Exercise:

Problem: parallel to $y = 2x + 5$ and passes through the point $(4, 3)$

Exercise:

Problem: perpendicular to $3y = x - 4$ and passes through the point $(-2, 1)$.

Solution:

$y = -3x - 5$

For the following exercises, find the equation of the line using the given information.

Exercise:

Problem: $(-2, 0)$ and $(-2, 5)$

Exercise:

Problem: $(1, 7)$ and $(3, 7)$

Solution:

$y = 7$

Exercise:

Problem: The slope is undefined and it passes through the point $(2, 3)$.

Exercise:

Problem: The slope equals zero and it passes through the point $(1, -4)$.

Solution:

$y=-4$

Exercise:

Problem: The slope is $\frac{3}{4}$ and it passes through the point $(1,4)$.

Exercise:

Problem: $(-1,3)$ and $(4,-5)$

Solution:

$8x+5y=7$

Graphical

For the following exercises, graph the pair of equations on the same axes, and state whether they are parallel, perpendicular, or neither.

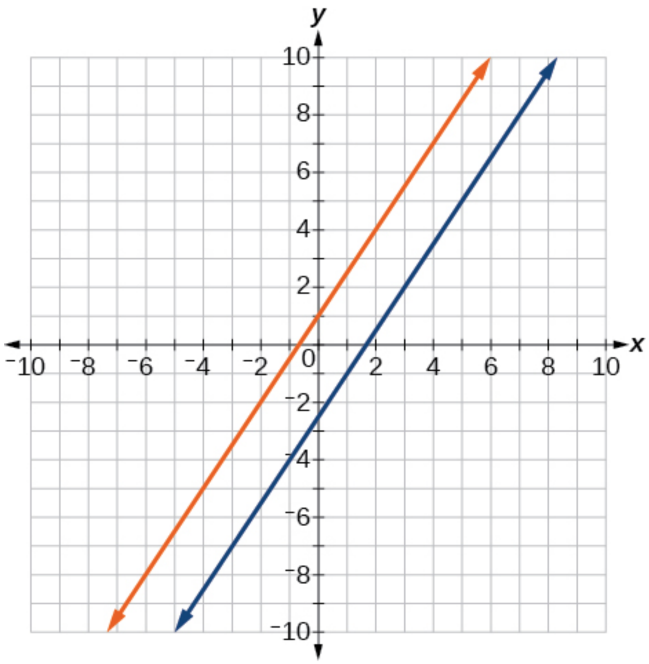
Exercise:

Problem:
$$y=2x+7$$
$$y=-\frac{1}{2}x-4$$

Exercise:

Problem:
$$3x-2y=5$$
$$6y-9x=6$$

Solution:



Parallel

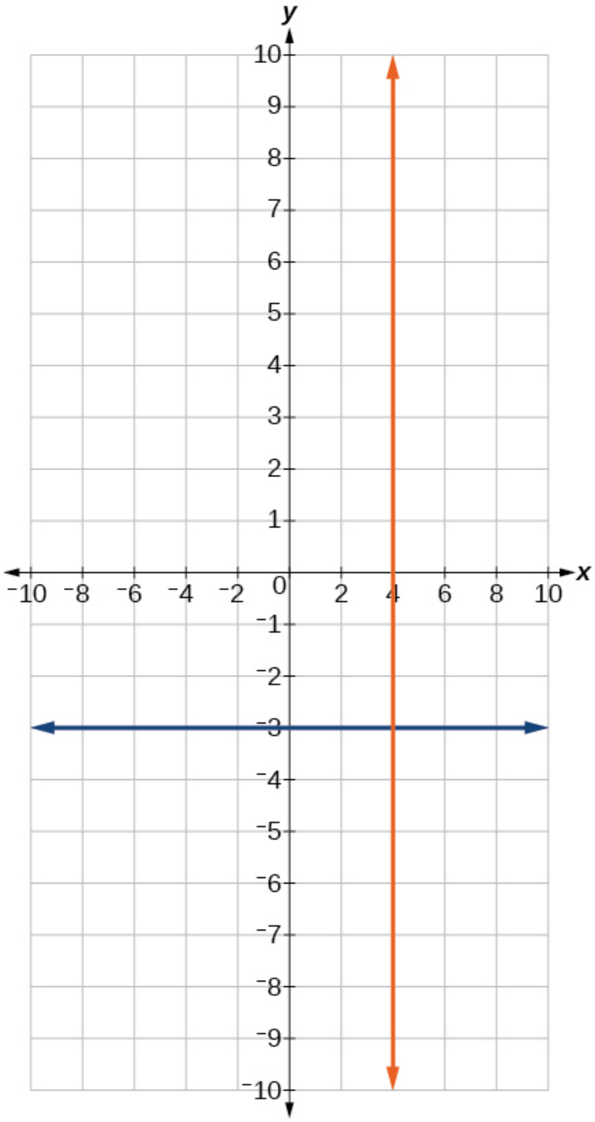
Exercise:

Problem: $y = \frac{3x+1}{4}$
$y = 3x + 2$

Exercise:

Problem: $x = 4$
$y = -3$

Solution:



Perpendicular

Numeric

For the following exercises, find the slope of the line that passes through the given points.

Exercise:

Problem: $(5,4)$ and $(7,9)$

Exercise:

Problem: $(-3,2)$ and $(4,-7)$

Solution:

$m=-\frac{9}{7}$

Exercise:

Problem: $(-5,4)$ and $(2,4)$

Exercise:

Problem: $(-1,-2)$ and $(3,4)$

Solution:

$m=\frac{3}{2}$

Exercise:

Problem: $(3,-2)$and $(3,-2)$

For the following exercises, find the slope of the lines that pass through each pair of points and determine whether the lines are parallel or perpendicular.

Exercise:

Problem: $\begin{array}{c}(-1,3)\ \text{ and }\ (5,1)\\(-2,3)\ \text{ and }\ (0,9)\end{array}$

Solution:

$m_1=-\frac{1}{3},\quad m_2=3;\quad \text{Perpendicular.}$

Exercise:

Problem: $\begin{array}{c}(2,5)\ \text{ and }\ (5,9)\\(-1,-1)\ \text{ and }\ (2,3)\end{array}$

Technology

For the following exercises, express the equations in slope intercept form (rounding each number to the thousandths place). Enter this into a graphing calculator as Y1, then adjust the ymin and ymax values for your window to include where the *y*-intercept occurs. State your ymin and ymax values.

Exercise:

Problem: $0.537x-2.19y=100$

Solution:

$y=0.245x-45.662$. Answers may vary. $y_{\min}=-50,\ y_{\max}=-40$

Exercise:

Problem: $4{,}500x-200y=9{,}528$

Exercise:

Problem: $\frac{200-30y}{x} = 70$

Solution:

$y = -2.333x + 6.667$. Answers may vary. $y_{\text{min}} = -10,\ y_{\text{max}} = 10$

Extensions

Exercise:

Problem:

Starting with the point-slope formula $y - y_1 = m(x - x_1)$,solve this expression for x in terms of x_1, y, y_1,and m.

Exercise:

Problem:

Starting with the standard form of an equation $\mathrm{A}x + \mathrm{B}y = \mathrm{C}$,solve this expression for y in terms of $A, B, C,$ and $x.$ Then put the expression in slope-intercept form.

Solution:

$y = -\frac{A}{B}x + \frac{C}{B}$

Exercise:

Problem:

Use the above derived formula to put the following standard equation in slope intercept form: $7x - 5y = 25.$

Exercise:

Problem:

Given that the following coordinates are the vertices of a rectangle, prove that this truly is a rectangle by showing the slopes of the sides that meet are perpendicular.

$(-1, 1)$, $(2, 0)$, $(3, 3)$,and $(0, 4)$

Solution:

The slope for $(-1, 1)$ to $(0, 4)$ is 3.
The slope for $(-1, 1)$ to $(2, 0)$ is $\frac{-1}{3}$.
The slope for $(2, 0)$ to $(3, 3)$ is 3.
The slope for $(0, 4)$ to $(3, 3)$ is $\frac{-1}{3}$.

Yes they are perpendicular.

Exercise:

Problem: Find the slopes of the diagonals in the previous exercise. Are they perpendicular?

Real-World Applications

Exercise:

Problem:

The slope for a wheelchair ramp for a home has to be $\frac{1}{12}$. If the vertical distance from the ground to the door bottom is 2.5 ft, find the distance the ramp has to extend from the home in order to comply with the needed slope.

Solution:

30 ft

Exercise:

Problem:

If the profit equation for a small business selling x number of item one and y number of item two is $p = 3x + 4y$, find the y value when $p = \$453$ and $x = 75$.

For the following exercises, use this scenario: The cost of renting a car is $45/wk plus $0.25/mi traveled during that week. An equation to represent the cost would be $y = 45 + .25x$, where x is the number of miles traveled.

Exercise:

Problem: What is your cost if you travel 50 mi?

Solution:

$57.50

Exercise:

Problem: If your cost were $\$63.75$, how many miles were you charged for traveling?

Exercise:

Problem:

Suppose you have a maximum of $100 to spend for the car rental. What would be the maximum number of miles you could travel?

Solution:

220 mi

Glossary

conditional equation
: an equation that is true for some values of the variable

identity equation

an equation that is true for all values of the variable

inconsistent equation

an equation producing a false result

linear equation

an algebraic equation in which each term is either a constant or the product of a constant and the first power of a variable

solution set

the set of all solutions to an equation

slope

the change in y-values over the change in x-values

rational equation

an equation consisting of a fraction of polynomials

Models and Applications
In this section you will:

- Set up a linear equation to solve a real-world application.
- Use a formula to solve a real-world application.

Credit: Kevin Dooley

Josh is hoping to get an A in his college algebra class. He has scores of 75, 82, 95, 91, and 94 on his first five tests. Only the final exam remains, and the maximum of points that can be earned is 100. Is it possible for Josh to end the course with an A? A simple linear equation will give Josh his answer.

Many real-world applications can be modeled by linear equations. For example, a cell phone package may include a monthly service fee plus a charge per minute of talk-time; it costs a widget manufacturer a certain amount to produce x widgets per month plus monthly operating charges; a

car rental company charges a daily fee plus an amount per mile driven. These are examples of applications we come across every day that are modeled by linear equations. In this section, we will set up and use linear equations to solve such problems.

Setting up a Linear Equation to Solve a Real-World Application

To set up or model a linear equation to fit a real-world application, we must first determine the known quantities and define the unknown quantity as a variable. Then, we begin to interpret the words as mathematical expressions using mathematical symbols. Let us use the car rental example above. In this case, a known cost, such as \$0.10/mi, is multiplied by an unknown quantity, the number of miles driven. Therefore, we can write $0.10x.$ This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it, according to the problem. As these amounts do not change, we call them fixed costs. Consider a car rental agency that charges \$0.10/mi plus a daily fee of \$50. We can use these quantities to model an equation that can be used to find the daily car rental cost C.

Equation:

$$C=0.10x+50$$

When dealing with real-world applications, there are certain expressions that we can translate directly into math. [link] lists some common verbal expressions and their equivalent mathematical expressions.

Verbal	Translation to Math Operations
One number exceeds another by *a*	$x,\ x+a$
Twice a number	$2x$
One number is *a* more than another number	$x,\ x+a$
One number is *a* less than twice another number	$x,\ 2x-a$
The product of a number and *a*, decreased by *b*	$ax-b$
The quotient of a number and the number plus *a* is three times the number	$\frac{x}{x+a}=3x$
The product of three times a number and the number decreased by *b* is *c*	$3x\left(x-b\right)=c$

Note:
Given a real-world problem, model a linear equation to fit it.

1. Identify known quantities.
2. Assign a variable to represent the unknown quantity.
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words as mathematical operations.
5. Solve the equation. Be sure the solution can be explained in words, including the units of measure.

Example:
Exercise:

Problem:
Modeling a Linear Equation to Solve an Unknown Number Problem

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution:

Let x equal the first number. Then, as the second number exceeds the first by 17, we can write the second number as $x+17$. The sum of the two numbers is 31. We usually interpret the word *is* as an equal sign.
Equation:

$$\begin{aligned} x+(x+17) &= 31 \\ 2x+17 &= 31 \qquad \text{Simplify and solve.} \\ 2x &= 14 \\ x &= 7 \\ \\ x+17 &= 7+17 \\ &= 24 \end{aligned}$$

The two numbers are 7 and 24.

Note:
Exercise:

Problem:

Find a linear equation to solve for the following unknown quantities: One number is three more than twice another number. If the sum of the two numbers is 36,find the numbers.

Solution:

11 and 25

Example:
Exercise:

Problem:
Setting Up a Linear Equation to Solve a Real-World Application

There are two cell phone companies that offer different packages. Company A charges a monthly service fee of \$34 plus \$.05/min talk-time. Company B charges a monthly service fee of \$40 plus \$.04/min talk-time.

a. Write a linear equation that models the packages offered by both companies.
b. If the average number of minutes used each month is 1,160, which company offers the better plan?
c. If the average number of minutes used each month is 420, which company offers the better plan?
d. How many minutes of talk-time would yield equal monthly statements from both companies?

Solution:

a. The model for Company A can be written as $A = 0.05x + 34$. This includes the variable cost of $0.05x$ plus the monthly service charge of \$34. Company B's package charges a higher monthly

fee of \$40, but a lower variable cost of $0.04x$. Company B's model can be written as $B = 0.04x + \$40$.

b. If the average number of minutes used each month is 1,160, we have the following:

Equation:

$$\begin{aligned}\text{Company } A &= 0.05(1,160) + 34 \\ &= 58 + 34 \\ &= 92\end{aligned}$$

$$\begin{aligned}\text{Company } B &= 0.04(1,160) + 40 \\ &= 46.4 + 40 \\ &= 86.4\end{aligned}$$

So, Company B offers the lower monthly cost of \$86.40 as compared with the \$92 monthly cost offered by Company A when the average number of minutes used each month is 1,160.

c. If the average number of minutes used each month is 420, we have the following:

Equation:

$$\begin{aligned}\text{Company } A &= 0.05(420) + 34 \\ &= 21 + 34 \\ &= 55\end{aligned}$$

$$\begin{aligned}\text{Company } B &= 0.04(420) + 40 \\ &= 16.8 + 40 \\ &= 56.8\end{aligned}$$

If the average number of minutes used each month is 420, then Company A offers a lower monthly cost of \$55 compared to

Company *B*'s monthly cost of $56.80.

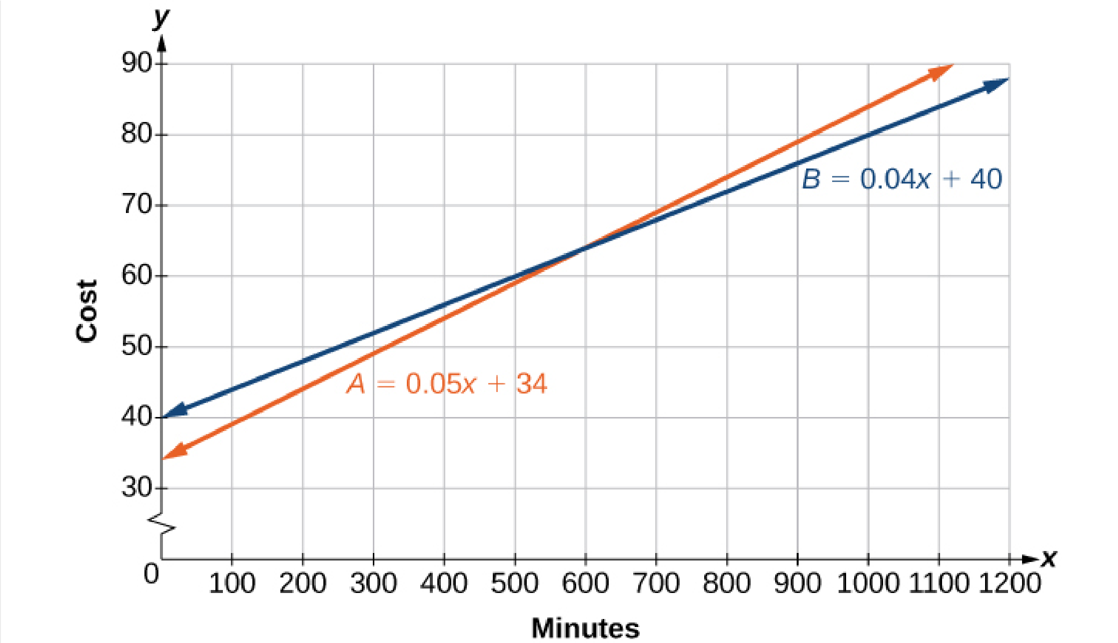
d. To answer the question of how many talk-time minutes would yield the same bill from both companies, we should think about the problem in terms of (x, y) coordinates: At what point are both the *x*-value and the *y*-value equal? We can find this point by setting the equations equal to each other and solving for *x*.
Equation:

$$\begin{aligned} 0.05x + 34 &= 0.04x + 40 \\ 0.01x &= 6 \\ x &= 600 \end{aligned}$$

Check the *x*-value in each equation.
Equation:

$$\begin{aligned} 0.05(600) + 34 &= 64 \\ 0.04(600) + 40 &= 64 \end{aligned}$$

Therefore, a monthly average of 600 talk-time minutes renders the plans equal. See [link]



Note:
Exercise:

Problem:

Find a linear equation to model this real-world application: It costs ABC electronics company $2.50 per unit to produce a part used in a popular brand of desktop computers. The company has monthly operating expenses of $350 for utilities and $3,300 for salaries. What are the company’s monthly expenses?

Solution:

$C = 2.5x + 3,650$

Using a Formula to Solve a Real-World Application

Many applications are solved using known formulas. The problem is stated, a formula is identified, the known quantities are substituted into the formula, the equation is solved for the unknown, and the problem's question is answered. Typically, these problems involve two equations representing two trips, two investments, two areas, and so on. Examples of formulas include the **area** of a rectangular region, $A = LW$;the **perimeter** of a rectangle, $P = 2L + 2W$;and the **volume** of a rectangular solid, $V = LWH.$ When there are two unknowns, we find a way to write one in terms of the other because we can solve for only one variable at a time.

Example:
Exercise:

Problem:
Solving an Application Using a Formula

It takes Andrew 30 min to drive to work in the morning. He drives home using the same route, but it takes 10 min longer, and he averages 10 mi/h less than in the morning. How far does Andrew drive to work?

Solution:

This is a distance problem, so we can use the formula $d = rt$,where distance equals rate multiplied by time. Note that when rate is given in mi/h, time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.

First, we identify the known and unknown quantities. Andrew's morning drive to work takes 30 min, or $\frac{1}{2}$ h at rate $r.$ His drive home takes 40 min, or $\frac{2}{3}$ h, and his speed averages 10 mi/h less than the morning drive. Both trips cover distance $d.$ A table, such as [link], is often helpful for keeping track of information in these types of problems.

	d	r	t
To Work	d	r	$\frac{1}{2}$
To Home	d	$r-10$	$\frac{2}{3}$

Write two equations, one for each trip.
Equation:

$$\begin{array}{rcll} d & = & r\left(\frac{1}{2}\right) & \qquad \text{To work} \\ d & = & (r-10)\left(\frac{2}{3}\right) & \qquad \text{To home} \end{array}$$

As both equations equal the same distance, we set them equal to each other and solve for *r*.
Equation:

$$\begin{array}{rcl} r\left(\frac{1}{2}\right) & = & (r-10)\left(\frac{2}{3}\right) \\ \frac{1}{2}r & = & \frac{2}{3}r-\frac{20}{3} \\ \frac{1}{2}r-\frac{2}{3}r & = & -\frac{20}{3} \\ -\frac{1}{6}r & = & -\frac{20}{3} \\ r & = & -\frac{20}{3}(-6) \\ r & = & 40 \end{array}$$

We have solved for the rate of speed to work, 40 mph. Substituting 40 into the rate on the return trip yields 30 mi/h. Now we can answer the question. Substitute the rate back into either equation and solve for *d*.
Equation:

$$\begin{array}{rcl} d & = & 40\left(\frac{1}{2}\right) \\ & = & 20 \end{array}$$

The distance between home and work is 20 mi.

Analysis

Note that we could have cleared the fractions in the equation by multiplying both sides of the equation by the LCD to solve for r.

Equation:

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r-10)\left(\frac{2}{3}\right) \\ 6 \times r\left(\frac{1}{2}\right) &= 6 \times (r-10)\left(\frac{2}{3}\right) \\ 3r &= 4(r-10) \\ 3r &= 4r-40 \\ -r &= -40 \\ r &= 40 \end{aligned}$$

Note:

Exercise:

Problem:

On Saturday morning, it took Jennifer 3.6 h to drive to her mother's house for the weekend. On Sunday evening, due to heavy traffic, it took Jennifer 4 h to return home. Her speed was 5 mi/h slower on Sunday than on Saturday. What was her speed on Sunday?

Solution:

45 mi/h

Example:

Exercise:

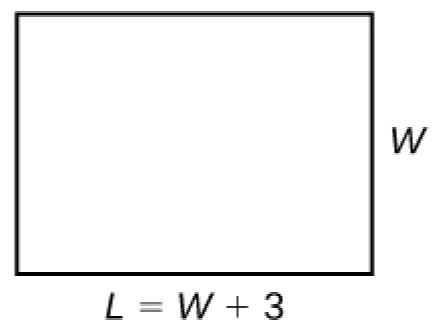
Problem:

Solving a Perimeter Problem

The perimeter of a rectangular outdoor patio is 54 ft. The length is 3 ft greater than the width. What are the dimensions of the patio?

Solution:

The perimeter formula is standard: $P = 2L + 2W.$ We have two unknown quantities, length and width. However, we can write the length in terms of the width as $L = W + 3$. Substitute the perimeter value and the expression for length into the formula. It is often helpful to make a sketch and label the sides as in [link].



Now we can solve for the width and then calculate the length.
Equation:

$$\begin{array}{rcl} P & = & 2L + 2W \\ 54 & = & 2(W + 3) + 2W \\ 54 & = & 2W + 6 + 2W \\ 54 & = & 4W + 6 \\ 48 & = & 4W \\ 12 & = & W \\ (12 + 3) & = & L \\ 15 & = & L \end{array}$$

The dimensions are $L = 15$ ft and $W = 12$ ft.

Note:
Exercise:

Problem:

Find the dimensions of a rectangle given that the perimeter is 110 cm and the length is 1 cm more than twice the width.

Solution:

$L = 37\text{ cm}, W = 18\text{ cm}$

Example:
Exercise:

Problem:
Solving an Area Problem

The perimeter of a tablet of graph paper is 48 in. The length is 6 in. more than the width. Find the area of the graph paper.

Solution:

The standard formula for area is $A = LW$;however, we will solve the problem using the perimeter formula. The reason we use the perimeter formula is because we know enough information about the perimeter that the formula will allow us to solve for one of the unknowns. As both perimeter and area use length and width as dimensions, they are often used together to solve a problem such as this one.

We know that the length is 6 in. more than the width, so we can write length as $L = W + 6$. Substitute the value of the perimeter and the expression for length into the perimeter formula and find the length.
Equation:

$$
\begin{aligned}
P &= 2L + 2W \\
48 &= 2(W+6) + 2W \\
48 &= 2W + 12 + 2W \\
48 &= 4W + 12 \\
36 &= 4W \\
9 &= W \\
(9+6) &= L \\
15 &= L
\end{aligned}
$$

Now, we find the area given the dimensions of $L = 15$ in. and $W = 9$ in.

Equation:

$$
\begin{aligned}
A &= LW \\
A &= 15(9) \\
&= 135 \text{ in.}^2
\end{aligned}
$$

The area is 135 in.^{2}.

Note:
Exercise:

Problem:

A game room has a perimeter of 70 ft. The length is five more than twice the width. How many ft^2 of new carpeting should be ordered?

Solution:

250 ft^2

Example:
Exercise:

Problem:
Solving a Volume Problem

Find the dimensions of a shipping box given that the length is twice the width, the height is 8 inches, and the volume is 1,600 in.^{3}.

Solution:

The formula for the volume of a box is given as $V = LWH$,the product of length, width, and height. We are given that $L = 2W$,and $H = 8$. The volume is $1{,}600$ cubic inches.
Equation:

$$\begin{aligned} V &= LWH \\ 1,600 &= (2W)W(8) \\ 1,600 &= 16W^2 \\ 100 &= W^2 \\ 10 &= W \end{aligned}$$

The dimensions are $L = 20\,\text{in.}$, $W = 10\,\text{in.}$, and $H = 8\,\text{in.}$

Analysis

Note that the square root of W^2 would result in a positive and a negative value. However, because we are describing width, we can use only the positive result.

Note:
Access these online resources for additional instruction and practice with models and applications of linear equations.

- Problem solving using linear equations

- Problem solving using equations
- Finding the dimensions of area given the perimeter
- Find the distance between the cities using the distance = rate * time formula
- Linear equation application (Write a cost equation)

Key Concepts

- A linear equation can be used to solve for an unknown in a number problem. See [link].
- Applications can be written as mathematical problems by identifying known quantities and assigning a variable to unknown quantities. See [link].
- There are many known formulas that can be used to solve applications. Distance problems, for example, are solved using the $d = rt$ formula. See [link].
- Many geometry problems are solved using the perimeter formula $P = 2L + 2W$, the area formula $A = LW$, or the volume formula $V = LWH$. See [link], [link], and [link].

Section Exercises

Verbal

Exercise:

Problem:

To set up a model linear equation to fit real-world applications, what should always be the first step?

Solution:

Answers may vary. Possible answers: We should define in words what our variable is representing. We should declare the variable. A

heading.

Exercise:

Problem:

Use your own words to describe this equation where n is a number:

$5(n+3)=2n$

Exercise:

Problem:

If the total amount of money you had to invest was $2,000 and you deposit x amount in one investment, how can you represent the remaining amount?

Solution:

$2,000-x$

Exercise:

Problem:

If a man sawed a 10-ft board into two sections and one section was n ft long, how long would the other section be in terms of n?

Exercise:

Problem:

If Bill was traveling v mi/h, how would you represent Daemon’s speed if he was traveling 10 mi/h faster?

Solution:

$v+10$

Real-World Applications

For the following exercises, use the information to find a linear algebraic equation model to use to answer the question being asked.

Exercise:

Problem:

Mark and Don are planning to sell each of their marble collections at a garage sale. If Don has 1 more than 3 times the number of marbles Mark has, how many does each boy have to sell if the total number of marbles is 113?

Exercise:

Problem:

Beth and Ann are joking that their combined ages equal Sam's age. If Beth is twice Ann's age and Sam is 69 yr old, what are Beth and Ann's ages?

Solution:

Ann: 23;Beth: 46

Exercise:

Problem:

Ben originally filled out 8 more applications than Henry. Then each boy filled out 3 additional applications, bringing the total to 28. How many applications did each boy originally fill out?

For the following exercises, use this scenario: Two different telephone carriers offer the following plans that a person is considering. Company A has a monthly fee of $20 and charges of $.05/min for calls. Company B has a monthly fee of $5 and charges $.10/min for calls.

Exercise:

Problem:

Find the model of the total cost of Company A’s plan, using m for the minutes.

Solution:

$20 + 0.05m$

Exercise:

Problem:

Find the model of the total cost of Company B’s plan, using m for the minutes.

Exercise:

Problem:

Find out how many minutes of calling would make the two plans equal.

Solution:

300 min

Exercise:

Problem:

If the person makes a monthly average of 200 min of calls, which plan should for the person choose?

For the following exercises, use this scenario: A wireless carrier offers the following plans that a person is considering. The Family Plan: $90 monthly fee, unlimited talk and text on up to 8 lines, and data charges of $40 for each device for up to 2 GB of data per device. The Mobile Share Plan: $120 monthly fee for up to 10 devices, unlimited talk and text for all the lines, and data charges of $35 for each device up to a shared total of 10 GB of

data. Use P for the number of devices that need data plans as part of their cost.

Exercise:

Problem: Find the model of the total cost of the Family Plan.

Solution:

$90 + 40P$

Exercise:

Problem: Find the model of the total cost of the Mobile Share Plan.

Exercise:

Problem:

Assuming they stay under their data limit, find the number of devices that would make the two plans equal in cost.

Solution:

6 devices

Exercise:

Problem:

If a family has 3 smart phones, which plan should they choose?

For exercises 17 and 18, use this scenario: A retired woman has $50,000 to invest but needs to make $6,000 a year from the interest to meet certain living expenses. One bond investment pays 15% annual interest. The rest of it she wants to put in a CD that pays 7%.

Exercise:

Problem:

If we let x be the amount the woman invests in the 15% bond, how much will she be able to invest in the CD?

Solution:

$50,000 - x$

Exercise:

Problem:

Set up and solve the equation for how much the woman should invest in each option to sustain a $6,000 annual return.

Exercise:

Problem:

Two planes fly in opposite directions. One travels 450 mi/h and the other 550 mi/h. How long will it take before they are 4,000 mi apart?

Solution:

4 h

Exercise:

Problem:

Ben starts walking along a path at 4 mi/h. One and a half hours after Ben leaves, his sister Amanda begins jogging along the same path at 6 mi/h. How long will it be before Amanda catches up to Ben?

Exercise:

Problem:

Fiora starts riding her bike at 20 mi/h. After a while, she slows down to 12 mi/h, and maintains that speed for the rest of the trip. The whole trip of 70 mi takes her 4.5 h. For what distance did she travel at 20 mi/h?

Solution:

She traveled for 2 h at 20 mi/h, or 40 miles.

Exercise:

Problem:

A chemistry teacher needs to mix a 30% salt solution with a 70% salt solution to make 20 qt of a 40% salt solution. How many quarts of each solution should the teacher mix to get the desired result?

Exercise:

Problem:

Paul has $20,000 to invest. His intent is to earn 11% interest on his investment. He can invest part of his money at 8% interest and part at 12% interest. How much does Paul need to invest in each option to make get a total 11% return on his $20,000?

Solution:

$5,000 at 8% and $15,000 at 12%

For the following exercises, use this scenario: A truck rental agency offers two kinds of plans. Plan A charges $75/wk plus $.10/mi driven. Plan B charges $100/wk plus $.05/mi driven.

Exercise:

Problem:

Write the model equation for the cost of renting a truck with plan A.

Exercise:

Problem:

Write the model equation for the cost of renting a truck with plan B.

Solution:

$B = 100 + .05x$

Exercise:

Problem:

Find the number of miles that would generate the same cost for both plans.

Exercise:

Problem:

If Tim knows he has to travel 300 mi, which plan should he choose?

Solution:

Plan A

For the following exercises, use the given formulas to answer the questions.

Exercise:

Problem:

$A = P(1 + rt)$ is used to find the principal amount *P* deposited, earning *r*% interest, for *t* years. Use this to find what principal amount *P* David invested at a 3% rate for 20 yr if $A = \$8,000$.

Exercise:

Problem:

The formula $F = \frac{mv^2}{R}$ relates force (*F*), velocity (*v*), mass (*m*), and resistance (*R*). Find R when $m = 45$,$v = 7$,and $F = 245$.

Solution:

$R = 9$

Exercise:

Problem:

$F = ma$ indicates that force (F) equals mass (m) times acceleration (a). Find the acceleration of a mass of 50 kg if a force of 12 N is exerted on it.

Exercise:

Problem:

$Sum = \frac{1}{1-r}$ is the formula for an infinite series sum. If the sum is 5, find r.

Solution:

$r = \frac{4}{5}$ or 0.8

For the following exercises, solve for the given variable in the formula. After obtaining a new version of the formula, you will use it to solve a question.

Exercise:

Problem: Solve for W: $P = 2L + 2W$

Exercise:

Problem:

Use the formula from the previous question to find the width, W, of a rectangle whose length is 15 and whose perimeter is 58.

Solution:

$W = \frac{P-2L}{2} = \frac{58-2(15)}{2} = 14$

Exercise:

Problem: Solve for f : $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$

Exercise:

Problem:

Use the formula from the previous question to find f when $p = 8$ and $q = 13$.

Solution:

$f = \frac{pq}{p+q} = \frac{8(13)}{8+13} = \frac{104}{21}$

Exercise:

Problem: Solve for m in the slope-intercept formula: $y = mx + b$

Exercise:

Problem:

Use the formula from the previous question to find m when the coordinates of the point are $(4, 7)$ and $b = 12$.

Solution:

$m = \frac{-5}{4}$

Exercise:

Problem:

The area of a trapezoid is given by $A = \frac{1}{2}h\left(b_1 + b_2\right)$. Use the formula to find the area of a trapezoid with $h = 6,\ b_1 = 14,\ \text{and}\ b_2 = 8$.

Exercise:

Problem: Solve for h: $A=\frac{1}{2}h\left(b_1+b_2\right)$

Solution:

$h=\frac{2A}{b_1+b_2}$

Exercise:

Problem:

Use the formula from the previous question to find the height of a trapezoid with $A=150,\ b_1=19,\ \text{and } b_2=11.$

Exercise:

Problem:

Find the dimensions of an American football field. The length is 200 ft more than the width, and the perimeter is 1,040 ft. Find the length and width. Use the perimeter formula $P=2L+2W.$

Solution:

length = 360 ft; width = 160 ft

Exercise:

Problem:

Distance equals rate times time, $d=rt.$ Find the distance Tom travels if he is moving at a rate of 55 mi/h for 3.5 h.

Exercise:

Problem:

Using the formula in the previous exercise, find the distance that Susan travels if she is moving at a rate of 60 mi/h for 6.75 h.

Solution:

405 mi

Exercise:

Problem:

What is the total distance that two people travel in 3 h if one of them is riding a bike at 15 mi/h and the other is walking at 3 mi/h?

Exercise:

Problem:

If the area model for a triangle is $A = \frac{1}{2}bh$,find the area of a triangle with a height of 16 in. and a base of 11 in.

Solution:

$A = 88 \text{ in.}^2$

Exercise:

Problem: Solve for *h:* $A = \frac{1}{2}bh$

Exercise:

Problem:

Use the formula from the previous question to find the height to the nearest tenth of a triangle with a base of 15 and an area of 215.

Solution:

28.7

Exercise:

Problem:

The volume formula for a cylinder is $V = \pi r^2 h.$ Using the symbol π in your answer, find the volume of a cylinder with a radius, r,of 4 cm and a height of 14 cm.

Exercise:

Problem: Solve for h: $V = \pi r^2 h$

Solution:

$h = \frac{V}{\pi r^2}$

Exercise:

Problem:

Use the formula from the previous question to find the height of a cylinder with a radius of 8 and a volume of 16π

Exercise:

Problem: Solve for r: $V = \pi r^2 h$

Solution:

$r = \sqrt{\frac{V}{\pi h}}$

Exercise:

Problem:

Use the formula from the previous question to find the radius of a cylinder with a height of 36 and a volume of 324π.

Exercise:

Problem:

The formula for the circumference of a circle is $C = 2\pi r.$ Find the circumference of a circle with a diameter of 12 in. (diameter = $2r$). Use the symbol π in your final answer.

Solution:

$C = 12\pi$

Exercise:

Problem:

Solve the formula from the previous question for π. Notice why π is sometimes defined as the ratio of the circumference to its diameter.

Glossary

area
: in square units, the area formula used in this section is used to find the area of any two-dimensional rectangular region: $A = LW$

perimeter
: in linear units, the perimeter formula is used to find the linear measurement, or outside length and width, around a two-dimensional regular object; for a rectangle: $P = 2L + 2W$

volume
: in cubic units, the volume measurement includes length, width, and depth: $V = LWH$

Quadratic Equations
In this section you will:

- Solve quadratic equations by factoring.
- Solve quadratic equations by the square root property.
- Solve quadratic equations by completing the square.
- Solve quadratic equations by using the quadratic formula.

The computer monitor on the left in [link] is a 23.6-inch model and the one on the right is a 27-inch model. Proportionally, the monitors appear very similar. If there is a limited amount of space and we desire the largest monitor possible, how do we decide which one to choose? In this section, we will learn how to solve problems such as this using four different methods.

Solving Quadratic Equations by Factoring

An equation containing a second-degree polynomial is called a quadratic equation. For example, equations such as $2x^2 + 3x - 1 = 0$ and $x^2 - 4 = 0$ are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

Often the easiest method of solving a quadratic equation is factoring. Factoring means finding expressions that can be multiplied together to give

the expression on one side of the equation.

If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property, which states that if $a \cdot b = 0$, then $a = 0$ or $b = 0$, where *a* and *b* are real numbers or algebraic expressions. In other words, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

Multiplying the factors expands the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication undoes the operation of factoring. For example, expand the factored expression $(x-2)(x+3)$ by multiplying the two factors together.

Equation:

$$\begin{array}{rcl}(x-2)(x+3) & = & x^2+3x-2x-6 \\ & = & x^2+x-6\end{array}$$

The product is a quadratic expression. Set equal to zero, $x^2+x-6=0$ is a quadratic equation. If we were to factor the equation, we would get back the factors we multiplied.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is 1 or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form, $ax^2+bx+c=0$, where *a*, *b*, and *c* are real numbers, and $a \neq 0$. The equation $x^2+x-6=0$ is in standard form.

We can use the zero-product property to solve quadratic equations in which we first have to factor out the greatest common factor (GCF), and for equations that have special factoring formulas as well, such as the difference of squares, both of which we will see later in this section.

Note:

The Zero-Product Property and Quadratic Equations

The **zero-product property** states

Equation:

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0,$$

where a and b are real numbers or algebraic expressions.

A **quadratic equation** is an equation containing a second-degree polynomial; for example

Equation:

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers, and if $a \neq 0$,it is in standard form.

Solving Quadratics with a Leading Coefficient of 1

In the quadratic equation $x^2 + x - 6 = 0$,the leading coefficient, or the coefficient of x^2,is 1. We have one method of factoring quadratic equations in this form.

Note:

Given a quadratic equation with the leading coefficient of 1, factor it.

1. Find two numbers whose product equals c and whose sum equals b.
2. Use those numbers to write two factors of the form $(x + k)$ or $(x - k)$,where k is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are 1 and -2,the factors are $(x + 1)(x - 2)$.
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

Example:
Exercise:

Problem:
Solving a Quadratic Equation by Factoring when the Leading Coefficient is not 1

Factor and solve the equation: $x^2 + x - 6 = 0$.

Solution:

To factor $x^2 + x - 6 = 0$,we look for two numbers whose product equals -6 and whose sum equals 1. Begin by looking at the possible factors of -6.
Equation:

$$\begin{array}{c} 1 \cdot (-6) \\ (-6) \cdot 1 \\ 2 \cdot (-3) \\ 3 \cdot (-2) \end{array}$$

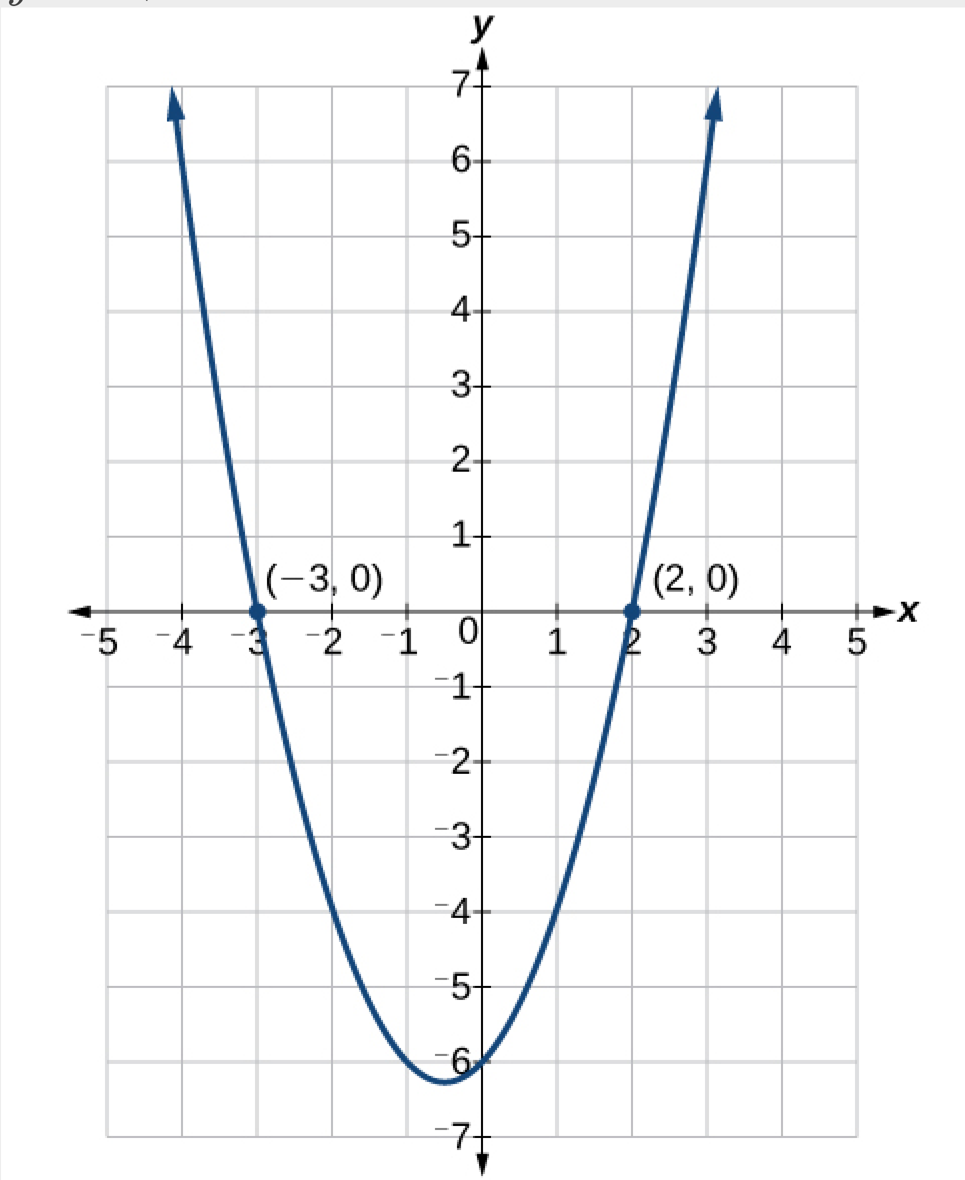
The last pair, $3 \cdot (-2)$ sums to 1, so these are the numbers. Note that only one pair of numbers will work. Then, write the factors.
Equation:

$$(x - 2)\,(x + 3) = 0$$

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.
Equation:

$$\begin{aligned}(x-2)(x+3) &= 0\\(x-2) &= 0\\x &= 2\\(x+3) &= 0\\x &= -3\end{aligned}$$

The two solutions are 2 and -3. We can see how the solutions relate to the graph in [link]. The solutions are the *x*-intercepts of $y = x^2 + x - 6 = 0$.



Note:
Exercise:

Problem: Factor and solve the quadratic equation: $x^2 - 5x - 6 = 0$.

Solution:

$(x - 6)(x + 1) = 0; x = 6, x = -1$

Example:
Exercise:

Problem:
Solve the Quadratic Equation by Factoring

Solve the quadratic equation by factoring: $x^2 + 8x + 15 = 0$.

Solution:

Find two numbers whose product equals 15 and whose sum equals 8. List the factors of 15.
Equation:

$$\begin{array}{l} 1 \cdot 15 \\ 3 \cdot 5 \\ (-1) \cdot (-15) \\ (-3) \cdot (-5) \end{array}$$

The numbers that add to 8 are 3 and 5. Then, write the factors, set each factor equal to zero, and solve.
Equation:

$$\begin{aligned} (x+3)(x+5) &= 0 \\ (x+3) &= 0 \\ x &= -3 \\ (x+5) &= 0 \\ x &= -5 \end{aligned}$$

The solutions are -3 and -5.

Note:
Exercise:

Problem:

Solve the quadratic equation by factoring: $x^2 - 4x - 21 = 0$.

Solution:

$(x-7)(x+3) = 0, x = 7, x = -3.$

Example:
Exercise:

Problem:
Using the Zero-Product Property to Solve a Quadratic Equation Written as the Difference of Squares

Solve the difference of squares equation using the zero-product property: $x^2 - 9 = 0$.

Solution:

Recognizing that the equation represents the difference of squares, we can write the two factors by taking the square root of each term, using a minus sign as the operator in one factor and a plus sign as the operator in the other. Solve using the zero-factor property.

Equation:

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \\ \\ (x-3) &= 0 \\ x &= 3 \\ \\ (x+3) &= 0 \\ x &= -3 \end{aligned}$$

The solutions are 3 and -3.

Note:
Exercise:

Problem: Solve by factoring: $x^2 - 25 = 0$.

Solution:

$(x+5)(x-5) = 0$, $x = -5$, $x = 5$.

Solving a Quadratic Equation by Factoring when the Leading Coefficient is not 1

When the leading coefficient is not 1, we factor a quadratic equation using the method called grouping, which requires four terms. With the equation in standard form, let's review the grouping procedures:

1. With the quadratic in standard form, $ax^2 + bx + c = 0$,multiply $a \cdot c$.
2. Find two numbers whose product equals ac and whose sum equals b.
3. Rewrite the equation replacing the bx term with two terms using the numbers found in step 1 as coefficients of *x*.
4. Factor the first two terms and then factor the last two terms. The expressions in parentheses must be exactly the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

Example:
Exercise:

Problem:
Solving a Quadratic Equation Using Grouping

Use grouping to factor and solve the quadratic equation: $4x^2 + 15x + 9 = 0$.

Solution:

First, multiply $ac : 4\left(9\right) = 36$. Then list the factors of 36.
Equation:

$$\begin{array}{c}1 \cdot 36\\ 2 \cdot 18\\ 3 \cdot 12\\ 4 \cdot 9\\ 6 \cdot 6\end{array}$$

The only pair of factors that sums to 15 is $3 + 12$. Rewrite the equation replacing the *b* term, $15x$,with two terms using 3 and 12 as

coefficients of *x*. Factor the first two terms, and then factor the last two terms.

Equation:

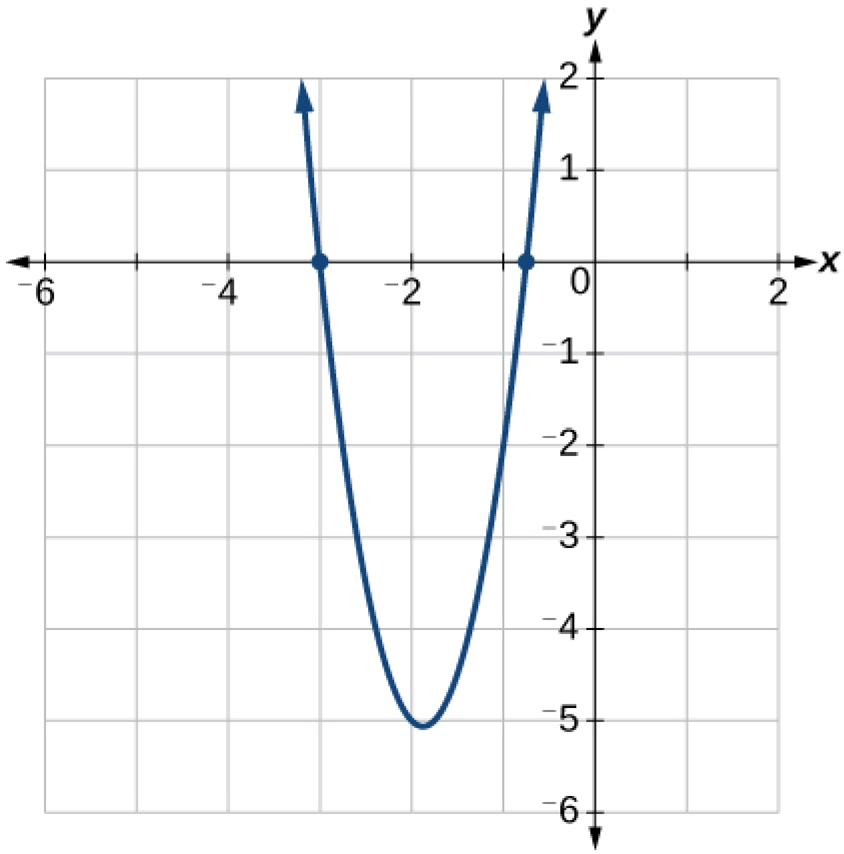
$$\begin{aligned} 4x^2+3x+12x+9 &= 0 \\ x(4x+3)+3(4x+3) &= 0 \\ (4x+3)(x+3) &= 0 \end{aligned}$$

Solve using the zero-product property.

Equation:

$$\begin{aligned} (4x+3)(x+3) &= 0 \\ \\ (4x+3) &= 0 \\ x &= -\tfrac{3}{4} \\ \\ (x+3) &= 0 \\ x &= -3 \end{aligned}$$

The solutions are $-\frac{3}{4}$, and -3. See [link].



Note:
Exercise:

Problem: Solve using factoring by grouping: $12x^2 + 11x + 2 = 0$.

Solution:

$(3x + 2)(4x + 1) = 0, x = -\frac{2}{3}, x = -\frac{1}{4}$

Example:
Exercise:

Problem:

Solving a Polynomial of Higher Degree by Factoring

Solve the equation by factoring: $-3x^3 - 5x^2 - 2x = 0$.

Solution:

This equation does not look like a quadratic, as the highest power is 3, not 2. Recall that the first thing we want to do when solving any equation is to factor out the GCF, if one exists. And it does here. We can factor out $-x$ from all of the terms and then proceed with grouping.

Equation:

$$\begin{aligned} -3x^3 - 5x^2 - 2x &= 0 \\ -x\left(3x^2 + 5x + 2\right) &= 0 \end{aligned}$$

Use grouping on the expression in parentheses.

Equation:

$$\begin{aligned} -x\left(3x^2 + 3x + 2x + 2\right) &= 0 \\ -x[3x(x+1) + 2(x+1)] &= 0 \\ -x(3x+2)(x+1) &= 0 \end{aligned}$$

Now, we use the zero-product property. Notice that we have three factors.

Equation:

$$\begin{aligned} -x &= 0 \\ x &= 0 \\ 3x + 2 &= 0 \\ x &= -\tfrac{2}{3} \\ x + 1 &= 0 \\ x &= -1 \end{aligned}$$

The solutions are $0, -\frac{2}{3}$, and -1.

Note:
Exercise:

Problem: Solve by factoring: $x^3 + 11x^2 + 10x = 0$.

Solution:

$x = 0, x = -10, x = -1$

Using the Square Root Property

When there is no linear term in the equation, another method of solving a quadratic equation is by using the **square root property**, in which we isolate the x^2 term and take the square root of the number on the other side of the equals sign. Keep in mind that sometimes we may have to manipulate the equation to isolate the x^2 term so that the square root property can be used.

Note:
The Square Root Property
With the x^2 term isolated, the square root property states that:
Equation:

$$\text{if } x^2 = k, \text{then } x = \pm\sqrt{k}$$

where *k* is a nonzero real number.

Note:
Given a quadratic equation with an x^2 term but no x term, use the square root property to solve it.

1. Isolate the x^2 term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a $\pm$ sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the $\pm$ sign.

Example:
Exercise:

Problem:
Solving a Simple Quadratic Equation Using the Square Root Property

Solve the quadratic using the square root property: $x^2 = 8$.

Solution:

Take the square root of both sides, and then simplify the radical. Remember to use a $\pm$ sign before the radical symbol.
Equation:

$$\begin{aligned} x^2 &= 8 \\ x &= \pm\sqrt{8} \\ &= \pm 2\sqrt{2} \end{aligned}$$

The solutions are $2\sqrt{2}, -2\sqrt{2}$.

Example:

Exercise:

Problem:
Solving a Quadratic Equation Using the Square Root Property

Solve the quadratic equation: $4x^2 + 1 = 7$.

Solution:

First, isolate the x^2 term. Then take the square root of both sides.
Equation:

$$\begin{aligned} 4x^2 + 1 &= 7 \\ 4x^2 &= 6 \\ x^2 &= \tfrac{6}{4} \\ x &= \pm\tfrac{\sqrt{6}}{2} \end{aligned}$$

The solutions are $\frac{\sqrt{6}}{2}$,and $-\frac{\sqrt{6}}{2}$.

Note:
Exercise:

Problem:

Solve the quadratic equation using the square root property: $3(x-4)^2 = 15$.

Solution:

$x = 4 \pm \sqrt{5}$

Completing the Square

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a quadratic equation known as **completing the square**. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, *a*, must equal 1. If it does not, then divide the entire equation by *a*. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example $x^2 + 4x + 1 = 0$ to illustrate each step.

1. Given a quadratic equation that cannot be factored, and with $a = 1$, first add or subtract the constant term to the right sign of the equal sign.
   Equation:

$$x^2 + 4x = -1$$

2. Multiply the *b* term by $\frac{1}{2}$ and square it.
   Equation:

$$\begin{aligned} \tfrac{1}{2}(4) &= 2 \\ 2^2 &= 4 \end{aligned}$$

3. Add $\left(\frac{1}{2}b\right)^2$ to both sides of the equal sign and simplify the right side. We have
   Equation:

$$\begin{aligned} x^2 + 4x + 4 &= -1 + 4 \\ x^2 + 4x + 4 &= 3 \end{aligned}$$

4. The left side of the equation can now be factored as a perfect square.
Equation:

$$\begin{aligned} x^2 + 4x + 4 &= 3 \\ (x+2)^2 &= 3 \end{aligned}$$

5. Use the square root property and solve.
Equation:

$$\begin{aligned} \sqrt{(x+2)^2} &= \pm\sqrt{3} \\ x + 2 &= \pm\sqrt{3} \\ x &= -2 \pm \sqrt{3} \end{aligned}$$

6. The solutions are $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$.

Example:
Exercise:

Problem:
Solving a Quadratic by Completing the Square

Solve the quadratic equation by completing the square: $x^2 - 3x - 5 = 0$.

Solution:

First, move the constant term to the right side of the equal sign.
Equation:

$$x^2 - 3x = 5$$

Then, take $\frac{1}{2}$ of the b term and square it.

Equation:

$$\begin{aligned} \tfrac{1}{2}(-3) &= -\tfrac{3}{2} \\ \left(-\tfrac{3}{2}\right)^2 &= \tfrac{9}{4} \end{aligned}$$

Add the result to both sides of the equal sign.

Equation:

$$\begin{aligned} x^2 - 3x + \left(-\tfrac{3}{2}\right)^2 &= 5 + \left(-\tfrac{3}{2}\right)^2 \\ x^2 - 3x + \tfrac{9}{4} &= 5 + \tfrac{9}{4} \end{aligned}$$

Factor the left side as a perfect square and simplify the right side.

Equation:

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

Use the square root property and solve.

Equation:

$$\begin{aligned} \sqrt{\left(x - \tfrac{3}{2}\right)^2} &= \pm\sqrt{\tfrac{29}{4}} \\ \left(x - \tfrac{3}{2}\right) &= \pm\tfrac{\sqrt{29}}{2} \\ x &= \tfrac{3}{2} \pm \tfrac{\sqrt{29}}{2} \end{aligned}$$

The solutions are $\frac{3}{2} + \frac{\sqrt{29}}{2}$,and $\frac{3}{2} - \frac{\sqrt{29}}{2}$.

Note:
Exercise:

Problem: Solve by completing the square: $x^2 - 6x = 13$.

Solution:

$x = 3 \pm \sqrt{22}$

Using the Quadratic Formula

The fourth method of solving a quadratic equation is by using the quadratic formula, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by completing the square. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by -1 and obtain a positive a. Given $ax^2 + bx + c = 0$,$a \neq 0$,we will complete the square as follows:

1. First, move the constant term to the right side of the equal sign:
   Equation:

$$ax^2 + bx = -c$$

2. As we want the leading coefficient to equal 1, divide through by a:
   Equation:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. Then, find $\frac{1}{2}$ of the middle term, and add $\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$ to both sides of the equal sign:
   Equation:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:
   Equation:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

5. Now, use the square root property, which gives
   Equation:

$$\begin{aligned} x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}} \\ x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2-4ac}}{2a} \end{aligned}$$

6. Finally, add $-\frac{b}{2a}$ to both sides of the equation and combine the terms on the right side. Thus,
   Equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note:
The Quadratic Formula

Written in standard form, $ax^2 + bx + c = 0$,any quadratic equation can be solved using the **quadratic formula**:

Equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a, b, and c are real numbers and $a \neq 0$.

Note:

Given a quadratic equation, solve it using the quadratic formula

1. Make sure the equation is in standard form: $ax^2 + bx + c = 0$.
2. Make note of the values of the coefficients and constant term, a, b,and c.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

Example:

Exercise:

Problem:

Solve the Quadratic Equation Using the Quadratic Formula

Solve the quadratic equation: $x^2 + 5x + 1 = 0$.

Solution:

Identify the coefficients: $a = 1, b = 5, c = 1$. Then use the quadratic formula.

Equation:

$$\begin{aligned} x &= \frac{-(5)\pm\sqrt{(5)^2-4(1)(1)}}{2(1)} \\ &= \frac{-5\pm\sqrt{25-4}}{2} \\ &= \frac{-5\pm\sqrt{21}}{2} \end{aligned}$$

Example:
Exercise:

Problem:
Solving a Quadratic Equation with the Quadratic Formula

Use the quadratic formula to solve $x^2 + x + 2 = 0$.

Solution:

First, we identify the coefficients: $a = 1, b = 1,$and $c = 2$.

Substitute these values into the quadratic formula.
Equation:

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-(1)\pm\sqrt{(1)^2-(4)\cdot(1)\cdot(2)}}{2\cdot 1} \\ &= \frac{-1\pm\sqrt{1-8}}{2} \\ &= \frac{-1\pm\sqrt{-7}}{2} \\ &= \frac{-1\pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $\frac{-1+i\sqrt{7}}{2}$ and $\frac{-1-i\sqrt{7}}{2}$ or $\frac{-1}{2} + \frac{i\sqrt{7}}{2}$ and $\frac{-1}{2} - \frac{i\sqrt{7}}{2}$.

Note:
Exercise:

Problem:

Solve the quadratic equation using the quadratic formula: $9x^2 + 3x - 2 = 0.$

Solution:

$x = -\frac{2}{3}, x = \frac{1}{3}$

The Discriminant

The quadratic formula not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the discriminant, or the expression under the radical, $b^2 - 4ac.$ The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. [link] relates the value of the discriminant to the solutions of a quadratic equation.

Value of Discriminant	Results
$b^2 - 4ac = 0$	One rational solution (double solution)
$b^2 - 4ac > 0$, perfect square	Two rational solutions
$b^2 - 4ac > 0$, not a perfect square	Two irrational solutions

Value of Discriminant	Results
$b^2 - 4ac < 0$	Two complex solutions

Note:
The Discriminant
For $ax^2 + bx + c = 0$, where a, b, and c are real numbers, the **discriminant** is the expression under the radical in the quadratic formula: $b^2 - 4ac$. It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

Example:
Exercise:

Problem:
Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Use the discriminant to find the nature of the solutions to the following quadratic equations:

a. $x^2 + 4x + 4 = 0$
b. $8x^2 + 14x + 3 = 0$
c. $3x^2 - 5x - 2 = 0$
d. $3x^2 - 10x + 15 = 0$

Solution:

Calculate the discriminant $b^2 - 4ac$ for each equation and state the expected type of solutions.

a. $x^2 + 4x + 4 = 0$

$b^2 - 4ac = (4)^2 - 4(1)(4) = 0$. There will be one rational double solution.

b. $8x^2 + 14x + 3 = 0$

$b^2 - 4ac = (14)^2 - 4(8)(3) = 100$. As 100 is a perfect square, there will be two rational solutions.

c. $3x^2 - 5x - 2 = 0$

$b^2 - 4ac = (-5)^2 - 4(3)(-2) = 49$. As 49 is a perfect square, there will be two rational solutions.

d. $3x^2 - 10x + 15 = 0$

$b^2 - 4ac = (-10)^2 - 4(3)(15) = -80$. There will be two complex solutions.

Using the Pythagorean Theorem

One of the most famous formulas in mathematics is the **Pythagorean Theorem**. It is based on a right triangle, and states the relationship among the lengths of the sides as $a^2 + b^2 = c^2$,where a and b refer to the legs of a right triangle adjacent to the 90° angle, and c refers to the hypotenuse. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

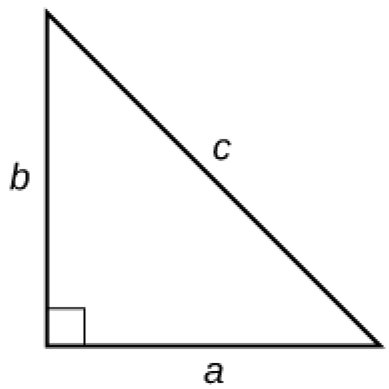
We use the Pythagorean Theorem to solve for the length of one side of a triangle when we have the lengths of the other two. Because each of the terms is squared in the theorem, when we are solving for a side of a triangle, we have a quadratic equation. We can use the methods for solving quadratic equations that we learned in this section to solve for the missing side.

The Pythagorean Theorem is given as

Equation:

$$a^2 + b^2 = c^2$$

where a and b refer to the legs of a right triangle adjacent to the 90° angle, and c refers to the hypotenuse, as shown in [link].



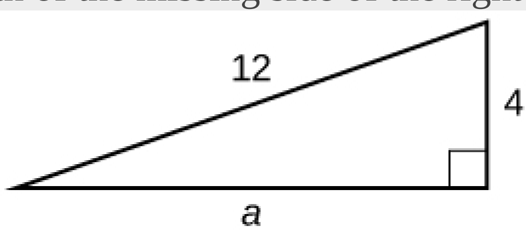
Example:

Exercise:

Problem:

Finding the Length of the Missing Side of a Right Triangle

Find the length of the missing side of the right triangle in [link].



Solution:

As we have measurements for side *b* and the hypotenuse, the missing side is *a*.

Equation:

$$\begin{aligned} a^2+b^2 &= c^2 \\ a^2+(4)^2 &= (12)^2 \\ a^2+16 &= 144 \\ a^2 &= 128 \\ a &= \sqrt{128} \\ &= 8\sqrt{2} \end{aligned}$$

Note:

Exercise:

Problem:

Use the Pythagorean Theorem to solve the right triangle problem: Leg *a* measures 4 units, leg *b* measures 3 units. Find the length of the hypotenuse.

Solution:

5 units

Note:

Access these online resources for additional instruction and practice with quadratic equations.

- Solving Quadratic Equations by Factoring
- The Zero-Product Property
- Completing the Square
- Quadratic Formula with Two Rational Solutions
- Length of a leg of a right triangle

Key Equations

quadratic formula	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Key Concepts

- Many quadratic equations can be solved by factoring when the equation has a leading coefficient of 1 or if the equation is a difference of squares. The zero-factor property is then used to find solutions. See [link], [link], and [link].
- Many quadratic equations with a leading coefficient other than 1 can be solved by factoring using the grouping method. See [link] and [link].
- Another method for solving quadratics is the square root property. The variable is squared. We isolate the squared term and take the square root of both sides of the equation. The solution will yield a positive and negative solution. See [link] and [link].
- Completing the square is a method of solving quadratic equations when the equation cannot be factored. See [link].
- A highly dependable method for solving quadratic equations is the quadratic formula, based on the coefficients and the constant term in the equation. See [link].

- The discriminant is used to indicate the nature of the roots that the quadratic equation will yield: real or complex, rational or irrational, and how many of each. See [link].
- The Pythagorean Theorem, among the most famous theorems in history, is used to solve right-triangle problems and has applications in numerous fields. Solving for the length of one side of a right triangle requires solving a quadratic equation. See [link].

Section Exercises

Verbal

Exercise:

Problem: How do we recognize when an equation is quadratic?

Solution:

It is a second-degree equation (the highest variable exponent is 2).

Exercise:

Problem:

When we solve a quadratic equation, how many solutions should we always start out seeking? Explain why when solving a quadratic equation in the form $ax^2 + bx + c = 0$ we may graph the equation $y = ax^2 + bx + c$ and have no zeroes (*x*-intercepts).

Exercise:

Problem:

When we solve a quadratic equation by factoring, why do we move all terms to one side, having zero on the other side?

Solution:

We want to take advantage of the zero property of multiplication in the fact that if $a \cdot b = 0$ then it must follow that each factor separately offers a solution to the product being zero: $a = 0 \; or$ b $= 0$.

Exercise:

Problem:

In the quadratic formula, what is the name of the expression under the radical sign $b^2 - 4ac$,and how does it determine the number of and nature of our solutions?

Exercise:

Problem:

Describe two scenarios where using the square root property to solve a quadratic equation would be the most efficient method.

Solution:

One, when no linear term is present (no *x* term), such as $x^2 = 16$. Two, when the equation is already in the form $(ax + b)^2 = d$.

Algebraic

For the following exercises, solve the quadratic equation by factoring.

Exercise:

Problem:$x^2 + 4x - 21 = 0$

Exercise:

Problem:$x^2 - 9x + 18 = 0$

Solution:

$x = 6$,$x = 3$

Exercise:

Problem: $2x^2 + 9x - 5 = 0$

Exercise:

Problem: $6x^2 + 17x + 5 = 0$

Solution:

$x = \frac{-5}{2}$, $x = \frac{-1}{3}$

Exercise:

Problem: $4x^2 - 12x + 8 = 0$

Exercise:

Problem: $3x^2 - 75 = 0$

Solution:

$x = 5$, $x = -5$

Exercise:

Problem: $8x^2 + 6x - 9 = 0$

Exercise:

Problem: $4x^2 = 9$

Solution:

$x = \frac{-3}{2}$, $x = \frac{3}{2}$

Exercise:

Problem: $2x^2 + 14x = 36$

Exercise:

Problem: $5x^2 = 5x + 30$

Solution:

$x = -2, 3$

Exercise:

Problem: $4x^2 = 5x$

Exercise:

Problem: $7x^2 + 3x = 0$

Solution:

$x = 0$, $x = \frac{-3}{7}$

Exercise:

Problem: $\frac{x}{3} - \frac{9}{x} = 2$

For the following exercises, solve the quadratic equation by using the square root property.

Exercise:

Problem: $x^2 = 36$

Solution:

$x = -6$, $x = 6$

Exercise:

Problem:$x^2 = 49$

Exercise:

Problem:$(x-1)^2 = 25$

Solution:

$x = 6$,$x = -4$

Exercise:

Problem:$(x-3)^2 = 7$

Exercise:

Problem:$(2x+1)^2 = 9$

Solution:

$x = 1$,$x = -2$

Exercise:

Problem:$(x-5)^2 = 4$

For the following exercises, solve the quadratic equation by completing the square. Show each step.

Exercise:

Problem:$x^2 - 9x - 22 = 0$

Solution:

$x = -2$,$x = 11$

Exercise:

Problem:$2x^2 - 8x - 5 = 0$

Exercise:

Problem:$x^2 - 6x = 13$

Solution:

$x = 3 \pm \sqrt{22}$

Exercise:

Problem:$x^2 + \frac{2}{3}x - \frac{1}{3} = 0$

Exercise:

Problem:$2 + z = 6z^2$

Solution:

$z = \frac{2}{3}$,$z = -\frac{1}{2}$

Exercise:

Problem:$6p^2 + 7p - 20 = 0$

Exercise:

Problem:$2x^2 - 3x - 1 = 0$

Solution:

$x = \frac{3 \pm \sqrt{17}}{4}$

For the following exercises, determine the discriminant, and then state how many solutions there are and the nature of the solutions. Do not solve.

Exercise:

Problem: $2x^2 - 6x + 7 = 0$

Exercise:

Problem: $x^2 + 4x + 7 = 0$

Solution:

Not real

Exercise:

Problem: $3x^2 + 5x - 8 = 0$

Exercise:

Problem: $9x^2 - 30x + 25 = 0$

Solution:

One rational

Exercise:

Problem: $2x^2 - 3x - 7 = 0$

Exercise:

Problem: $6x^2 - x - 2 = 0$

Solution:

Two real; rational

For the following exercises, solve the quadratic equation by using the quadratic formula. If the solutions are not real, state *No Real Solution*.

Exercise:

Problem: $2x^2 + 5x + 3 = 0$

Exercise:

Problem: $x^2 + x = 4$

Solution:

$x = \frac{-1 \pm \sqrt{17}}{2}$

Exercise:

Problem: $2x^2 - 8x - 5 = 0$

Exercise:

Problem: $3x^2 - 5x + 1 = 0$

Solution:

$x = \frac{5 \pm \sqrt{13}}{6}$

Exercise:

Problem: $x^2 + 4x + 2 = 0$

Exercise:

Problem: $4 + \frac{1}{x} - \frac{1}{x^2} = 0$

Solution:

$x = \frac{-1 \pm \sqrt{17}}{8}$

Technology

For the following exercises, enter the expressions into your graphing utility and find the zeroes to the equation (the *x*-intercepts) by using 2nd CALC 2:zero. Recall finding zeroes will ask left bound (move your cursor to the left of the zero,enter), then right bound (move your cursor to the right of the zero,enter), then guess (move your cursor between the bounds near the zero, enter). Round your answers to the nearest thousandth.

Exercise:

Problem: $\mathrm{Y}_1 = 4x^2 + 3x - 2$

Exercise:

Problem: $\mathrm{Y}_1 = -3x^2 + 8x - 1$

Solution:

$x \approx 0.131$ and $x \approx 2.535$

Exercise:

Problem: $\mathrm{Y}_1 = 0.5x^2 + x - 7$

Exercise:

Problem:

To solve the quadratic equation $x^2 + 5x - 7 = 4$,we can graph these two equations

$$\mathrm{Y}_1 = x^2 + 5x - 7$$
$$\mathrm{Y}_2 = 4$$

and find the points of intersection. Recall 2^{nd} CALC 5:intersection. Do this and find the solutions to the nearest tenth.

Solution:

$x \approx -6.7$ and $x \approx 1.7$

Exercise:

Problem:

To solve the quadratic equation $0.3x^2 + 2x - 4 = 2$,we can graph these two equations

$\mathrm{Y}_1 = 0.3x^2 + 2x - 4$
$\mathrm{Y}_2 = 2$

and find the points of intersection. Recall 2^{nd} CALC 5:intersection. Do this and find the solutions to the nearest tenth.

Extensions

Exercise:

Problem:

Beginning with the general form of a quadratic equation, $ax^2 + bx + c = 0$,solve for *x* by using the completing the square method, thus deriving the quadratic formula.

Solution:

$$\begin{aligned} ax^2+bx+c &= 0 \\ x^2+\frac{b}{a}x &= \frac{-c}{a} \\ x^2+\frac{b}{a}x+\frac{b^2}{4a^2} &= \frac{-c}{a}+\frac{b}{4a^2} \\ \left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2} \\ x+\frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}} \\ x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \end{aligned}$$

Exercise:

Problem:

Show that the sum of the two solutions to the quadratic equation is $\frac{-b}{a}$.

Exercise:

Problem:

A person has a garden that has a length 10 feet longer than the width. Set up a quadratic equation to find the dimensions of the garden if its area is 119 ft.^{2}. Solve the quadratic equation to find the length and width.

Solution:

$x(x+10)=119$;7 ft. and 17 ft.

Exercise:

Problem:

Abercrombie and Fitch stock had a price given as $P=0.2t^2-5.6t+50.2$,where t is the time in months from 1999 to 2001. ($t=1$ is January 1999). Find the two months in which the price of the stock was $30.

Exercise:

Problem:

Suppose that an equation is given $p = -2x^2 + 280x - 1000$,where x represents the number of items sold at an auction and p is the profit made by the business that ran the auction. How many items sold would make this profit a maximum? Solve this by graphing the expression in your graphing utility and finding the maximum using 2nd CALC maximum. To obtain a good window for the curve, set x [0,200] and y [0,10000].

Solution:

maximum at $x = 70$

Real-World Applications

Exercise:

Problem:

A formula for the normal systolic blood pressure for a man age A, measured in mmHg, is given as $P = 0.006A^2 - 0.02A + 120$. Find the age to the nearest year of a man whose normal blood pressure measures 125 mmHg.

Exercise:

Problem:

The cost function for a certain company is $C = 60x + 300$ and the revenue is given by $R = 100x - 0.5x^2$. Recall that profit is revenue minus cost. Set up a quadratic equation and find two values of *x* (production level) that will create a profit of $300.

Solution:

The quadratic equation would be $(100x–0.5x^2) - (60x + 300) = 300$. The two values of x are 20 and 60.

Exercise:

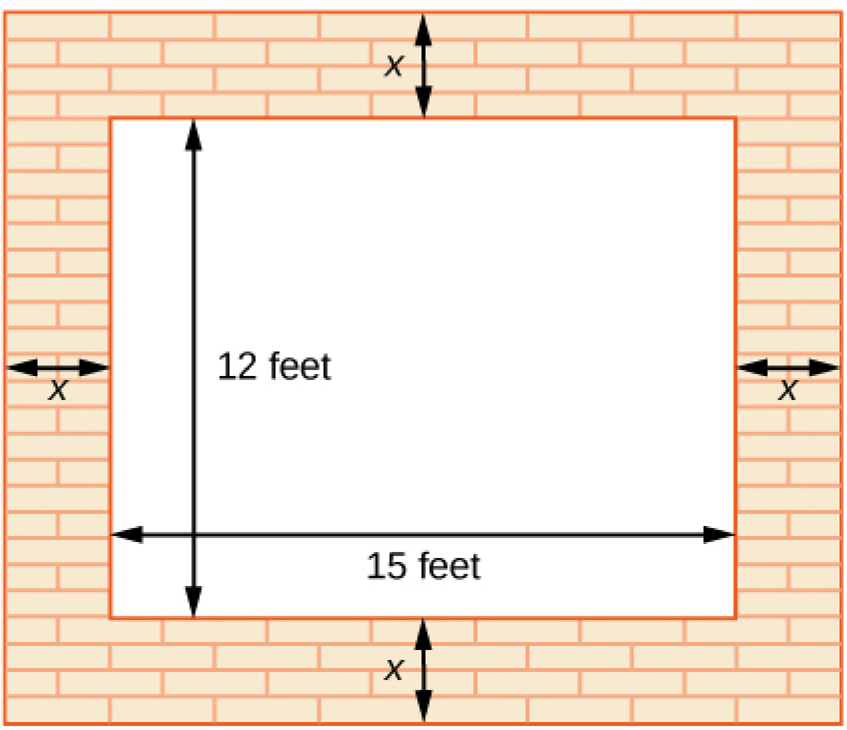
Problem:

A falling object travels a distance given by the formula $d = 5t + 16t^2$ ft, where t is measured in seconds. How long will it take for the object to traveled 74 ft?

Exercise:

Problem:

A vacant lot is being converted into a community garden. The garden and the walkway around its perimeter have an area of 378 ft^2. Find the width of the walkway if the garden is 12 ft. wide by 15 ft. long.



Solution:

3 feet

Exercise:

Problem:

An epidemiological study of the spread of a certain influenza strain that hit a small school population found that the total number of students, P,who contracted the flu t days after it broke out is given by the model $P = -t^2 + 13t + 130$,where $1 \leq t \leq 6$. Find the day that 160 students had the flu. Recall that the restriction on t is at most 6.

Glossary

completing the square
: a process for solving quadratic equations in which terms are added to or subtracted from both sides of the equation in order to make one side a perfect square

discriminant
: the expression under the radical in the quadratic formula that indicates the nature of the solutions, real or complex, rational or irrational, single or double roots.

Pythagorean Theorem
: a theorem that states the relationship among the lengths of the sides of a right triangle, used to solve right triangle problems

quadratic equation
: an equation containing a second-degree polynomial; can be solved using multiple methods

quadratic formula
: a formula that will solve all quadratic equations

square root property
: one of the methods used to solve a quadratic equation, in which the x^2 term is isolated so that the square root of both sides of the equation can

be taken to solve for *x*

zero-product property

the property that formally states that multiplication by zero is zero, so that each factor of a quadratic equation can be set equal to zero to solve equations

Other Types of Equations
In this section you will:

- Solve equations involving rational exponents.
- Solve equations using factoring.
- Solve radical equations.
- Solve absolute value equations.
- Solve other types of equations.

We have solved linear equations, rational equations, and quadratic equations using several methods. However, there are many other types of equations, and we will investigate a few more types in this section. We will look at equations involving rational exponents, polynomial equations, radical equations, absolute value equations, equations in quadratic form, and some rational equations that can be transformed into quadratics. Solving any equation, however, employs the same basic algebraic rules. We will learn some new techniques as they apply to certain equations, but the algebra never changes.

Solving Equations Involving Rational Exponents

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, $16^{\frac{1}{2}}$ is another way of writing $\sqrt{16}$;$8^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{8}$. The ability to work with rational exponents is a useful skill, as it is highly applicable in calculus.

We can solve equations in which a variable is raised to a rational exponent by raising both sides of the equation to the reciprocal of the exponent. The reason we raise the equation to the reciprocal of the exponent is because we want to eliminate the exponent on the variable term, and a number multiplied by its reciprocal equals 1. For example, $\frac{2}{3}\left(\frac{3}{2}\right)=1$,$3\left(\frac{1}{3}\right)=1$,and so on.

Note:
Rational Exponents
A rational exponent indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:
Equation:

$$a^{\frac{m}{n}}=\left(a^{\frac{1}{n}}\right)^{m}=\left(a^{m}\right)^{\frac{1}{n}}=\sqrt[n]{a^{m}}=\left(\sqrt[n]{a}\right)^{m}$$

Example:
Exercise:

Problem:

Evaluating a Number Raised to a Rational Exponent

Evaluate $8^{\frac{2}{3}}$.

Solution:

Whether we take the root first or the power first depends on the number. It is easy to find the cube root of 8, so rewrite $8^{\frac{2}{3}}$ as $\left(8^{\frac{1}{3}}\right)^2$.

Equation:

$$\begin{aligned}\left(8^{\frac{1}{3}}\right)^2 &= (2)^2 \\ &= 4\end{aligned}$$

Note:

Exercise:

Problem: Evaluate $64^{-\frac{1}{3}}$.

Solution:

$\frac{1}{4}$

Example:

Exercise:

Problem:

Solve the Equation Including a Variable Raised to a Rational Exponent

Solve the equation in which a variable is raised to a rational exponent: $x^{\frac{5}{4}} = 32$.

Solution:

The way to remove the exponent on *x* is by raising both sides of the equation to a power that is the reciprocal of $\frac{5}{4}$,which is $\frac{4}{5}$.

Equation:

$$\begin{aligned} x^{\frac{5}{4}} &= 32 \\ \left(x^{\frac{5}{4}}\right)^{\frac{4}{5}} &= (32)^{\frac{4}{5}} \\ x &= (2)^4 && \text{The fifth root of 32 is 2.} \\ &= 16 \end{aligned}$$

Note:
Exercise:

Problem: Solve the equation $x^{\frac{3}{2}} = 125.$

Solution:

25

Example:
Exercise:

Problem:
Solving an Equation Involving Rational Exponents and Factoring

Solve $3x^{\frac{3}{4}} = x^{\frac{1}{2}}.$

Solution:

This equation involves rational exponents as well as factoring rational exponents. Let us take this one step at a time. First, put the variable terms on one side of the equal sign and set the equation equal to zero.
Equation:

$$\begin{aligned} 3x^{\frac{3}{4}} - \left(x^{\frac{1}{2}}\right) &= x^{\frac{1}{2}} - \left(x^{\frac{1}{2}}\right) \\ 3x^{\frac{3}{4}} - x^{\frac{1}{2}} &= 0 \end{aligned}$$

Now, it looks like we should factor the left side, but what do we factor out? We can always factor the term with the lowest exponent. Rewrite $x^{\frac{1}{2}}$ as $x^{\frac{2}{4}}.$ Then, factor out $x^{\frac{2}{4}}$ from both terms on the left.
Equation:

$$\begin{aligned} 3x^{\frac{3}{4}} - x^{\frac{2}{4}} &= 0 \\ x^{\frac{2}{4}}\left(3x^{\frac{1}{4}} - 1\right) &= 0 \end{aligned}$$

Where did $x^{\frac{1}{4}}$ come from? Remember, when we multiply two numbers with the same base, we add the exponents. Therefore, if we multiply $x^{\frac{2}{4}}$ back in using the distributive property, we get the expression we had before the factoring, which is what should happen. We need an exponent such that when added to $\frac{2}{4}$ equals $\frac{3}{4}$. Thus, the exponent on *x* in the parentheses is $\frac{1}{4}$.

Let us continue. Now we have two factors and can use the zero factor theorem.
Equation:

$$\begin{aligned} x^{\frac{2}{4}}\left(3x^{\frac{1}{4}} - 1\right) &= 0 \\ x^{\frac{2}{4}} &= 0 \\ x &= 0 \\ 3x^{\frac{1}{4}} - 1 &= 0 \\ 3x^{\frac{1}{4}} &= 1 \\ x^{\frac{1}{4}} &= \frac{1}{3} && \text{Divide both sides by 3.} \\ \left(x^{\frac{1}{4}}\right)^4 &= \left(\frac{1}{3}\right)^4 && \text{Raise both sides to the reciprocal of } \frac{1}{4}. \\ x &= \frac{1}{81} \end{aligned}$$

The two solutions are 0 and $\frac{1}{81}$.

Note:
Exercise:

Problem: Solve: $(x+5)^{\frac{3}{2}} = 8$.

Solution:

$\{-1\}$

Solving Equations Using Factoring

We have used factoring to solve quadratic equations, but it is a technique that we can use with many types of polynomial equations, which are equations that contain a string of terms including numerical coefficients and variables. When we are faced with an equation containing polynomials of degree higher than 2, we can often solve them by factoring.

Note:
Polynomial Equations
A polynomial of degree *n* is an expression of the type
Equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

where *n* is a positive integer and $a_n, \ldots, a_0$ are real numbers and $a_n \neq 0$.
Setting the polynomial equal to zero gives a **polynomial equation**. The total number of solutions (real and complex) to a polynomial equation is equal to the highest exponent *n*.

Example:
Exercise:

Problem:
Solving a Polynomial by Factoring

Solve the polynomial by factoring: $5x^4 = 80x^2$.

Solution:

First, set the equation equal to zero. Then factor out what is common to both terms, the GCF.
Equation:

$$\begin{aligned} 5x^4 - 80x^2 &= 0 \\ 5x^2\left(x^2 - 16\right) &= 0 \end{aligned}$$

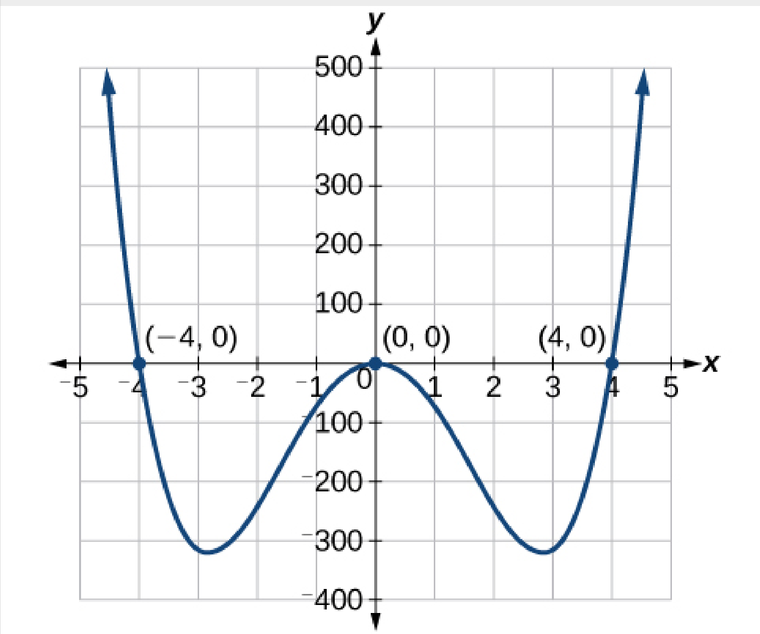
Notice that we have the difference of squares in the factor $x^2 - 16$,which we will continue to factor and obtain two solutions. The first term, $5x^2$,generates, technically, two solutions as the exponent is 2, but they are the same solution.
Equation:

$$\begin{aligned}5x^2 &= 0\\ x &= 0\\ x^2-16 &= 0\\ (x-4)(x+4) &= 0\\ x &= 4\\ x &= -4\end{aligned}$$

The solutions are 0 (double solution)$,4,$ and $-4.$

Analysis

We can see the solutions on the graph in [link]. The *x*-coordinates of the points where the graph crosses the *x*-axis are the solutions—the *x*-intercepts. Notice on the graph that at the solution $0,$ the graph touches the *x*-axis and bounces back. It does not cross the *x*-axis. This is typical of double solutions.



Note:
Exercise:

Problem: Solve by factoring: $12x^4=3x^2.$

Solution:

$x=0,x=\frac{1}{2},x=-\frac{1}{2}$

Example:
Exercise:

Problem:
Solve a Polynomial by Grouping

Solve a polynomial by grouping: $x^3 + x^2 - 9x - 9 = 0.$

Solution:

This polynomial consists of 4 terms, which we can solve by grouping. Grouping procedures require factoring the first two terms and then factoring the last two terms. If the factors in the parentheses are identical, we can continue the process and solve, unless more factoring is suggested.

Equation:

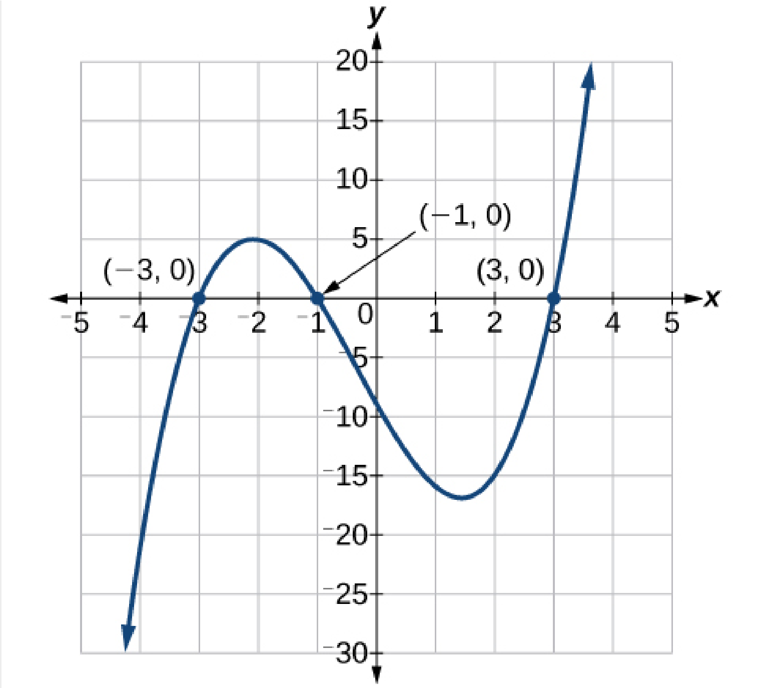
$$\begin{array}{rcl} x^3 + x^2 - 9x - 9 & = & 0 \\ x^2(x+1) - 9(x+1) & = & 0 \\ \left(x^2 - 9\right)(x+1) & = & 0 \end{array}$$

The grouping process ends here, as we can factor $x^2 - 9$ using the difference of squares formula.

Equation:

$$\begin{array}{rcl} \left(x^2 - 9\right)(x+1) & = & 0 \\ (x-3)(x+3)(x+1) & = & 0 \\ x & = & 3 \\ x & = & -3 \\ x & = & -1 \end{array}$$

The solutions are $3, -3,$ and $-1.$ Note that the highest exponent is 3 and we obtained 3 solutions. We can see the solutions, the *x*-intercepts, on the graph in [link].



Analysis

We looked at solving quadratic equations by factoring when the leading coefficient is 1. When the leading coefficient is not 1, we solved by grouping. Grouping requires four terms, which we obtained by splitting the linear term of quadratic equations. We can also use grouping for some polynomials of degree higher than 2, as we saw here, since there were already four terms.

Solving Radical Equations

Radical equations are equations that contain variables in the radicand (the expression under a radical symbol), such as
Equation:

$$\begin{aligned} \sqrt{3x+18} &= x \\ \sqrt{x+3} &= x-3 \\ \sqrt{x+5}-\sqrt{x-3} &= 2 \end{aligned}$$

Radical equations may have one or more radical terms, and are solved by eliminating each radical, one at a time. We have to be careful when solving radical equations, as it is not unusual to find **extraneous solutions**, roots that are not, in fact, solutions to the equation. These solutions are not due to a mistake in the solving method, but result from the process of raising both sides of an equation to a power. However, checking each answer in the original equation will confirm the true solutions.

Note:
Radical Equations
An equation containing terms with a variable in the radicand is called a **radical equation**.

Note:
Given a radical equation, solve it.

1. Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side.
2. If the radical is a square root, then square both sides of the equation. If it is a cube root, then raise both sides of the equation to the third power. In other words, for an *n*th root radical, raise both sides to the *n*th power. Doing so eliminates the radical symbol.
3. Solve the remaining equation.
4. If a radical term still remains, repeat steps 1–2.
5. Confirm solutions by substituting them into the original equation.

Example:
Exercise:

Problem:
Solving an Equation with One Radical

Solve $\sqrt{15-2x}=x$.

Solution:

The radical is already isolated on the left side of the equal side, so proceed to square both sides.
Equation:

$$\begin{aligned}\sqrt{15-2x} &= x \\ \left(\sqrt{15-2x}\right)^2 &= (x)^2 \\ 15-2x &= x^2\end{aligned}$$

We see that the remaining equation is a quadratic. Set it equal to zero and solve.
Equation:

$$\begin{aligned} 0 &= x^2 + 2x - 15 \\ &= (x+5)(x-3) \\ x &= -5 \\ x &= 3 \end{aligned}$$

The proposed solutions are -5 and 3. Let us check each solution back in the original equation. First, check $x = -5$.

Equation:

$$\begin{aligned} \sqrt{15 - 2x} &= x \\ \sqrt{15 - 2(-5)} &= -5 \\ \sqrt{25} &= -5 \\ 5 &\neq -5 \end{aligned}$$

This is an extraneous solution. While no mistake was made solving the equation, we found a solution that does not satisfy the original equation.

Check $x = 3$.

Equation:

$$\begin{aligned} \sqrt{15 - 2x} &= x \\ \sqrt{15 - 2(3)} &= 3 \\ \sqrt{9} &= 3 \\ 3 &= 3 \end{aligned}$$

The solution is 3.

Note:
Exercise:

Problem: Solve the radical equation: $\sqrt{x + 3} = 3x - 1$

Solution:

$x = 1$;extraneous solution $x = -\frac{2}{9}$

Example:

Exercise:

Problem:
Solving a Radical Equation Containing Two Radicals

Solve $\sqrt{2x+3}+\sqrt{x-2}=4.$

Solution:

As this equation contains two radicals, we isolate one radical, eliminate it, and then isolate the second radical.
Equation:

$$\begin{array}{rcll} \sqrt{2x+3}+\sqrt{x-2} & = & 4 & \\ \sqrt{2x+3} & = & 4-\sqrt{x-2} & \text{Subtract } \sqrt{x-2} \text{ from both sides.} \\ \left(\sqrt{2x+3}\right)^2 & = & \left(4-\sqrt{x-2}\right)^2 & \text{Square both sides.} \end{array}$$

Use the perfect square formula to expand the right side: $(a-b)^2=a^2-2ab+b^2.$
Equation:

$$\begin{array}{rcll} 2x+3 & = & (4)^2-2(4)\sqrt{x-2}+\left(\sqrt{x-2}\right)^2 & \\ 2x+3 & = & 16-8\sqrt{x-2}+(x-2) & \\ 2x+3 & = & 14+x-8\sqrt{x-2} & \text{Combine like terms.} \\ x-11 & = & -8\sqrt{x-2} & \text{Isolate the second radical.} \\ (x-11)^2 & = & \left(-8\sqrt{x-2}\right)^2 & \text{Square both sides.} \\ x^2-22x+121 & = & 64(x-2) & \end{array}$$

Now that both radicals have been eliminated, set the quadratic equal to zero and solve.
Equation:

$$\begin{array}{rcll} x^2-22x+121 & = & 64x-128 & \\ x^2-86x+249 & = & 0 & \\ (x-3)(x-83) & = & 0 & \text{Factor and solve.} \\ x & = & 3 & \\ x & = & 83 & \end{array}$$

The proposed solutions are 3 and 83. Check each solution in the original equation.
Equation:

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2} &= 4 \\ \sqrt{2x+3} &= 4-\sqrt{x-2} \\ \sqrt{2(3)+3} &= 4-\sqrt{(3)-2} \\ \sqrt{9} &= 4-\sqrt{1} \\ 3 &= 3\end{aligned}$$

One solution is 3.

Check $x=83$.
Equation:

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2} &= 4 \\ \sqrt{2x+3} &= 4-\sqrt{x-2} \\ \sqrt{2(83)+3} &= 4-\sqrt{(83-2)} \\ \sqrt{169} &= 4-\sqrt{81} \\ 13 &\neq -5\end{aligned}$$

The only solution is 3. We see that $x=83$ is an extraneous solution.

Note:
Exercise:

Problem: Solve the equation with two radicals: $\sqrt{3x+7}+\sqrt{x+2}=1$.

Solution:

$x=-2$; extraneous solution $x=-1$

Solving an Absolute Value Equation

Next, we will learn how to solve an absolute value equation. To solve an equation such as $|2x-6|=8$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value bars is 8 or -8. This leads to two different equations we can solve independently.
Equation:

$$\begin{array}{rclcrcl} 2x-6 & = & 8 & \quad\text{or}\quad & 2x-6 & = & -8 \\ 2x & = & 14 & & 2x & = & -2 \\ x & = & 7 & & x & = & -1 \end{array}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

Note:
Absolute Value Equations
The absolute value of x is written as $|x|$. It has the following properties:
Equation:

$$\begin{array}{l} \text{If } x \geq 0, \text{ then } |x| = x. \\ \text{If } x < 0, \text{ then } |x| = -x. \end{array}$$

For real numbers A and B,an equation of the form $|A| = B$,with $B \geq 0$,will have solutions when $A = B$ or $A = -B$. If $B < 0$,the equation $|A| = B$ has no solution.
An **absolute value equation** in the form $|ax+b| = c$ has the following properties:
Equation:

$$\begin{array}{l} \text{If } \ c < 0, |ax+b| = c \text{ has no solution.} \\ \text{If } \ c = 0, |ax+b| = c \text{ has one solution.} \\ \text{If } \ c > 0, |ax+b| = c \text{ has two solutions.} \end{array}$$

Note:
Given an absolute value equation, solve it.

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c > 0$,write and solve two equations: $ax+b = c$ and $ax+b = -c$.

Example:
Exercise:

Problem:
Solving Absolute Value Equations

Solve the following absolute value equations:

- (a) $|6x+4|=8$
- (b) $|3x+4|=-9$
- (c) $|3x-5|-4=6$
- (d) $|-5x+10|=0$

Solution:

- (a) $|6x+4|=8$

  Write two equations and solve each:
  Equation:

  $$\begin{aligned} 6x+4 &= 8 \\ 6x &= 4 \\ x &= \tfrac{2}{3} \end{aligned} \qquad\qquad \begin{aligned} 6x+4 &= -8 \\ 6x &= -12 \\ x &= -2 \end{aligned}$$

  The two solutions are $\frac{2}{3}$ and -2.

- (b) $|3x+4|=-9$

  There is no solution as an absolute value cannot be negative.

- (c) $|3x-5|-4=6$

  Isolate the absolute value expression and then write two equations.
  Equation:

  $$\begin{aligned} |3x-5|-4 &= 6 \\ |3x-5| &= 10 \end{aligned}$$

  $$\begin{aligned} 3x-5 &= 10 \\ 3x &= 15 \\ x &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x-5 &= -10 \\ 3x &= -5 \\ x &= -\tfrac{5}{3} \end{aligned}$$

  There are two solutions: 5, and $-\frac{5}{3}$.

- (d) $|-5x+10|=0$

  The equation is set equal to zero, so we have to write only one equation.
  Equation:

  $$\begin{aligned} -5x+10 &= 0 \\ -5x &= -10 \\ x &= 2 \end{aligned}$$

There is one solution: 2.

Note:
Exercise:

Problem: Solve the absolute value equation: $|1-4x|+8=13$.

Solution:

$x=-1$, $x=\frac{3}{2}$

Solving Other Types of Equations

There are many other types of equations in addition to the ones we have discussed so far. We will see more of them throughout the text. Here, we will discuss equations that are in quadratic form, and rational equations that result in a quadratic.

Solving Equations in Quadratic Form

Equations in quadratic form are equations with three terms. The first term has a power other than 2. The middle term has an exponent that is one-half the exponent of the leading term. The third term is a constant. We can solve equations in this form as if they were quadratic. A few examples of these equations include $x^4-5x^2+4=0$, $x^6+7x^3-8=0$, and $x^{\frac{2}{3}}+4x^{\frac{1}{3}}+2=0$. In each one, doubling the exponent of the middle term equals the exponent on the leading term. We can solve these equations by substituting a variable for the middle term.

Note:
Quadratic Form
If the exponent on the middle term is one-half of the exponent on the leading term, we have an **equation in quadratic form**, which we can solve as if it were a quadratic. We substitute a variable for the middle term to solve equations in quadratic form.

Note:
Given an equation quadratic in form, solve it.

1. Identify the exponent on the leading term and determine whether it is double the exponent on the middle term.
2. If it is, substitute a variable, such as *u*, for the variable portion of the middle term.
3. Rewrite the equation so that it takes on the standard form of a quadratic.
4. Solve using one of the usual methods for solving a quadratic.
5. Replace the substitution variable with the original term.
6. Solve the remaining equation.

Example:
Exercise:

Problem:
Solving a Fourth-degree Equation in Quadratic Form

Solve this fourth-degree equation: $3x^4 - 2x^2 - 1 = 0$.

Solution:

This equation fits the main criteria, that the power on the leading term is double the power on the middle term. Next, we will make a substitution for the variable term in the middle. Let $u = x^2$.Rewrite the equation in *u*.
Equation:

$$3u^2 - 2u - 1 = 0$$

Now solve the quadratic.
Equation:

$$\begin{array}{rcl} 3u^2 - 2u - 1 & = & 0 \\ (3u+1)(u-1) & = & 0 \end{array}$$

Solve each factor and replace the original term for *u*.
Equation:

$$\begin{array}{rcl} 3u+1 & = & 0 \\ 3u & = & -1 \\ u & = & -\frac{1}{3} \\ x^2 & = & -\frac{1}{3} \\ x & = & \pm i\sqrt{\frac{1}{3}} \end{array}$$

Equation:

$$\begin{aligned} u-1 &= 0 \\ u &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

The solutions are $\pm\, i\sqrt{\frac{1}{3}}$ and $\pm\, 1$.

Note:
Exercise:

Problem: Solve using substitution: $x^4 - 8x^2 - 9 = 0$.

Solution:

$x = -3, 3, -i, i$

Example:
Exercise:

Problem:
Solving an Equation in Quadratic Form Containing a Binomial

Solve the equation in quadratic form: $(x+2)^2 + 11\,(x+2) - 12 = 0$.

Solution:

This equation contains a binomial in place of the single variable. The tendency is to expand what is presented. However, recognizing that it fits the criteria for being in quadratic form makes all the difference in the solving process. First, make a substitution, letting $u = x + 2$. Then rewrite the equation in *u*.
Equation:

$$\begin{aligned} u^2 + 11u - 12 &= 0 \\ (u+12)(u-1) &= 0 \end{aligned}$$

Solve using the zero-factor property and then replace *u* with the original expression.
Equation:

$$\begin{aligned} u+12 &= 0 \\ u &= -12 \\ x+2 &= -12 \\ x &= -14 \end{aligned}$$

The second factor results in
Equation:

$$\begin{aligned} u-1 &= 0 \\ u &= 1 \\ x+2 &= 1 \\ x &= -1 \end{aligned}$$

We have two solutions: -14, and -1.

Note:
Exercise:

Problem: Solve: $(x-5)^2 - 4(x-5) - 21 = 0$.

Solution:

$x = 2, x = 12$

Solving Rational Equations Resulting in a Quadratic

Earlier, we solved rational equations. Sometimes, solving a rational equation results in a quadratic. When this happens, we continue the solution by simplifying the quadratic equation by one of the methods we have seen. It may turn out that there is no solution.

Example:
Exercise:

Problem:
Solving a Rational Equation Leading to a Quadratic

Solve the following rational equation: $\frac{-4x}{x-1} + \frac{4}{x+1} = \frac{-8}{x^2-1}$.

Solution:

We want all denominators in factored form to find the LCD. Two of the denominators cannot be factored further. However, $x^2-1=(x+1)(x-1).$ Then, the LCD is $(x+1)(x-1).$ Next, we multiply the whole equation by the LCD.

Equation:

$$\begin{aligned}(x+1)(x-1)\left[\frac{-4x}{x-1}+\frac{4}{x+1}\right] &= \left[\frac{-8}{(x+1)(x-1)}\right](x+1)(x-1)\\ -4x(x+1)+4(x-1) &= -8\\ -4x^2-4x+4x-4 &= -8\\ -4x^2+4 &= 0\\ -4\left(x^2-1\right) &= 0\\ -4(x+1)(x-1) &= 0\\ x &= -1\\ x &= 1\end{aligned}$$

In this case, either solution produces a zero in the denominator in the original equation. Thus, there is no solution.

Note:

Exercise:

Problem: Solve $\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}.$

Solution:

$x=-1$, $x=0$ is not a solution.

Note:

Access these online resources for additional instruction and practice with different types of equations.

- Rational Equation with no Solution
- Solving equations with rational exponents using reciprocal powers
- Solving radical equations part 1 of 2
- Solving radical equations part 2 of 2

Key Concepts

- Rational exponents can be rewritten several ways depending on what is most convenient for the problem. To solve, both sides of the equation are raised to a power that will render the exponent on the variable equal to 1. See [link], [link], and [link].
- Factoring extends to higher-order polynomials when it involves factoring out the GCF or factoring by grouping. See [link] and [link].
- We can solve radical equations by isolating the radical and raising both sides of the equation to a power that matches the index. See [link] and [link].
- To solve absolute value equations, we need to write two equations, one for the positive value and one for the negative value. See [link].
- Equations in quadratic form are easy to spot, as the exponent on the first term is double the exponent on the second term and the third term is a constant. We may also see a binomial in place of the single variable. We use substitution to solve. See [link] and [link].
- Solving a rational equation may also lead to a quadratic equation or an equation in quadratic form. See [link].

Section Exercises

Verbal

Exercise:

Problem: In a radical equation, what does it mean if a number is an extraneous solution?

Solution:

This is not a solution to the radical equation, it is a value obtained from squaring both sides and thus changing the signs of an equation which has caused it not to be a solution in the original equation.

Exercise:

Problem: Explain why possible solutions *must* be checked in radical equations.

Exercise:

Problem:

Your friend tries to calculate the value $-9^{\frac{3}{2}}$and keeps getting an ERROR message. What mistake is he or she probably making?

Solution:

He or she is probably trying to enter negative 9, but taking the square root of -9 is not a real number. The negative sign is in front of this, so your friend should be taking the square root of 9, cubing it, and then putting the negative sign in front, resulting in -27.

Exercise:

Problem: Explain why $|2x+5|=-7$ has no solutions.

Exercise:

Problem: Explain how to change a rational exponent into the correct radical expression.

Solution:

A rational exponent is a fraction: the denominator of the fraction is the root or index number and the numerator is the power to which it is raised.

Algebraic

For the following exercises, solve the rational exponent equation. Use factoring where necessary.

Exercise:

Problem:$x^{\frac{2}{3}}=16$

Exercise:

Problem:$x^{\frac{3}{4}}=27$

Solution:

$x=81$

Exercise:

Problem:$2x^{\frac{1}{2}}-x^{\frac{1}{4}}=0$

Exercise:

Problem:$(x-1)^{\frac{3}{4}}=8$

Solution:

$x=17$

Exercise:

Problem:$(x+1)^{\frac{2}{3}}=4$

Exercise:

Problem:$x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$

Solution:

$x = 8, \;\; x = 27$

Exercise:

Problem:$x^{\frac{7}{3}} - 3x^{\frac{4}{3}} - 4x^{\frac{1}{3}} = 0$

For the following exercises, solve the following polynomial equations by grouping and factoring.

Exercise:

Problem:$x^3 + 2x^2 - x - 2 = 0$

Solution:

$x = -2, 1, -1$

Exercise:

Problem:$3x^3 - 6x^2 - 27x + 54 = 0$

Exercise:

Problem:$4y^3 - 9y = 0$

Solution:

$y = 0, \;\; \frac{3}{2}, \;\; \frac{-3}{2}$

Exercise:

Problem:$x^3 + 3x^2 - 25x - 75 = 0$

Exercise:

Problem:$m^3 + m^2 - m - 1 = 0$

Solution:

$m = 1, -1$

Exercise:

Problem:$2x^5 - 14x^3 = 0$

Exercise:

Problem: $5x^3 + 45x = 2x^2 + 18$

Solution:

$x = \frac{2}{5}, \pm 3i$

For the following exercises, solve the radical equation. Be sure to check all solutions to eliminate extraneous solutions.

Exercise:

Problem: $\sqrt{3x-1} - 2 = 0$

Exercise:

Problem: $\sqrt{x-7} = 5$

Solution:

$x = 32$

Exercise:

Problem: $\sqrt{x-1} = x - 7$

Exercise:

Problem: $\sqrt{3t+5} = 7$

Solution:

$t = \frac{44}{3}$

Exercise:

Problem: $\sqrt{t+1} + 9 = 7$

Exercise:

Problem: $\sqrt{12-x} = x$

Solution:

$x = 3$

Exercise:

Problem: $\sqrt{2x+3} - \sqrt{x+2} = 2$

Exercise:

Problem: $\sqrt{3x+7} + \sqrt{x+2} = 1$

Solution:

$x = -2$

Exercise:

Problem: $\sqrt{2x+3} - \sqrt{x+1} = 1$

For the following exercises, solve the equation involving absolute value.

Exercise:

Problem: $|3x-4| = 8$

Solution:

$x = 4, \frac{-4}{3}$

Exercise:

Problem: $|2x-3| = -2$

Exercise:

Problem: $|1-4x| - 1 = 5$

Solution:

$x = \frac{-5}{4}, \frac{7}{4}$

Exercise:

Problem: $|4x+1| - 3 = 6$

Exercise:

Problem: $|2x-1| - 7 = -2$

Solution:

$x = 3, -2$

Exercise:

Problem: $|2x+1|-2=-3$

Exercise:

Problem: $|x+5|=0$

Solution:

$x=-5$

Exercise:

Problem: $-|2x+1|=-3$

For the following exercises, solve the equation by identifying the quadratic form. Use a substitute variable and find all real solutions by factoring.

Exercise:

Problem: $x^4-10x^2+9=0$

Solution:

$x=1,-1,3,-3$

Exercise:

Problem: $4(t-1)^2-9(t-1)=-2$

Exercise:

Problem: $\left(x^2-1\right)^2+\left(x^2-1\right)-12=0$

Solution:

$x=2,-2$

Exercise:

Problem: $(x+1)^2-8(x+1)-9=0$

Exercise:

Problem: $(x-3)^2-4=0$

Solution:

$x = 1, 5$

Extensions

For the following exercises, solve for the unknown variable.

Exercise:

Problem: $x^{-2} - x^{-1} - 12 = 0$

Exercise:

Problem: $\sqrt{|x|^2} = x$

Solution:

All real numbers

Exercise:

Problem: $t^{10} - t^5 + 1 = 0$

Exercise:

Problem: $\left|x^2 + 2x - 36\right| = 12$

Solution:

$x = 4, 6, -6, -8$

Real-World Applications

For the following exercises, use the model for the period of a pendulum, T, such that $T = 2\pi\sqrt{\frac{L}{g}}$, where the length of the pendulum is L and the acceleration due to gravity is g.

Exercise:

Problem:

If the acceleration due to gravity is 9.8 m/s^2 and the period equals 1 s, find the length to the nearest cm (100 cm = 1 m).

Exercise:

Problem:

If the gravity is 32 ft/s^2 and the period equals 1 s, find the length to the nearest in. (12 in. = 1 ft). Round your answer to the nearest in.

Solution:

10 in.

For the following exercises, use a model for body surface area, BSA, such that $BSA=\sqrt{\frac{wh}{3600}}$, where *w* = weight in kg and *h* = height in cm.

Exercise:

Problem: Find the height of a 72-kg female to the nearest cm whose $BSA=1.8$.

Exercise:

Problem: Find the weight of a 177-cm male to the nearest kg whose $BSA=2.1$.

Solution:

90 kg

Glossary

absolute value equation
: an equation in which the variable appears in absolute value bars, typically with two solutions, one accounting for the positive expression and one for the negative expression

equations in quadratic form
: equations with a power other than 2 but with a middle term with an exponent that is one-half the exponent of the leading term

extraneous solutions
: any solutions obtained that are not valid in the original equation

polynomial equation
: an equation containing a string of terms including numerical coefficients and variables raised to whole-number exponents

radical equation
: an equation containing at least one radical term where the variable is part of the radicand

Linear Inequalities and Absolute Value Inequalities
In this section you will:

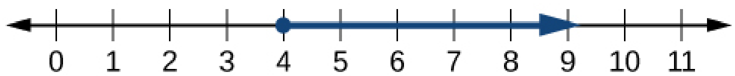
- Use interval notation.
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.
- Solve absolute value inequalities.

It is not easy to make the honor role at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

We can use a number line as shown in [link]. The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.



We can use set-builder notation: $\{x|x \geq 4\}$,which translates to "all real numbers *x* such that *x* is greater than or equal to 4." Notice that braces are used to indicate a set.

The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2, 6)$,or all numbers between -2 and 6,including -2,but not including 6;$(-1, 0)$,all real numbers between, but not including -1 and 0;and $(-\infty, 1]$,all real numbers less than and including 1. [link] outlines the possibilities.

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between *a* and *b*, but not including *a* or *b*	$\{x\|a < x < b\}$	(a, b)
All real numbers greater than *a*, but not including *a*	$\{x\|x > a\}$	(a, ∞)
All real numbers less than *b*, but not including *b*	$\{x\|x < b\}$	$(-\infty, b)$
All real numbers greater than *a*, including *a*	$\{x\|x \geq a\}$	$[a, \infty)$
All real numbers less than *b*, including *b*	$\{x\|x \leq b\}$	$(-\infty, b]$
All real numbers between *a* and *b*, including *a*	$\{x\|a \leq x < b\}$	$[a, b)$
All real numbers between *a* and *b*, including *b*	$\{x\|a < x \leq b\}$	$(a, b]$
All real numbers between *a* and *b*, including *a* and *b*	$\{x\|a \leq x \leq b\}$	$[a, b]$
All real numbers less than *a* or greater than *b*	$\{x\|x < a \text{ and } x > b\}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{x\|x \text{ is all real numbers}\}$	$(-\infty, \infty)$

Example:
Exercise:

Problem:
Using Interval Notation to Express All Real Numbers Greater Than or Equal to *a*

Use interval notation to indicate all real numbers greater than or equal to -2.

Solution:

Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$.The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

Note:
Exercise:

Problem: Use interval notation to indicate all real numbers between and including -3 and 5.

Solution:

$[-3, 5]$

Example:
Exercise:

Problem:
Using Interval Notation to Express All Real Numbers Less Than or Equal to *a* or Greater Than or Equal to *b*

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1.

Solution:

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1. So, this interval begins at $-\infty$ and ends at -1,which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to 1,which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol,$\cup$,between the two intervals.
Equation:

$$(-\infty, -1] \cup [1, \infty)$$

Note:
Exercise:

Problem:

Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

Solution:

$(-\infty,-2)\cup[3,\infty)$

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

Note:
Properties of Inequalities
Equation:

$$\begin{array}{ll}\textbf{Addition Property} & \text{If } a<b, \text{ then } a+c<b+c.\\ \\ \textbf{Multiplication Property} & \text{If } a<b \text{ and } c>0, \text{ then } ac<bc.\\ & \text{If } a<b \text{ and } c<0, \text{ then } ac>bc.\end{array}$$

These properties also apply to $a\le b$, $a>b$, and $a\ge b$.

Example:
Exercise:

Problem:
Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following:

- (a) $x-15<4$
- (b) $6\ge x-1$
- (c) $x+7>9$

Solution:

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

a.

$$\begin{array}{ll} x-15<4 & \\ x-15+15<4+15 & \text{Add 15 to both sides.} \\ x<19 & \end{array}$$

b.

$$\begin{array}{ll} 6\geq x-1 & \\ 6+1\geq x-1+1 & \text{Add 1 to both sides.} \\ 7\geq x & \end{array}$$

c.

$$\begin{array}{ll} x+7>9 & \\ x+7-7>9-7 & \text{Subtract 7 from both sides.} \\ x>2 & \end{array}$$

Note:
Exercise:

Problem: Solve: $3x-2<1$.

Solution:

$x<1$

Example:
Exercise:

Problem:
Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

a. $3x<6$
b. $-2x-1\geq 5$
c. $5-x>10$

Solution:

a.

$$\begin{aligned} 3x &< 6 \\ \tfrac{1}{3}(3x) &< (6)\tfrac{1}{3} \\ x &< 2 \end{aligned}$$

b.

$$\begin{aligned} -2x-1 &\geq 5 \\ -2x &\geq 6 \\ \left(-\tfrac{1}{2}\right)(-2x) &\geq (6)\left(-\tfrac{1}{2}\right) && \text{Multiply by } -\tfrac{1}{2}. \\ x &\leq -3 && \text{Reverse the inequality.} \end{aligned}$$

c.

$$\begin{aligned} 5-x &> 10 \\ -x &> 5 \\ (-1)(-x) &> (5)(-1) && \text{Multiply by } -1. \\ x &< -5 && \text{Reverse the inequality.} \end{aligned}$$

Note:
Exercise:

Problem: Solve: $4x+7 \geq 2x-3$.

Solution:

$x \geq -5$

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

Example:
Exercise:

Problem:
Solving an Inequality Algebraically

Solve the inequality: $13-7x \geq 10x-4$.

Solution:

Solving this inequality is similar to solving an equation up until the last step.
Equation:

$$\begin{array}{ll} 13-7x \geq 10x-4 & \\ 13-17x \geq -4 & \text{Move variable terms to one side of the inequality.} \\ -17x \geq -17 & \text{Isolate the variable term.} \\ x \leq 1 & \text{Dividing both sides by } -17 \text{ reverses the inequality.} \end{array}$$

The solution set is given by the interval $(-\infty, 1]$,or all real numbers less than and including 1.

Note:
Exercise:

Problem:

Solve the inequality and write the answer using interval notation: $-x+4<\frac{1}{2}x+1$.

Solution:

$(2, \infty)$

Example:
Exercise:

Problem:
Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8}+\frac{2}{3}x$.

Solution:

We begin solving in the same way we do when solving an equation.
Equation:

$$\begin{array}{ll} -\frac{3}{4}x \geq -\frac{5}{8}+\frac{2}{3}x & \\ -\frac{3}{4}x-\frac{2}{3}x \geq -\frac{5}{8} & \text{Put variable terms on one side.} \\ -\frac{9}{12}x-\frac{8}{12}x \geq -\frac{5}{8} & \text{Write fractions with common denominator.} \\ -\frac{17}{12}x \geq -\frac{5}{8} & \\ x \leq -\frac{5}{8}\left(-\frac{12}{17}\right) & \text{Multiplying by a negative number reverses the inequality.} \\ x \leq \frac{15}{34} & \end{array}$$

The solution set is the interval $\left(-\infty, \frac{15}{34}\right]$.

Note:
Exercise:

Problem: Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x \le \frac{3}{4} + \frac{8}{3}x$.

Solution:

$\left[-\frac{3}{14}, \infty\right)$

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4 < x \le 6$ means $4 < x$ and $x \le 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

Example:
Exercise:

Problem:
Solving a Compound Inequality

Solve the compound inequality: $3 \le 2x + 2 < 6$.

Solution:

The first method is to write two separate inequalities: $3 \le 2x + 2$ and $2x + 2 < 6$. We solve them independently.
Equation:

$$\begin{array}{lcl} 3 \le 2x + 2 & \text{and} & 2x + 2 < 6 \\ 1 \le 2x & & \quad 2x < 4 \\ \frac{1}{2} \le x & & \quad x < 2 \end{array}$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.
Equation:

$$\frac{1}{2} \le x < 2$$

In interval notation, the solution is written as $\left[\frac{1}{2}, 2\right)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

Equation:

$$\begin{array}{ll} 3 \le 2x + 2 < 6 & \\ 1 \le 2x < 4 & \text{Isolate the variable term, and subtract 2 from all three parts.} \\ \frac{1}{2} \le x < 2 & \text{Divide through all three parts by 2.} \end{array}$$

We get the same solution: $\left[\frac{1}{2}, 2\right)$.

Note:
Exercise:

Problem: Solve the compound inequality: $4 < 2x - 8 \le 10$.

Solution:

$6 < x \le 9$ or $(6, 9]$

Example:
Exercise:

Problem:
Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3 + x > 7x - 2 > 5x - 10$.

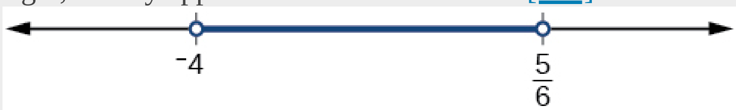
Solution:

Let's try the first method. Write two inequalities:

Equation:

$$\begin{array}{rcl} 3 + x > 7x - 2 & \text{and} & 7x - 2 > 5x - 10 \\ 3 > 6x - 2 & & 2x - 2 > -10 \\ 5 > 6x & & 2x > -8 \\ \frac{5}{6} > x & & x > -4 \\ x < \frac{5}{6} & & -4 < x \end{array}$$

The solution set is $-4 < x < \frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See [link].



Note:
Exercise:

Problem: Solve the compound inequality: $3y < 4 - 5y < 5 + 3y$.

Solution:

$\left(-\frac{1}{8}, \frac{1}{2}\right)$

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x, 0)$ has an absolute value of x, as it is *x* units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form
Equation:

$$|A| < B,\ |A| \leq B,\ |A| > B, \text{ or } |A| \geq B,$$

Where *A*, and sometimes *B*, represents an algebraic expression dependent on a variable *x*. Solving the inequality means finding the set of all x-values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of *x*-values such that the distance between x and 600 is less than 200. We represent the distance between x and 600 as $|x - 600|$, and therefore, $|x - 600| \leq 200$ or
Equation:

$$-200 \le x - 600 \le 200$$
$$-200 + 600 \le x - 600 + 600 \le 200 + 600$$
$$400 \le x \le 800$$

This means our returns would be between $400 and $800.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

Note:
Absolute Value Inequalities
For an algebraic expression X, and $k > 0$,an **absolute value inequality** is an inequality of the form
Equation:

$$|X| < k \text{ is equivalent to } -k < X < k$$
$$|X| > k \text{ is equivalent to } X < -k \text{ or } X > k$$

These statements also apply to $|X| \le k$ and $|X| \ge k$.

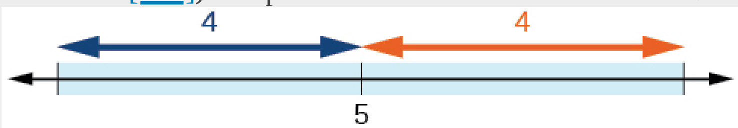
Example:
Exercise:

Problem:
Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

Solution:

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in [link], to represent the condition to be satisfied.



The distance from x to 5 can be represented using an absolute value symbol, $|x - 5|$. Write the values of x that satisfy the condition as an absolute value inequality.
Equation:

$$|x - 5| \le 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.
Equation:

$$\begin{array}{ccc} x-5 \le 4 & \text{and} & x-5 \ge -4 \\ x \le 9 & & x \ge 1 \end{array}$$

If the solution set is $x \le 9$ and $x \ge 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x-5| \le 4$ is equivalent to $[1, 9]$ in interval notation.

Note:
Exercise:

Problem: Describe all *x*-values within a distance of 3 from the number 2.

Solution:

$|x-2| \le 3$

Example:
Exercise:

Problem:
Solving an Absolute Value Inequality

Solve $|x-1| \le 3$.

Solution:
Equation:

$$|x-1| \le 3$$

$$-3 \le x-1 \le 3$$

$$-2 \le x \le 4$$

$$[-2, 4]$$

Example:
Exercise:

Problem:

Using a Graphical Approach to Solve Absolute Value Inequalities

Given the equation $y = -\frac{1}{2}\left|4x - 5\right| + 3$,determine the *x*-values for which the *y*-values are negative.

Solution:

We are trying to determine where $y < 0$,which is when $-\frac{1}{2}\left|4x - 5\right| + 3 < 0$. We begin by isolating the absolute value.

Equation:

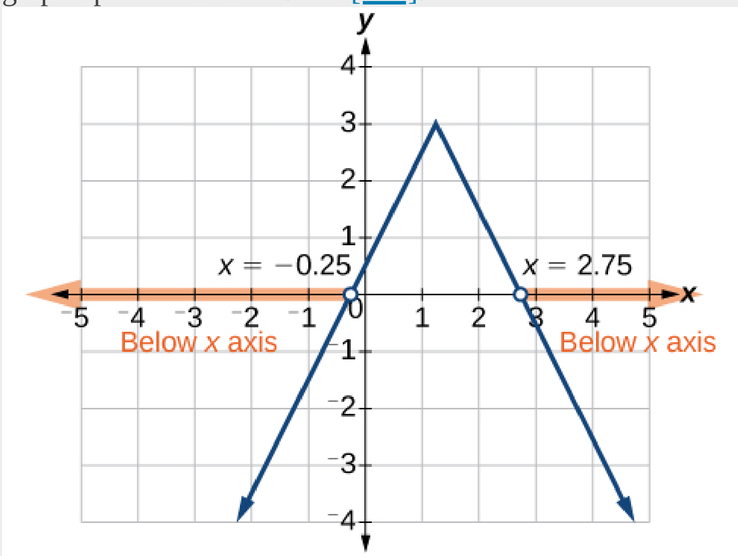
$$\begin{array}{cc} -\frac{1}{2}\left|4x-5\right| < -3 & \text{Multiply both sides by –2, and reverse the inequality.} \\ \left|4x-5\right| > 6 & \end{array}$$

Next, we solve for the equality $\left|4x - 5\right| = 6$.

Equation:

$$\begin{array}{ccc} 4x - 5 = 6 & & 4x - 5 = -6 \\ 4x = 11 & \text{or} & 4x = -1 \\ x = \frac{11}{4} & & x = -\frac{1}{4} \end{array}$$

Now, we can examine the graph to observe where the *y*-values are negative. We observe where the branches are below the *x*-axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$,and that the graph opens downward. See [link].

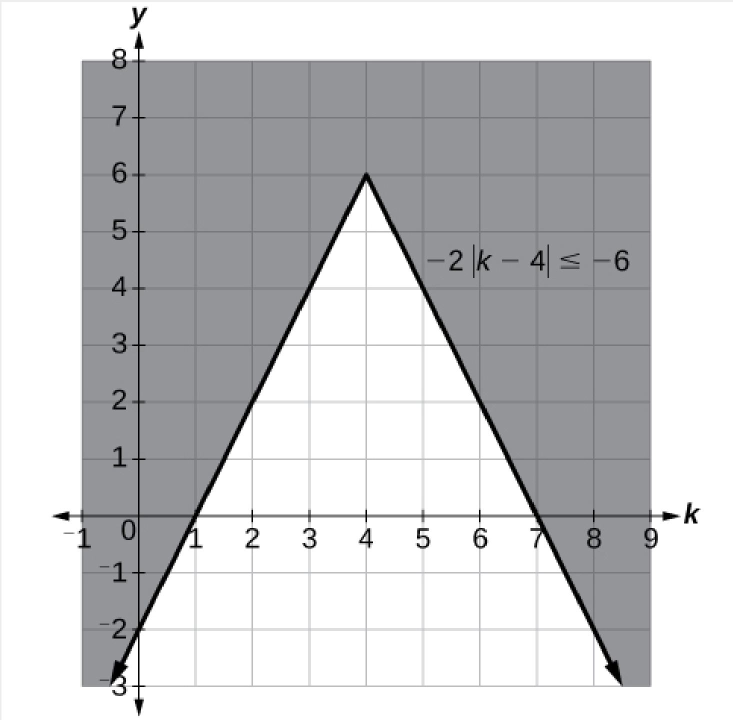


Note:
Exercise:

Problem: Solve $-2\left|k-4\right| \leq -6.$

Solution:

$k \leq 1$ or $k \geq 7;$in interval notation, this would be $\left(-\infty, 1\right] \cup \left[7, \infty\right).$



Note:
Access these online resources for additional instruction and practice with linear inequalities and absolute value inequalities.

- [Interval notation]
- [How to solve linear inequalities]
- [How to solve an inequality]
- [Absolute value equations]
- [Compound inequalities]
- [Absolute value inequalities]

Key Concepts

- Interval notation is a method to indicate the solution set to an inequality. Highly applicable in calculus, it is a system of parentheses and brackets that indicate what numbers are included in a

set and whether the endpoints are included as well. See [link] and [link].
- Solving inequalities is similar to solving equations. The same algebraic rules apply, except for one: multiplying or dividing by a negative number reverses the inequality. See [link], [link], [link], and [link].
- Compound inequalities often have three parts and can be rewritten as two independent inequalities. Solutions are given by boundary values, which are indicated as a beginning boundary or an ending boundary in the solutions to the two inequalities. See [link] and [link].
- Absolute value inequalities will produce two solution sets due to the nature of absolute value. We solve by writing two equations: one equal to a positive value and one equal to a negative value. See [link] and [link].
- Absolute value inequalities can also be solved by graphing. At least we can check the algebraic solutions by graphing, as we cannot depend on a visual for a precise solution. See [link].

Section Exercises

Verbal

Exercise:

Problem: When solving an inequality, explain what happened from Step 1 to Step 2:

$$\begin{array}{ll}\text{Step 1} & -2x > 6 \\ \text{Step 2} & \quad x < -3\end{array}$$

Solution:

When we divide both sides by a negative it changes the sign of both sides so the sense of the inequality sign changes.

Exercise:

Problem: When solving an inequality, we arrive at:

$$\begin{array}{l}x + 2 < x + 3 \\ \quad 2 < 3\end{array}$$

Explain what our solution set is.

Exercise:

Problem:

When writing our solution in interval notation, how do we represent all the real numbers?

Solution:

$(-\infty, \infty)$

Exercise:

Problem: When solving an inequality, we arrive at:

$$\begin{aligned} x+2 &> x+3 \\ 2 &> 3 \end{aligned}$$

Explain what our solution set is.

Exercise:

Problem: Describe how to graph $y=|x-3|$

Solution:

We start by finding the *x*-intercept, or where the function = 0. Once we have that point, which is $(3,0)$,we graph to the right the straight line graph $y=x{-}3$,and then when we draw it to the left we plot positive *y* values, taking the absolute value of them.

Algebraic

For the following exercises, solve the inequality. Write your final answer in interval notation.
Exercise:

Problem:$4x-7\le 9$

Exercise:

Problem:$3x+2\ge 7x-1$

Solution:

$\left(-\infty,\frac{3}{4}\right]$

Exercise:

Problem:$-2x+3>x-5$

Exercise:

Problem:$4(x+3)\ge 2x-1$

Solution:

$\left[\frac{-13}{2},\infty\right)$

Exercise:

Problem: $-\frac{1}{2}x \leq \frac{-5}{4} + \frac{2}{5}x$

Exercise:

Problem: $-5(x-1) + 3 > 3x - 4 - 4x$

Solution:

$(-\infty, 3)$

Exercise:

Problem: $-3(2x+1) > -2(x+4)$

Exercise:

Problem: $\frac{x+3}{8} - \frac{x+5}{5} \geq \frac{3}{10}$

Solution:

$\left(-\infty, -\frac{37}{3}\right]$

Exercise:

Problem: $\frac{x-1}{3} + \frac{x+2}{5} \leq \frac{3}{5}$

For the following exercises, solve the inequality involving absolute value. Write your final answer in interval notation.

Exercise:

Problem: $|x+9| \geq -6$

Solution:

All real numbers $(-\infty, \infty)$

Exercise:

Problem: $|2x+3| < 7$

Exercise:

Problem: $|3x-1| > 11$

Solution:

$\left(-\infty, \frac{-10}{3}\right) \cup (4, \infty)$

Exercise:

Problem: $|2x+1|+1 \leq 6$

Exercise:

Problem: $|x-2|+4 \geq 10$

Solution:

$(-\infty, -4] \cup [8, +\infty)$

Exercise:

Problem: $|-2x+7| \leq 13$

Exercise:

Problem: $|x-7| < -4$

Solution:

No solution

Exercise:

Problem: $|x-20| > -1$

Exercise:

Problem: $\left|\frac{x-3}{4}\right| < 2$

Solution:

$(-5, 11)$

For the following exercises, describe all the *x*-values within or including a distance of the given values.

Exercise:

Problem: Distance of 5 units from the number 7

Exercise:

Problem: Distance of 3 units from the number 9

Solution:

$[6, 12]$

Exercise:

Problem: Distance of10 units from the number 4

Exercise:

Problem: Distance of 11 units from the number 1

Solution:

$[-10, 12]$

For the following exercises, solve the compound inequality. Express your answer using inequality signs, and then write your answer using interval notation.

Exercise:

Problem:$-4 < 3x + 2 \leq 18$

Exercise:

Problem:$3x + 1 > 2x - 5 > x - 7$

Solution:

$$\begin{array}{ll} x > -6 \text{ and } x > -2 & \text{Take the intersection of two sets.} \\ x > -2, \quad (-2, +\infty) & \end{array}$$

Exercise:

Problem:$3y < 5 - 2y < 7 + y$

Exercise:

Problem:$2x - 5 < -11 \quad \text{or} \quad 5x + 1 \geq 6$

Solution:

$$\begin{array}{ll} x < -3 \quad \text{or} \quad x \geq 1 & \text{Take the union of the two sets.} \\ (-\infty, -3) \cup [1, \infty) & \end{array}$$

Exercise:

Problem:$x + 7 < x + 2$

Graphical

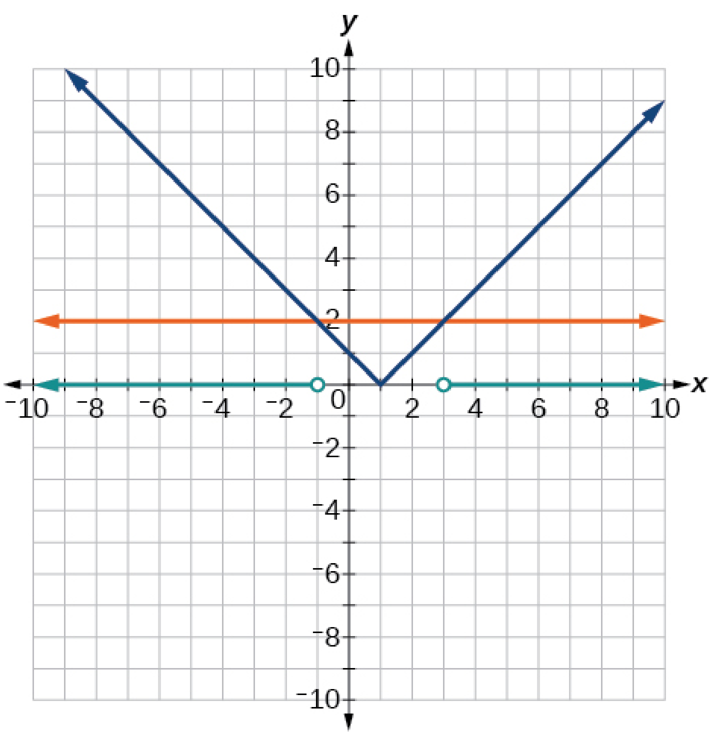
For the following exercises, graph the function. Observe the points of intersection and shade the *x*-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.

Exercise:

Problem: $|x-1|>2$

Solution:

$(-\infty,-1)\cup(3,\infty)$



Exercise:

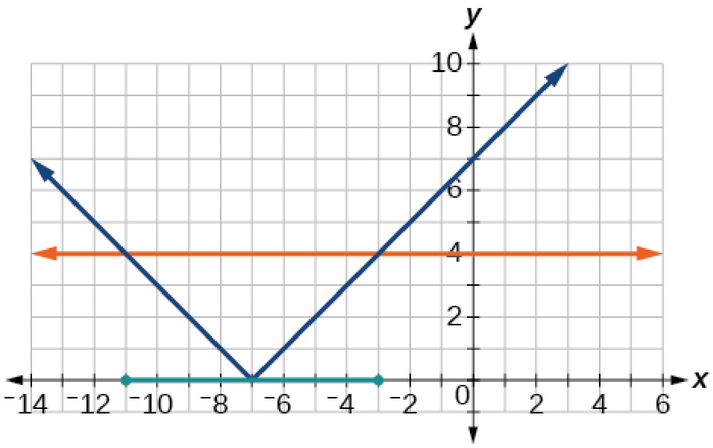
Problem: $|x+3|\geq 5$

Exercise:

Problem: $|x+7|\leq 4$

Solution:

$[-11,-3]$



Exercise:

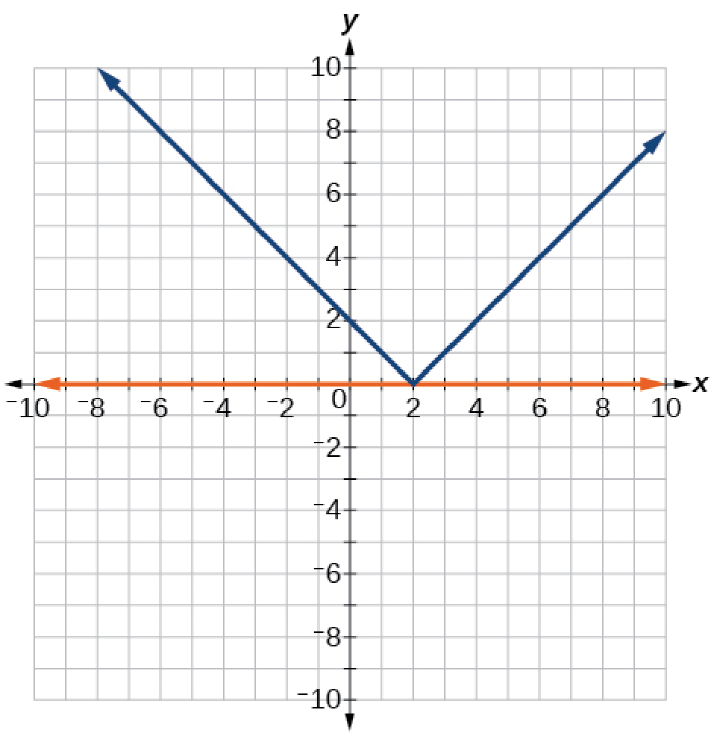
Problem: $|x-2|<7$

Exercise:

Problem: $|x-2|<0$

Solution:

It is never less than zero. No solution.



For the following exercises, graph both straight lines (left-hand side being y1 and right-hand side being y2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the *y*-values of the lines.

Exercise:

Problem: $x + 3 < 3x - 4$

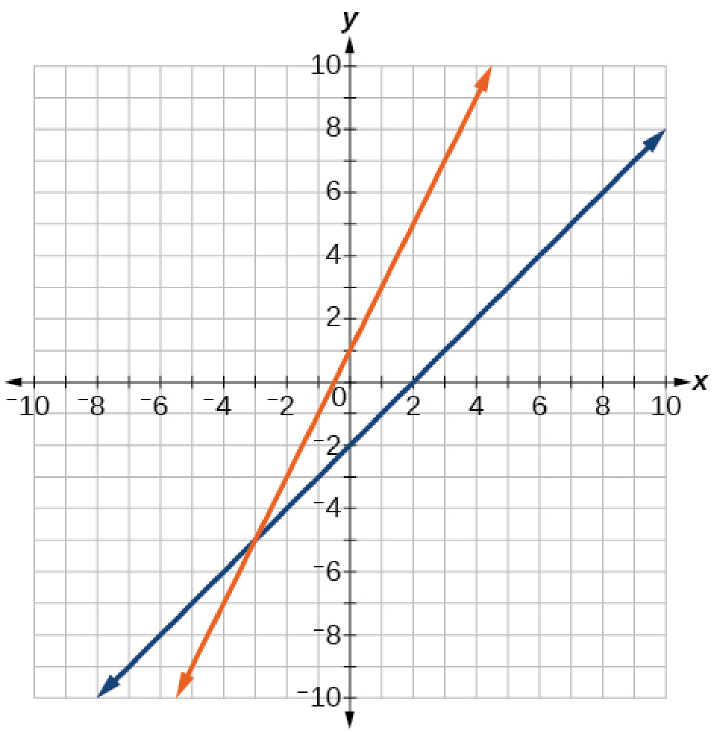
Exercise:

Problem: $x - 2 > 2x + 1$

Solution:

Where the blue line is above the orange line; point of intersection is $x = -3$.

$(-\infty, -3)$



Exercise:

Problem: $x + 1 > x + 4$

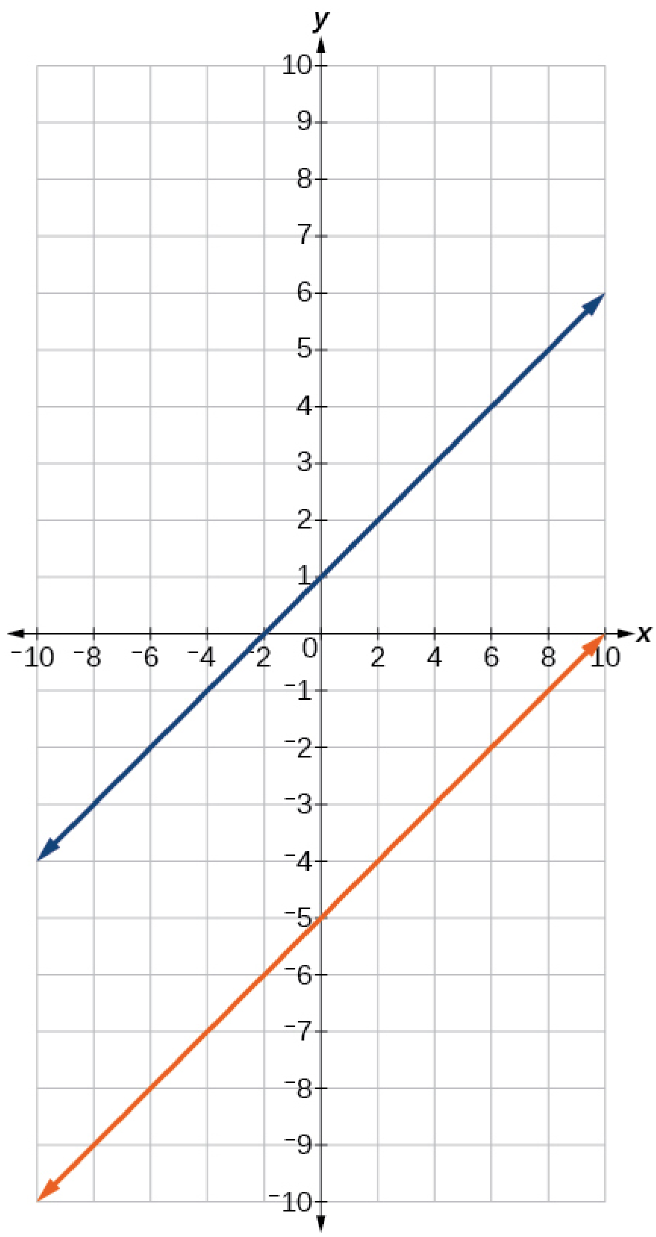
Exercise:

Problem: $\frac{1}{2}x + 1 > \frac{1}{2}x - 5$

Solution:

Where the blue line is above the orange line; always. All real numbers.

$(-\infty, -\infty)$



Exercise:

Problem: $4x + 1 < \frac{1}{2}x + 3$

Numeric

For the following exercises, write the set in interval notation.
Exercise:

Problem: $\{x|-1 < x < 3\}$

Solution:

$(-1, 3)$

Exercise:

Problem: $\{x|x \geq 7\}$

Exercise:

Problem: $\{x|x < 4\}$

Solution:

$(-\infty, 4)$

Exercise:

Problem: $\{\, x|\, x \text{ is all real numbers}\}$

For the following exercises, write the interval in set-builder notation.
Exercise:

Problem: $(-\infty, 6)$

Solution:

$\{x|x < 6\}$

Exercise:

Problem: $(4, +\infty)$

Exercise:

Problem: $[-3, 5)$

Solution:

$\{x|-3 \leq x < 5\}$

Exercise:

Problem: $[-4,1] \cup [9,\infty)$

For the following exercises, write the set of numbers represented on the number line in interval notation.

Exercise:

Problem:

Solution:

$(-2,1]$

Exercise:

Problem:

Exercise:

Problem:

Solution:

$(-\infty,4]$

Technology

For the following exercises, input the left-hand side of the inequality as a Y1 graph in your graphing utility. Enter y2 = the right-hand side. Entering the absolute value of an expression is found in the MATH menu, Num, 1:abs(. Find the points of intersection, recall (2nd CALC 5:intersection, 1st curve, enter, 2nd curve, enter, guess, enter). Copy a sketch of the graph and shade the *x*-axis for your solution set to the inequality. Write final answers in interval notation.

Exercise:

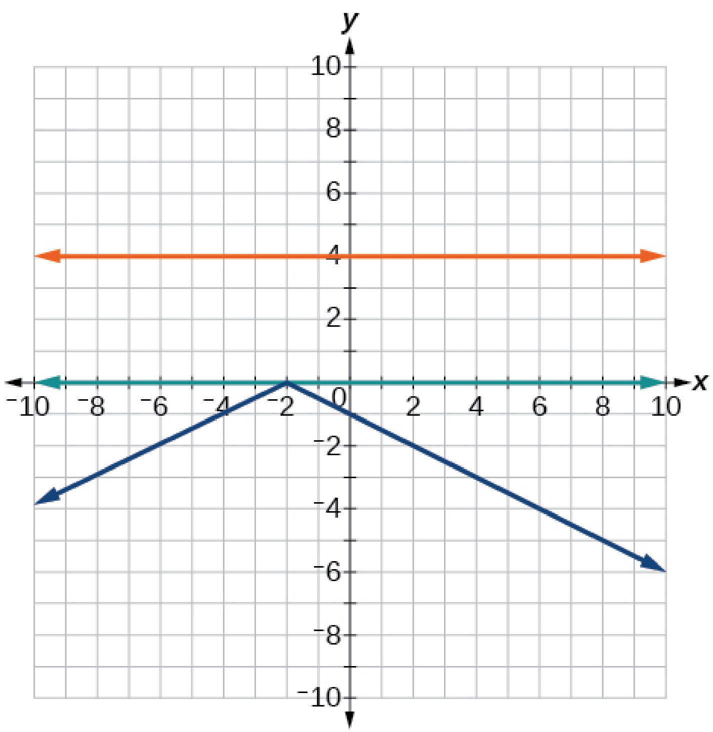
Problem: $|x+2|-5<2$

Exercise:

Problem: $\frac{-1}{2}|x+2|<4$

Solution:

Where the blue is below the orange; always. All real numbers. $(-\infty, +\infty)$.



Exercise:

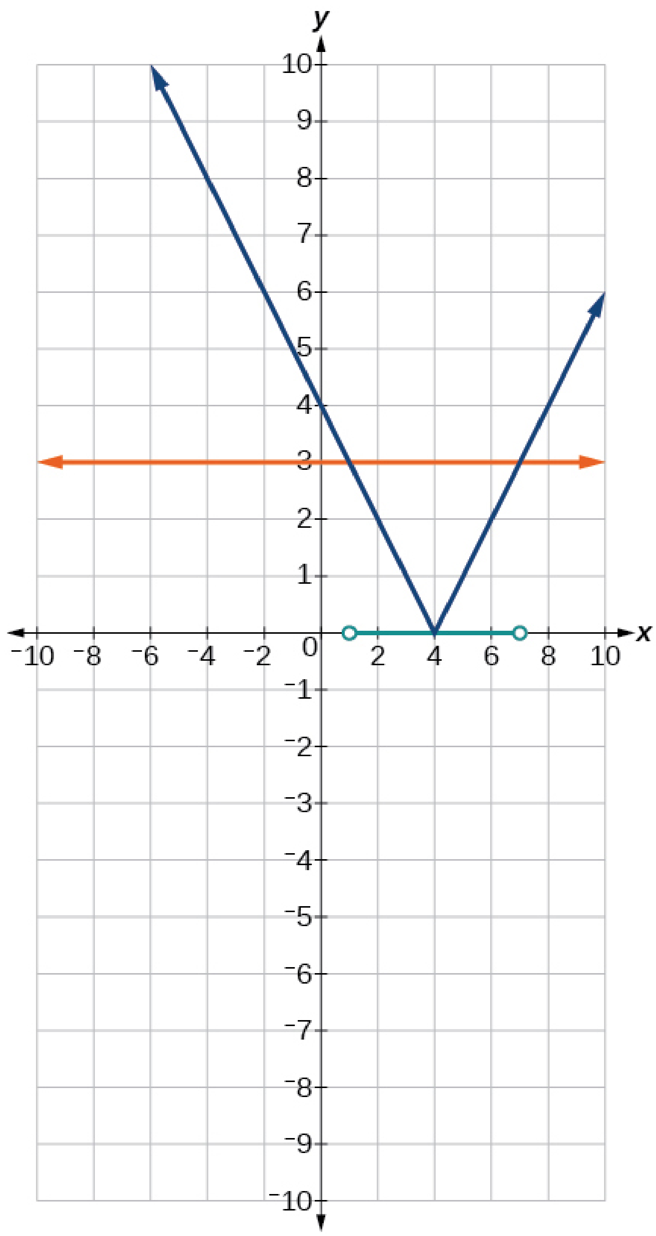
Problem: $|4x+1|-3>2$

Exercise:

Problem: $|x-4|<3$

Solution:

Where the blue is below the orange; $(1, 7)$.



Exercise:

Problem: $|x + 2| \geq 5$

Extensions

Exercise:

Problem: Solve $|3x + 1| = |2x + 3|$

Solution:

$x = 2, \frac{-4}{5}$

Exercise:

Problem: Solve $x^2 - x > 12$

Exercise:

Problem: $\frac{x-5}{x+7} \leq 0, x \neq -7$

Solution:

$(-7, 5]$

Exercise:

Problem:

$p = -x^2 + 130x - 3000$ is a profit formula for a small business. Find the set of *x*-values that will keep this profit positive.

Real-World Applications

Exercise:

Problem:

In chemistry the volume for a certain gas is given by $V = 20T$, where *V* is measured in cc and *T* is temperature in ºC. If the temperature varies between 80ºC and 120ºC, find the set of volume values.

Solution:

$80 \leq T \leq 120$
$1,600 \leq 20T \leq 2,400$

$[1,600,\ 2,400]$

Exercise:

Problem:

A basic cellular package costs $20/mo. for 60 min of calling, with an additional charge of $.30/min beyond that time.. The cost formula would be

$$C = \$20 + .30(x - 60).$$

If you have to keep your bill lower than $50, what is the maximum calling minutes you can use?

Glossary

compound inequality
: a problem or a statement that includes two inequalities

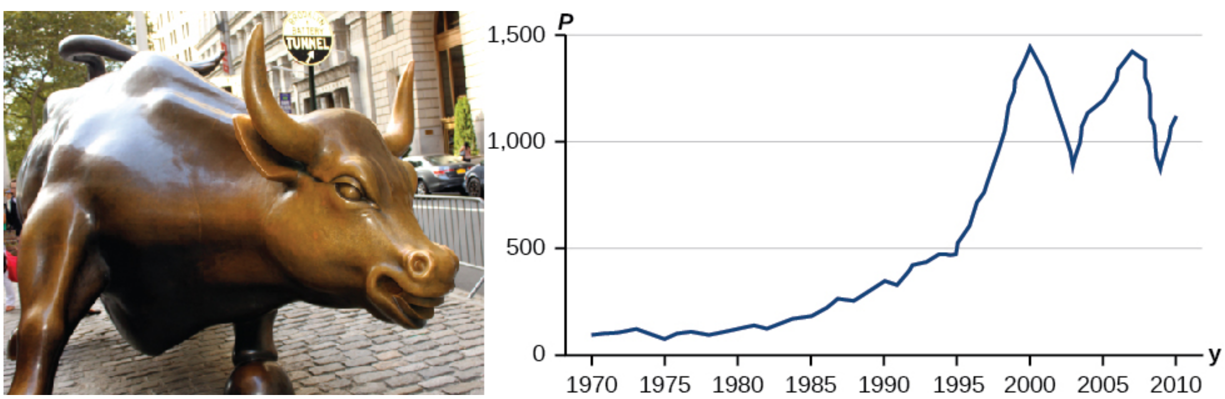
interval
: an interval describes a set of numbers within which a solution falls

interval notation
: a mathematical statement that describes a solution set and uses parentheses or brackets to indicate where an interval begins and ends

linear inequality
: similar to a linear equation except that the solutions will include sets of numbers

# Introduction to Functions
class="introduction"

Standard and Poor’s Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)



Toward the end of the twentieth century, the values of stocks of Internet and technology companies rose dramatically. As a result, the Standard and Poor’s stock market average rose as well. [link] tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1100 by the beginning of 2000. That five-year period became known as the “dot-com bubble” because so many Internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of

business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

Functions and Function Notation
In this section, you will:

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A **relation** is a set of ordered pairs. The set consisting of the first components of each ordered pair is called the **domain** and the set consisting of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.
Equation:

$$\{(1,\,2),\,(2,\,4),\,(3,\,6),\,(4,\,8),\,(5,\,10)\}$$

The domain is $\{1,\,2,\,3,\,4,\,5\}$. The range is $\{2,\,4,\,6,\,8,\,10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an **output** value, or **dependent variable**, and is often labeled lowercase letter y.

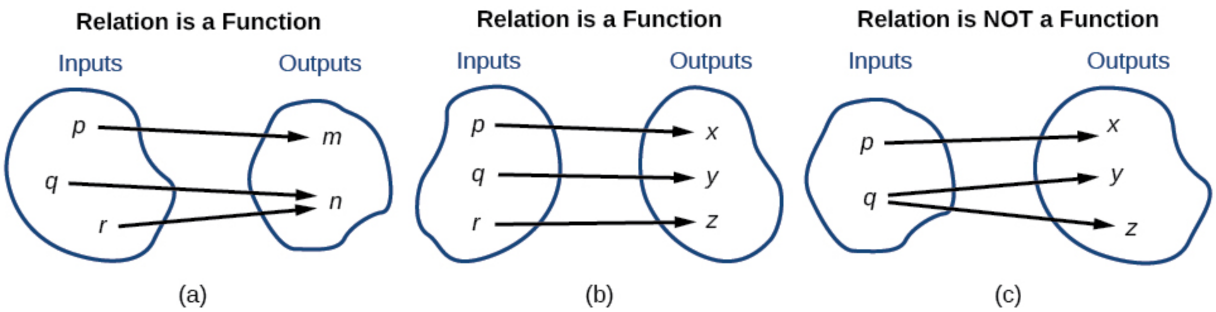
A function *f* is a relation that assigns a single element in the range to each element in the domain. In other words, no *x*-values are repeated. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, $\{1,\,2,\,3,\,4,\,5\}$, is paired with exactly one element in the range, $\{2,\,4,\,6,\,8,\,10\}$.

Now let’s consider the set of ordered pairs that relates the terms “even” and “odd” to the first five natural numbers. It would appear as
Equation:

$$\{(\text{odd},\,1),\,(\text{even},\,2),\,(\text{odd},\,3),\,(\text{even},\,4),\,(\text{odd},\,5)\}$$

Notice that each element in the domain, $\{\text{even}, \text{odd}\}$ is *not* paired with exactly one element in the range, $\{1,\,2,\,3,\,4,\,5\}$. For example, the term “odd” corresponds to three values from the domain, $\{1,\,3,\,5\}$ and the term “even” corresponds to two values from the range, $\{2,\,4\}$. This violates the definition of a function, so this relation is not a function.

[link] compares relations that are functions and not functions.



(a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Note:
Function
A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."
The **input** values make up the **domain**, and the **output** values make up the **range**.

Note:
Given a relationship between two quantities, determine whether the relationship is a function.

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example:
Exercise:

Problem:
Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in [link] consists of items and their prices.

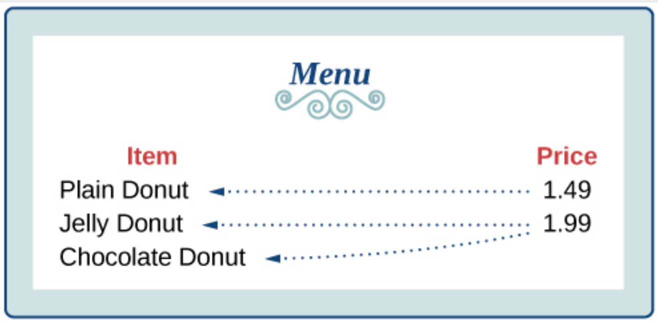
a. Is price a function of the item?
b. Is the item a function of the price?

Solution:

a. Let's begin by considering the input as the items on the menu. The output values are then the prices.

   Each item on the menu has only one price, so the price is a function of the item.

b. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See [link].



   Therefore, the item is a not a function of price.

Example:
Exercise:

Problem:
Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade-point average. Is grade-point average a function of the percent grade? Is the percent grade a function of the grade-point average? [link] shows a possible rule for assigning grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade-point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution:

For any percent grade earned, there is an associated grade-point average, so the grade-point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade-point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade-point average.

Note:
Exercise:

Problem:[link][footnote] lists the five greatest baseball players of all time in order of rank.
http://www.baseball-almanac.com/legendary/lisn100.shtml. Accessed 3/24/2014.

Player	**Rank**
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Solution:

a. yes; b. yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into graphing calculators and computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent “height is a function of age,” we start by identifying the descriptive variables h for height and a for age. The letters $f,\ g,$and h are often used to represent functions just as we use $x,\ y,$and$z$ to represent numbers and

A, B, and C to represent sets.
Equation:

$$\begin{array}{ll} h \text{ is } f \text{ of } a & \text{We name the function } f;\ \text{height is a function of age.} \\ h = f(a) & \text{We use parentheses to indicate the function input.} \\ f(a) & \text{We name the function } f;\ \text{the expression is read as "} f \text{ of } a\text{."} \end{array}$$

Remember, we can use any letter to name the function; the notation $h\left(a\right)$ shows us that h depends on $a.$ The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f\left(a+b\right)$ means "first add *a* and *b*, and the result is the input for the function *f*." The operations must be performed in this order to obtain the correct result.

Note:
Function Notation
The notation $y=f\left(x\right)$ defines a function named $f.$ This is read as "y is a function of $x.$" The letter x represents the input value, or independent variable. The letter y, or $f\left(x\right),$ represents the output value, or dependent variable.

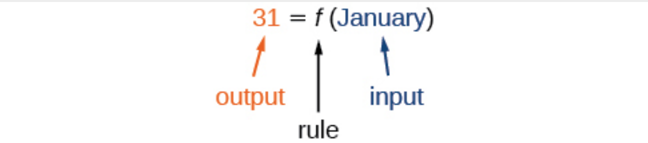
Example:
Exercise:

Problem:
Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

Solution:

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days}=f(\text{month})$ or $d=f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.



For example, $f\left(\text{March}\right)=31,$ because March has 31 days. The notation $d=f\left(m\right)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example:

Exercise:

Problem:
Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, $N,$ in a town in year $y.$ What does $f(2005) = 300$ represent?

Solution:

When we read $f(2005) = 300,$ we see that the input year is 2005. The value for the output, the number of police officers $(N),$ is 300. Remember, $N = f(y).$ The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Note:
Exercise:

Problem: Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Solution:

$w = f(d)$

Note:
Instead of a notation such as $y = f(x),$ could we use the same symbol for the output as for the function, such as $y = y(x),$ meaning "y is a function of x?"
Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x), P = W(d),$ and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

[link] lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Month number, m **(input)**	1	2	3	4	5	6	7	8	9	10	11	12

Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

[link] defines a function $Q = g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q .

n	1	2	3	4	5
Q	8	6	7	6	8

[link] displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Note:
Given a table of input and output values, determine whether the table represents a function.

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example:
Exercise:

Problem:
Identifying Tables that Represent Functions

Which table, [link], [link], or [link], represents a function (if any)?

Input	**Output**
2	1

Input	Output
5	3
8	6

Input	Output
–3	5
0	1
4	5

Input	Output
1	0
5	2
5	4

Solution:

[link] and [link] define functions. In both, each input value corresponds to exactly one output value. [link] does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by [link] can be represented by writing
Equation:

$$f(2)=1,\, f(5)=3, \text{and } f(8)=6$$

Similarly, the statements
Equation:

$$g\left(-3\right)=5,\; g\left(0\right)=1, \text{and } g\left(4\right)=5$$

represent the function in [link].

[link] cannot be expressed in a similar way because it does not represent a function.

Note:
Exercise:

Problem: Does [link] represent a function?

Input	Output
1	10
2	100
3	1000

Solution:

yes

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f\left(x\right)=5-3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Note:
Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example:
Exercise:

Problem:
Evaluating Functions at Specific Values

Evaluate $f\left(x\right)=x^2+3x-4$ at

a. 2
b. a
c. $a+h$
d. $\frac{f(a+h)-f(a)}{h}$

Solution:

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.
Equation:

$$\begin{array}{ccc}f(2) & = & 2^2+3(2)-4 \\ & = & 4+6-4 \\ & = & 6\end{array}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.
Equation:

$$f\left(a\right)=a^2+3a-4$$

c. With an input value of $a+h,$ we must use the distributive property.
Equation:

$$\begin{array}{ccc}f(a+h) & = & (a+h)^2+3(a+h)-4 \\ & = & a^2+2ah+h^2+3a+3h-4\end{array}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that
Equation:

$$f\left(a+h\right)=a^2+2ah+h^2+3a+3h-4$$

and we know that
Equation:

$$f\left(a\right)=a^2+3a-4$$

Now we combine the results and simplify.
Equation:

$$\begin{array}{rcll} \frac{f(a+h)-f(a)}{h} & = & \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} & \\ & = & \frac{2ah+h^2+3h}{h} & \\ & = & \frac{h(2a+h+3)}{h} & \text{Factor out } h. \\ & = & 2a+h+3 & \text{Simplify.} \end{array}$$

Example:
Exercise:

Problem:
Evaluating Functions

Given the function $h(p) = p^2 + 2p,$ evaluate $h(4).$

Solution:

To evaluate $h(4),$ we substitute the value 4 for the input variable p in the given function.
Equation:

$$\begin{array}{rcl} h(p) & = & p^2 + 2p \\ h(4) & = & (4)^2 + 2(4) \\ & = & 16 + 8 \\ & = & 24 \end{array}$$

Therefore, for an input of 4, we have an output of 24.

Note:
Exercise:

Problem: Given the function $g(m) = \sqrt{m-4},$ evaluate $g(5).$

Solution:

$g(5) = 1$

Example:
Exercise:

Problem:
Solving Functions

Given the function $h(p) = p^2 + 2p,$ solve for $h(p) = 3.$

Solution:
Equation:

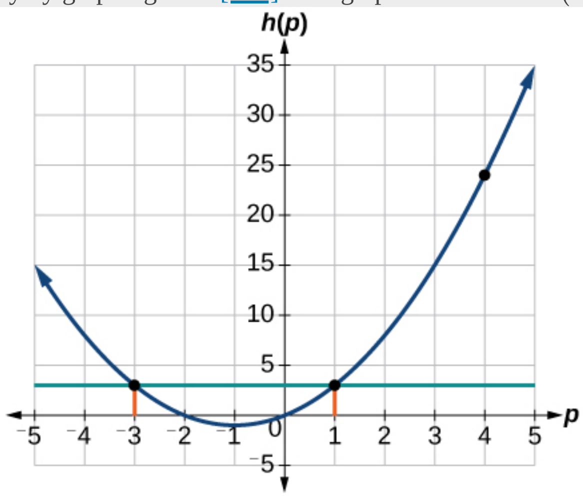
$$\begin{aligned} h(p) &= 3 \\ p^2+2p &= 3 && \text{Substitute the original function } h(p)=p^2+2p. \\ p^2+2p-3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.} \end{aligned}$$

If $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

Equation:

$$\begin{aligned} (p+3) &= 0, & p &= -3 \\ (p-1) &= 0, & p &= 1 \end{aligned}$$

This gives us two solutions. The output $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in [link]. The graph verifies that $h(1)=h(-3)=3$ and $h(4)=24$.



p	−3	−2	0	1	4
$h(p)$	3	0	0	3	24

Note:
Exercise:

Problem: Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Solution:

$m=8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic

form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

Note:
Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves *only* the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example:
Exercise:

Problem:
Finding an Equation of a Function

Express the relationship $2n+6p=12$ as a function $p=f\left(n\right),$ if possible.

Solution:

To express the relationship in this form, we need to be able to write the relationship where p is a function of $n,$ which means writing it as $p=\left[\text{expression involving }n\right].$
Equation:

$$\begin{array}{rcll} 2n+6p & = & 12 & \\ 6p & = & 12-2n & \text{Subtract }2n\text{ from both sides.} \\ p & = & \frac{12-2n}{6} & \text{Divide both sides by 6 and simplify.} \\ p & = & \frac{12}{6}-\frac{2n}{6} & \\ p & = & 2-\frac{1}{3}n & \end{array}$$

Therefore, p as a function of n is written as
Equation:

$$p=f\left(n\right)=2-\frac{1}{3}n$$

Example:
Exercise:

Problem:
Expressing the Equation of a Circle as a Function

Does the equation $x^2+y^2=1$ represent a function with x as input and y as output? If so, express the relationship as a function $y=f\left(x\right).$

Solution:

First we subtract x^2 from both sides.
Equation:

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation.
Equation:

$$\begin{aligned} y &= \pm\sqrt{1-x^2} \\ &= +\sqrt{1-x^2} \text{ and } -\sqrt{1-x^2} \end{aligned}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$. If we graph both functions on a graphing calculator, we will get the upper and lower semicircles.

Note:
Exercise:

Problem: If $x - 8y^3 = 0$, express y as a function of x.

Solution:

$y = f(x) = \frac{\sqrt[3]{x}}{2}$

Note:
Are there relationships expressed by an equation that do represent a function but that still cannot be represented by an algebraic formula?
Yes, this can happen. For example, given the equation $x = y + 2^y$, *if we want to express* y *as a function of* x, *there is no simple algebraic formula involving only* x *that equals* y. *However, each* x *does determine a unique value for* y, *and there are mathematical procedures by which* y *can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for* y *as a function of* x, *even though the formula cannot be written explicitly.*

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table. See [link].[footnote]
http://www.kgbanswers.com/how-long-is-a-dogs-memory-span/4221590. Accessed 3/24/2014.

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

Note:
Given a function represented by a table, identify specific output and input values.

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example:
Exercise:

Problem:
Evaluating and Solving a Tabular Function

Using [link],

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution:

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. [link] shows two solutions: 2 and 4.

n	1	**2**	3	**4**	5
$g\left(n\right)$	8	**6**	7	**6**	8

When we input 2 into the function $g,$ our output is 6. When we input 4 into the function $g,$ our output is also 6.

Note:
Exercise:

Problem: Using [link], evaluate $g\left(1\right).$

Solution:

$g\left(1\right)=8$

Finding Function Values from a Graph

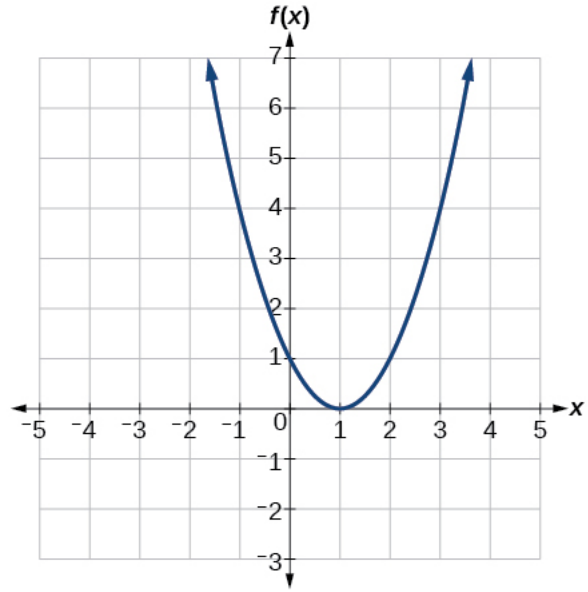
Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example:
Exercise:

Problem:
Reading Function Values from a Graph

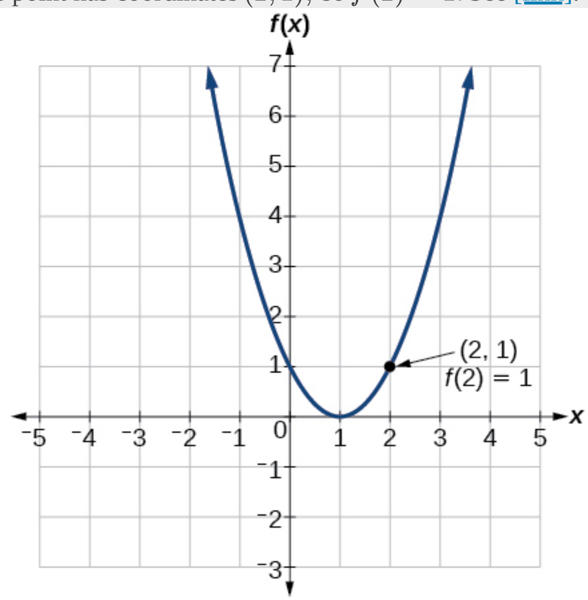
Given the graph in [link],

a. Evaluate $f\left(2\right).$
b. Solve $f\left(x\right)=4.$

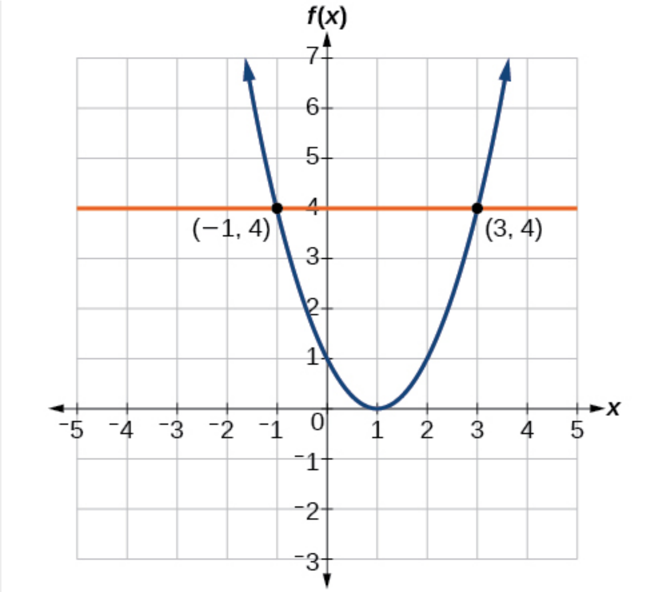


Solution:

a. To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See [link].



b. To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4. See [link].



Note:
Exercise:

Problem: Using [link], solve $f\left(x\right)=1.$

Solution:

$x=0\text{ or }x=2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in [link] at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in [link].

Letter grade	Grade point average
A	4.0
B	3.0

Letter grade	Grade point average
C	2.0
D	1.0

This grading system represents a one-to-one function because each letter input yields one particular grade-point average output and each grade-point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in [link]**(a)** and [link]**(b)**. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

Note:
One-to-One Function
A **one-to-one function** is a function in which each output value corresponds to exactly one input value. There are no repeated *x*- or *y*-values.

Example:
Exercise:

Problem:
Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution:

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Note:
Exercise:

Problem:

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Solution:

a. yes, because each bank account has a single balance at any given time; b. no, because several bank account numbers may have the same balance; c. no, because the same output may correspond to more than one input.

Note:
Exercise:

Problem:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

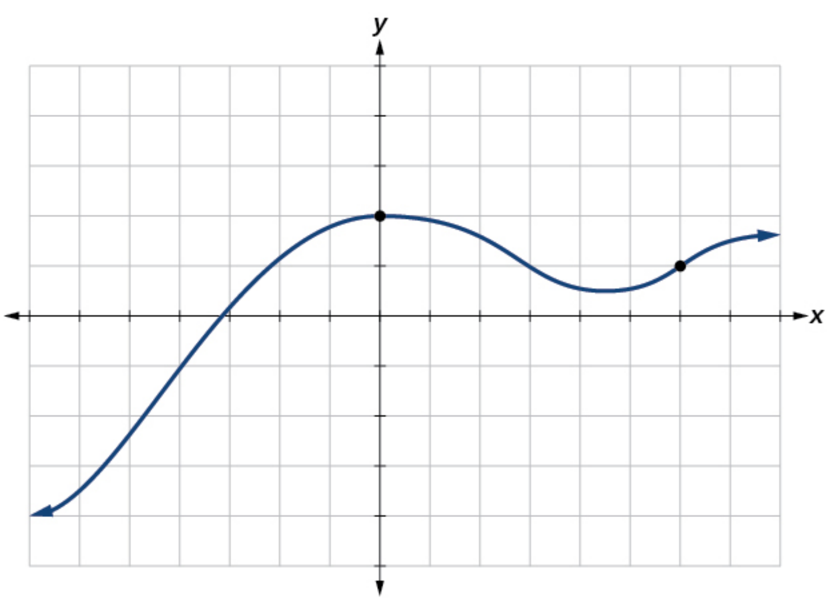
Solution:

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

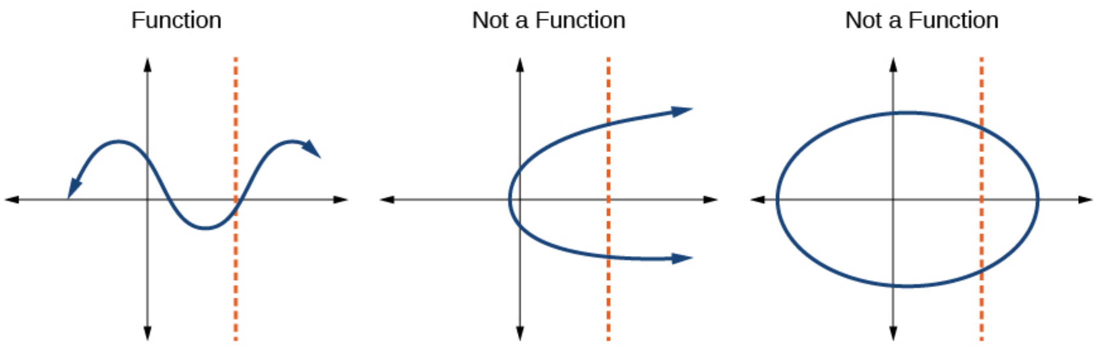
Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value $y,$ and we say y is a function of $x,$ or $y=f\left(x\right)$ when the function is named $f.$ The graph of the function is the set of all points $\left(x,y\right)$ in the plane that satisfies the equation$y=f\left(x\right).$ If the function is defined for only a few input values, then the graph of the function consists of only a few points, where the *x*-coordinate of each point is an input value and the *y*-coordinate of each point is the corresponding output value. For example, the black dots on the graph in [link] tell us that $f\left(0\right)=2$ and $f\left(6\right)=1.$ However, the set of all points $\left(x,y\right)$ satisfying $y=f\left(x\right)$ is a curve. The curve shown includes $\left(0,2\right)$ and $\left(6,1\right)$ because the curve passes through those points.



The **vertical line test** can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does *not* define a function because a function has only one output value for each input value. See [link].



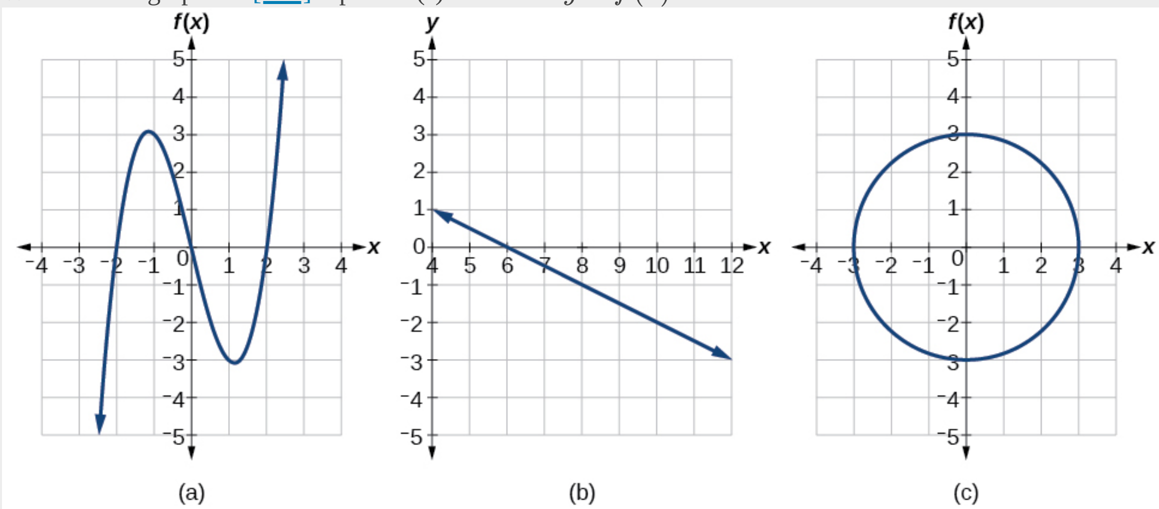
Note:
Given a graph, use the vertical line test to determine if the graph represents a function.

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example:
Exercise:

Problem:
Applying the Vertical Line Test

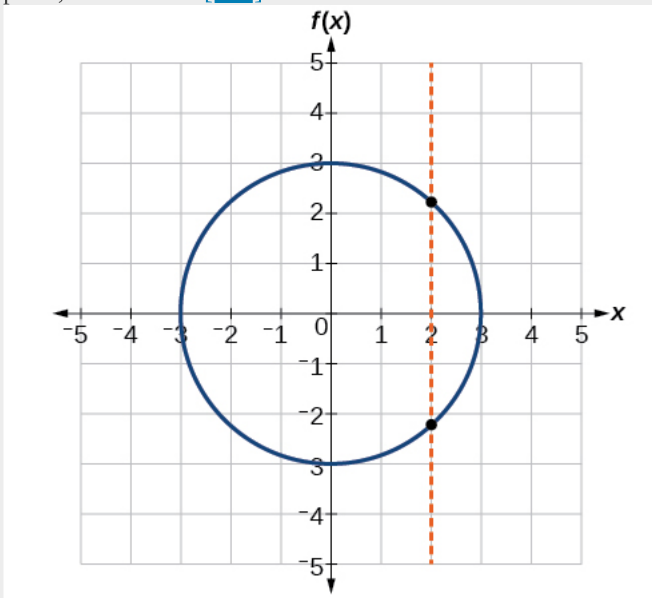
Which of the graphs in [link] represent(s) a function $y = f(x)$?



Solution:

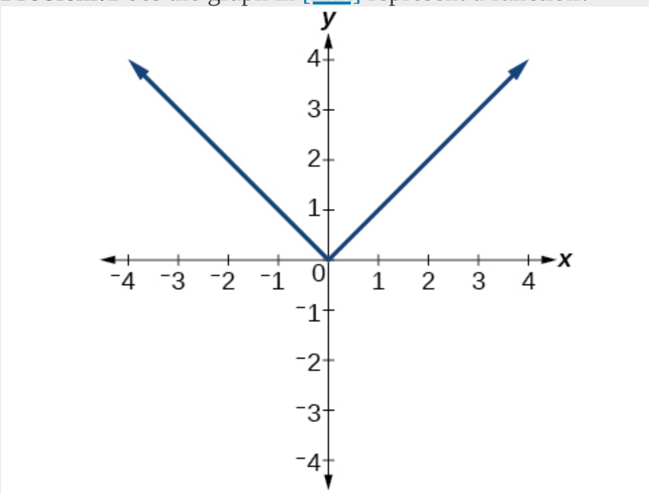
If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b)

of [link]. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in [link].



Note:
Exercise:

Problem: Does the graph in [link] represent a function?



Solution:

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Note:
Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

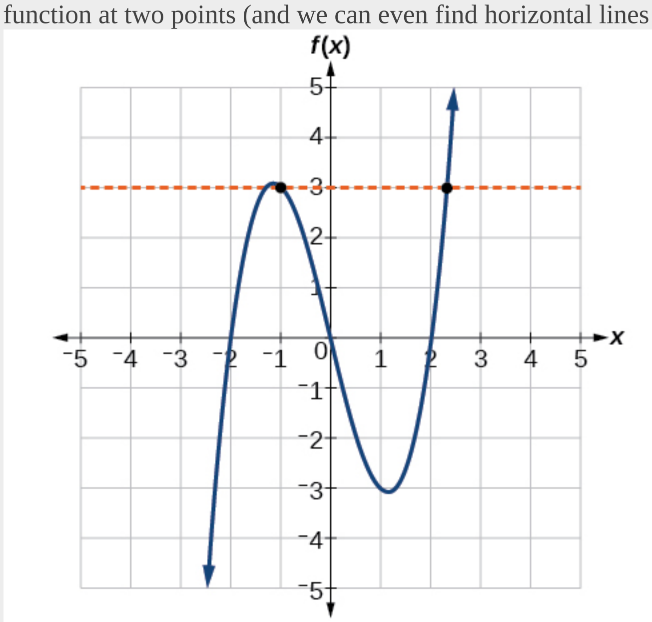
Example:
Exercise:

Problem:
Applying the Horizontal Line Test

Consider the functions shown in [link]**(a)** and [link]**(b)**. Are either of the functions one-to-one?

Solution:

The function in [link]**(a)** is not one-to-one. The horizontal line shown in [link] intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)



The function in [link]**(b)** is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Note:
Exercise:

Problem: Is the graph shown in [link] one-to-one?

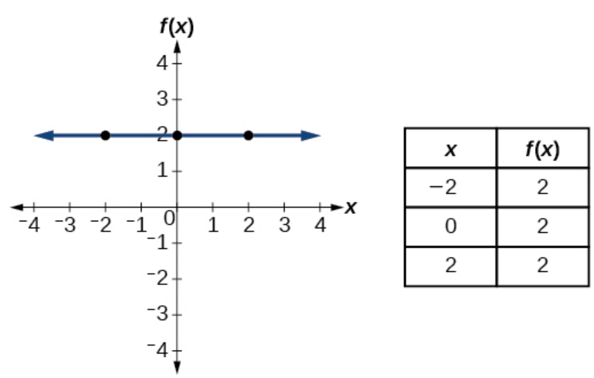
Solution:

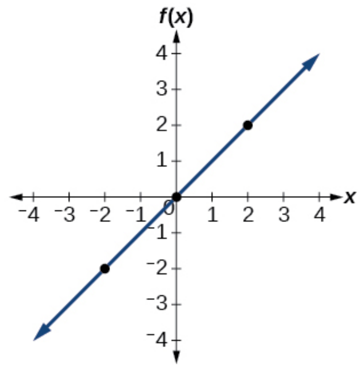
No, because it does not pass the horizontal line test.

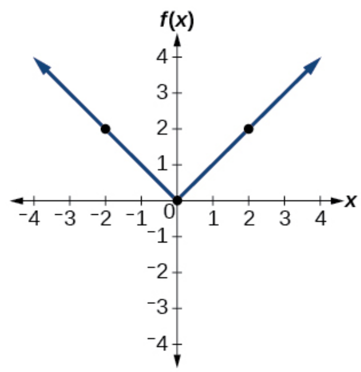
Identifying Basic Toolkit Functions

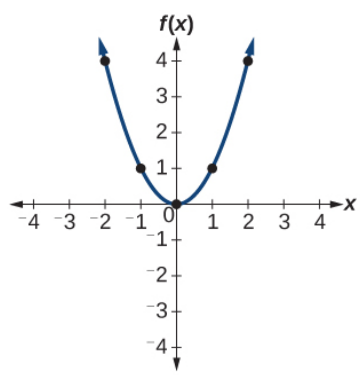
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in [link].

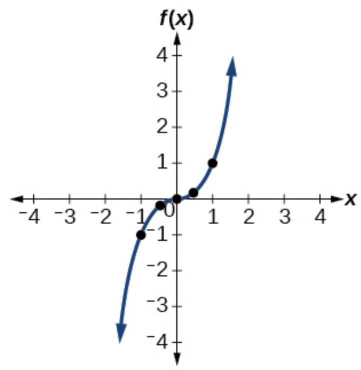
<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Constant</td><td>$f(x) = c$, where c is a constant</td><td>
</td></tr>
</table>

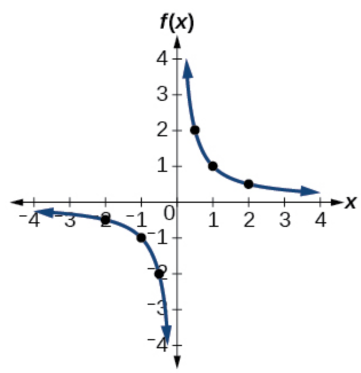
<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Identity</td><td>$f\left(x\right)=x$</td><td>

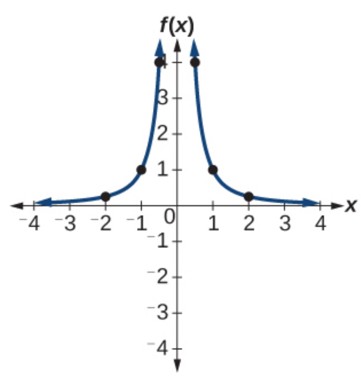
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Absolute value</td><td>$f\left(x\right)=\left|x\right|$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Quadratic</td><td>$f\left(x\right)=x^2$</td><td>

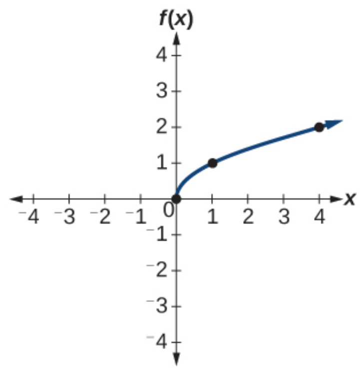
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>4</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table></td></tr>
</table>

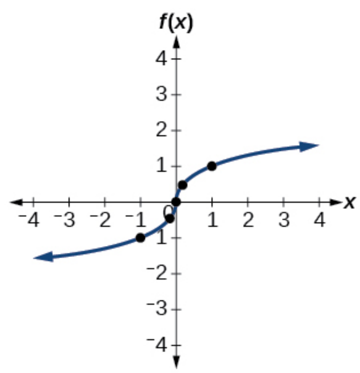
<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Cubic</td><td>$f\left(x\right)=x^3$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
<tr><td>Reciprocal</td><td>$f\left(x\right)=\frac{1}{x}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table></td></tr>
<tr><td>Reciprocal squared</td><td>$f\left(x\right)=\frac{1}{x^2}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>0.25</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>−0.5</td><td>4</td></tr>
<tr><td>0.5</td><td>4</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.25</td></tr>
</table></td></tr>
</table>

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Square root</td><td>$f\left(x\right)=\sqrt{x}$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table></td></tr>
<tr><td>Cube root</td><td>$f\left(x\right)=\sqrt[3]{x}$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.125</td><td>−0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table></td></tr>
</table>

Note:
Access the following online resources for additional instruction and practice with functions.

- Determine if a Relation is a Function
- Vertical Line Test
- Introduction to Functions
- Vertical Line Test on Graph
- One-to-one Functions
- Graphs as One-to-one Functions

Key Equations

Constant function	$f\left(x\right)=c$,where c is a constant
Identity function	$f\left(x\right)=x$
Absolute value function	$f\left(x\right)=\left\|x\right\|$
Quadratic function	$f\left(x\right)={x}^{2}$
Cubic function	$f\left(x\right)={x}^{3}$
Reciprocal function	$f\left(x\right)=\frac{1}{x}$
Reciprocal squared function	$f\left(x\right)=\frac{1}{{x}^{2}}$
Square root function	$f\left(x\right)=\sqrt{x}$
Cube root function	$f\left(x\right)=\sqrt[3]{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output. See [link] and [link].
- Function notation is a shorthand method for relating the input to the output in the form $y=f\left(x\right).$ See [link] and [link].
- In tabular form, a function can be represented by rows or columns that relate to input and output values. See [link].
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value. See [link] and [link].
- To solve for a specific function value, we determine the input values that yield the specific output value. See [link].
- An algebraic form of a function can be written from an equation. See [link] and [link].
- Input and output values of a function can be identified from a table. See [link].
- Relating input values to output values on a graph is another way to evaluate a function. See [link].
- A function is one-to-one if each output value corresponds to only one input value. See [link].
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point. See [link].
- The graph of a one-to-one function passes the horizontal line test. See [link].

Section Exercises

Verbal

Exercise:

Problem: What is the difference between a relation and a function?

Solution:

A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

Exercise:

Problem: What is the difference between the input and the output of a function?

Exercise:

Problem: Why does the vertical line test tell us whether the graph of a relation represents a function?

Solution:

When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

Exercise:

Problem: How can you determine if a relation is a one-to-one function?

Exercise:

Problem: Why does the horizontal line test tell us whether the graph of a function is one-to-one?

Solution:

When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

Algebraic

For the following exercises, determine whether the relation represents a function.

Exercise:

Problem: $\{(a,b),\ \ (c,d),\ \ (a,c)\}$

Exercise:

Problem: $\{(a,b),(b,c),(c,c)\}$

Solution:

function

For the following exercises, determine whether the relation represents y as a function of x.

Exercise:

Problem: $5x+2y=10$

Exercise:

Problem: $y=x^2$

Solution:

function

Exercise:

Problem: $x = y^2$

Exercise:

Problem: $3x^2 + y = 14$

Solution:

function

Exercise:

Problem: $2x + y^2 = 6$

Exercise:

Problem: $y = -2x^2 + 40x$

Solution:

function

Exercise:

Problem: $y = \frac{1}{x}$

Exercise:

Problem: $x = \frac{3y+5}{7y-1}$

Solution:

function

Exercise:

Problem: $x = \sqrt{1 - y^2}$

Exercise:

Problem: $y = \frac{3x+5}{7x-1}$

Solution:

function

Exercise:

Problem: $x^2 + y^2 = 9$

Exercise:

Problem: $2xy = 1$

Solution:

function

Exercise:

Problem: $x = y^3$

Exercise:

Problem: $y = x^3$

Solution:

function

Exercise:

Problem: $y = \sqrt{1 - x^2}$

Exercise:

Problem: $x = \pm\sqrt{1 - y}$

Solution:

function

Exercise:

Problem: $y = \pm\sqrt{1 - x}$

Exercise:

Problem: $y^2 = x^2$

Solution:

not a function

Exercise:

Problem: $y^3 = x^2$

For the following exercises, evaluate the function f at the indicated values $f(-3), f(2), f(-a), -f(a), f(a + h)$.

Exercise:

Problem: $f(x) = 2x - 5$

Solution:

$f(-3) = -11; \quad f(2) = -1; \quad f(-a) = -2a - 5; \quad -f(a) = -2a + 5; \quad f(a + h) = 2a + 2h - 5$

Exercise:

Problem: $f(x) = -5x^2 + 2x - 1$

Exercise:

Problem: $f(x) = \sqrt{2-x}+5$

Solution:

$f(-3) = \sqrt{5}+5;\quad f(2) = 5;\quad f(-a) = \sqrt{2+a}+5;\quad -f(a) = -\sqrt{2-a}-5;\quad f(a+h) = \sqrt{2-a-h}+5$

Exercise:

Problem: $f(x) = \frac{6x-1}{5x+2}$

Exercise:

Problem: $f(x) = |x-1| - |x+1|$

Solution:

$f(-3) = 2;\quad f(2) = 1-3 = -2;\quad f(-a) = |-a-1| - |-a+1|;\quad -f(a) = -|a-1| + |a+1|;\quad f(a$ -

Exercise:

Problem: Given the function $g(x) = 5-x^2$, simplify $\frac{g(x+h)-g(x)}{h}$, $h \neq 0$.

Exercise:

Problem: Given the function $g(x) = x^2+2x$, simplify $\frac{g(x)-g(a)}{x-a}$, $x \neq a$.

Solution:

$\frac{g(x)-g(a)}{x-a} = x+a+2,\ x \neq a$

Exercise:

Problem: Given the function $k(t) = 2t-1$:

a. Evaluate $k(2)$.
b. Solve $k(t) = 7$.

Exercise:

Problem: Given the function $f(x) = 8-3x$:

a. Evaluate $f(-2)$.
b. Solve $f(x) = -1$.

Solution:

a. $f(-2) = 14$; b. $x = 3$

Exercise:

Problem: Given the function $p(c) = c^2+c$:

a. Evaluate $p(-3)$.
b. Solve $p(c) = 2$.

Exercise:

Problem: Given the function $f(x) = x^2 - 3x$:

a. Evaluate $f(5)$.
b. Solve $f(x) = 4$.

Solution:

a. $f(5) = 10$; b. $x = -1$ or $x = 4$

Exercise:

Problem: Given the function $f(x) = \sqrt{x+2}$:

a. Evaluate $f(7)$.
b. Solve $f(x) = 4$.

Exercise:

Problem: Consider the relationship $3r + 2t = 18$.

a. Write the relationship as a function $r = f(t)$.
b. Evaluate $f(-3)$.
c. Solve $f(t) = 2$.

Solution:

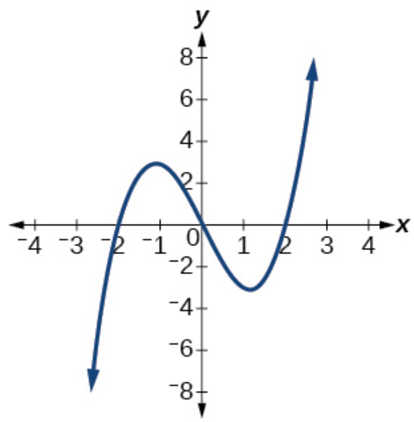
a. $f(t) = 6 - \frac{2}{3}t$; b. $f(-3) = 8$; c. $t = 6$

Graphical

For the following exercises, use the vertical line test to determine which graphs show relations that are functions.

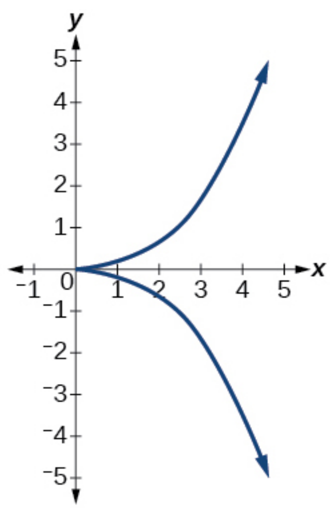
Exercise:

Problem:



Exercise:

Problem:

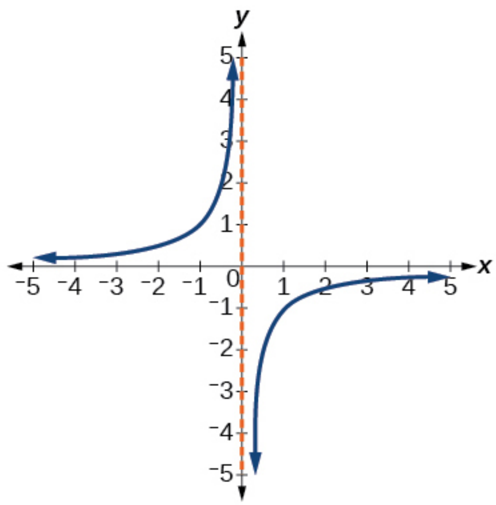


Solution:

not a function

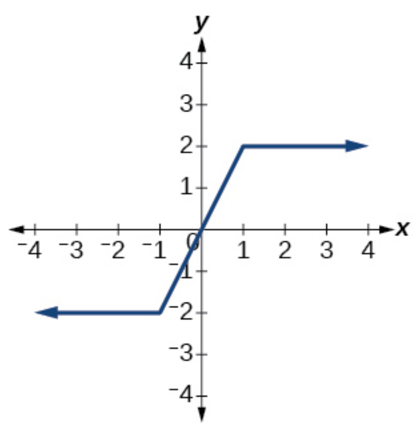
Exercise:

Problem:



Exercise:

Problem:

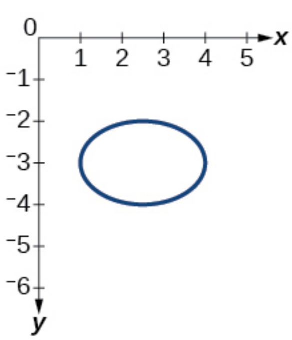


Solution:

function

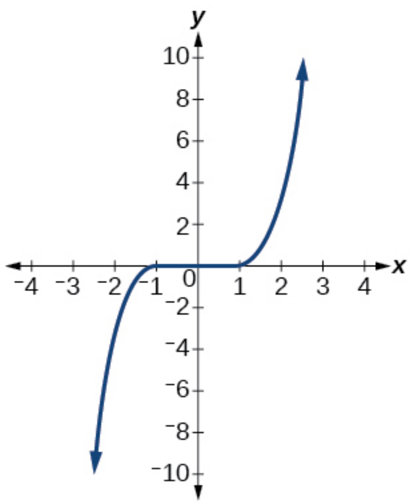
Exercise:

Problem:



Exercise:

Problem:

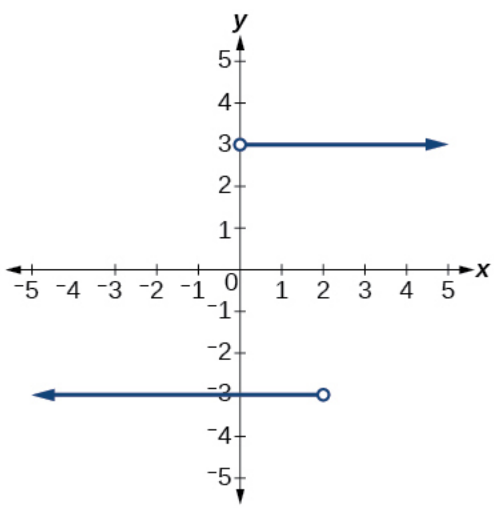


Solution:

function

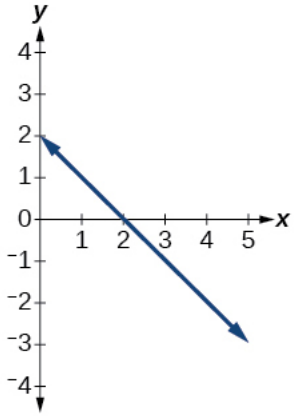
Exercise:

Problem:



Exercise:

Problem:

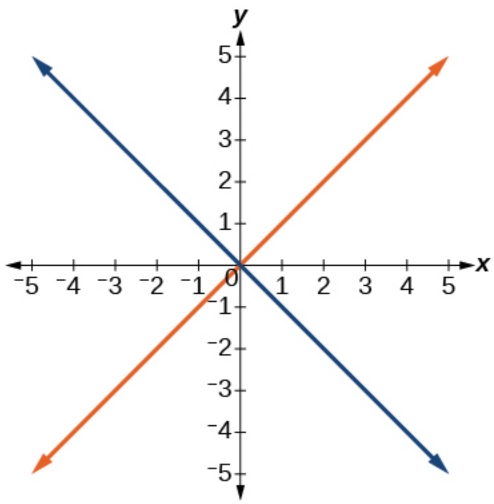


Solution:

function

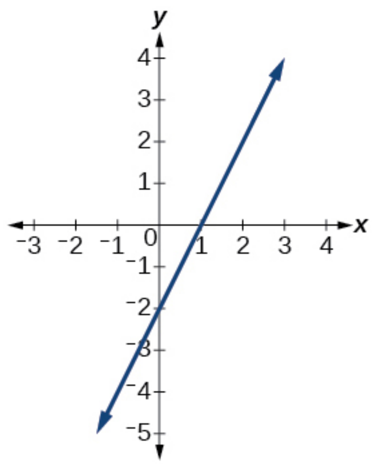
Exercise:

Problem:



Exercise:

Problem:

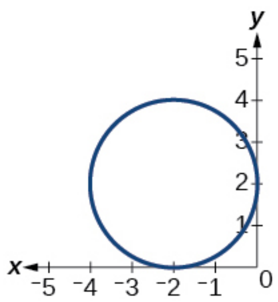


Solution:

function

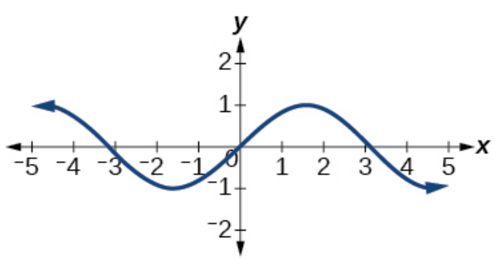
Exercise:

Problem:



Exercise:

Problem:



Solution:

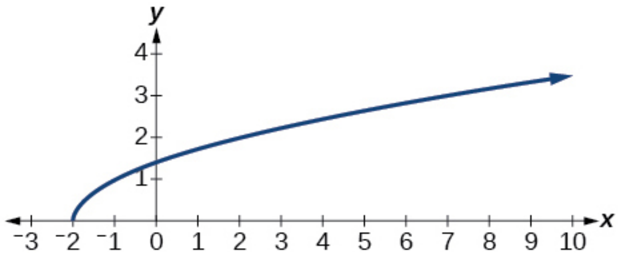
function

Exercise:

Problem: Given the following graph,

- Evaluate $f(-1)$.

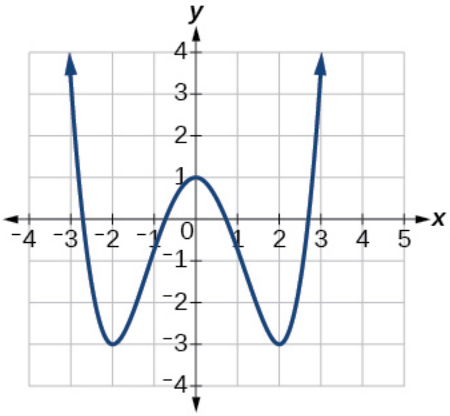
- Solve for $f(x) = 3$.



Exercise:

Problem: Given the following graph,

- Evaluate $f(0)$.
- Solve for $f(x) = -3$.



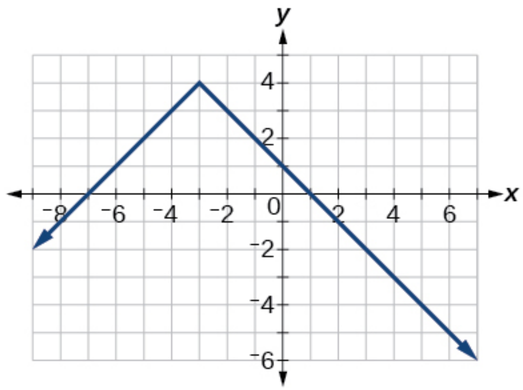
Solution:

a. $f(0) = 1$; b. $f(x) = -3$, $x = -2$ or $x = 2$

Exercise:

Problem: Given the following graph,

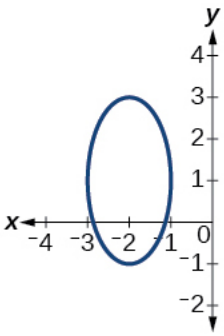
- Evaluate $f(4)$.
- Solve for $f(x) = 1$.



For the following exercises, determine if the given graph is a one-to-one function.

Exercise:

Problem:

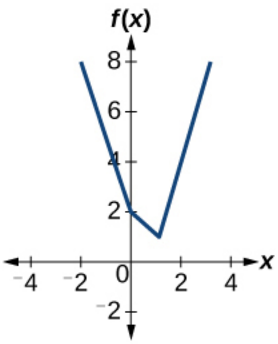


Solution:

not a function so it is also not a one-to-one function

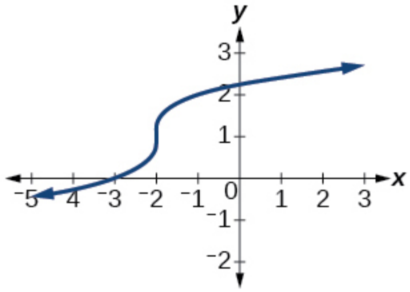
Exercise:

Problem:



Exercise:

Problem:

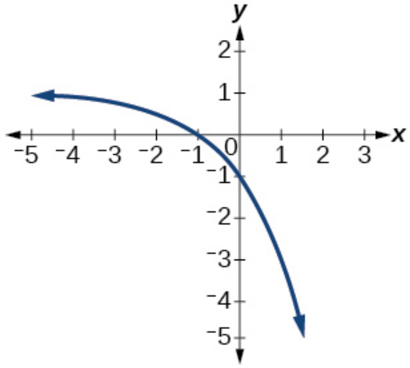


Solution:

one-to- one function

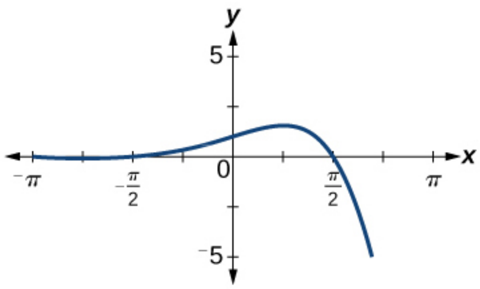
Exercise:

Problem:



Exercise:

Problem:



Solution:

function, but not one-to-one

Numeric

For the following exercises, determine whether the relation represents a function.

Exercise:

Problem: $\{(-1,-1),(-2,-2),(-3,-3)\}$

Exercise:

Problem: $\{(3,4),(4,5),(5,6)\}$

Solution:

function

Exercise:

Problem: $\{(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

Exercise:

Problem:

x	5	10	15
y	3	8	14

Solution:

function

Exercise:

Problem:

x	5	10	15
y	3	8	8

Exercise:

Problem:

x	5	10	10

y	3	8	14

Solution:

not a function

For the following exercises, use the function f represented in [link].

x	$f\left(x\right)$
0	74
1	28
2	1
3	53
4	56
5	3
6	36
7	45
8	14
9	47

Exercise:

Problem: Evaluate $f(3)$.

Exercise:

Problem: Solve $f(x)=1$.

Solution:

$f(x)=1,\; x=2$

For the following exercises, evaluate the function f at the values $f\left(-2\right),\; f(-1),\; f(0),\; f(1)$,and $f(2)$.

Exercise:

Problem: $f\left(x\right)=4-2x$

Exercise:

Problem: $f\left(x\right)=8-3x$

Solution:

$f(-2)=14;\quad f(-1)=11;\quad f(0)=8;\quad f(1)=5;\quad f(2)=2$

Exercise:

Problem: $f\left(x\right)=8{x}^{2}-7x+3$

Exercise:

Problem: $f\left(x\right)=3+\sqrt{x+3}$

Solution:

$f(-2)=4;\qquad f(-1)=4.414;\quad f(0)=4.732;\quad f(1)=5;\quad f(2)=5.236$

Exercise:

Problem: $f(x)=\frac{x-2}{x+3}$

Exercise:

Problem: $f\left(x\right)={3}^{x}$

Solution:

$f(-2)=\frac{1}{9};\quad f(-1)=\frac{1}{3};\quad f(0)=1;\quad f(1)=3;\quad f(2)=9$

For the following exercises, evaluate the expressions, given functions $f,\ g,$and h:

- $f(x)=3x-2$
- $g(x)=5-{x}^{2}$
- $h(x)=-2{x}^{2}+3x-1$

Exercise:

Problem: $3f\left(1\right)-4g\left(-2\right)$

Exercise:

Problem: $f\left(\frac{7}{3}\right)-h\left(-2\right)$

Solution:

20

Technology

For the following exercises, graph $y={x}^{2}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

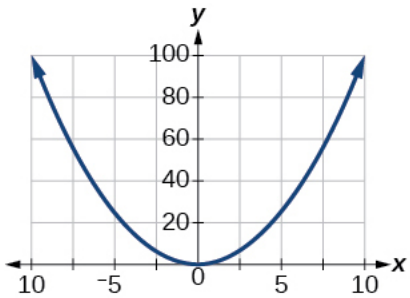
Problem: $[-0.1,\ 0.1]$

Exercise:

Problem:$[-10,\ 10]$

Solution:

$[0,\ 100]$



Exercise:

Problem: $[-100, 100]$

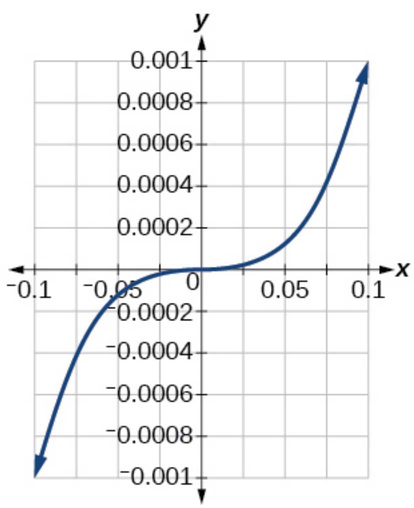
For the following exercises, graph $y = x^3$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

Problem: $[-0.1,\ 0.1]$

Solution:

$[-0.001,\ 0.001]$



Exercise:

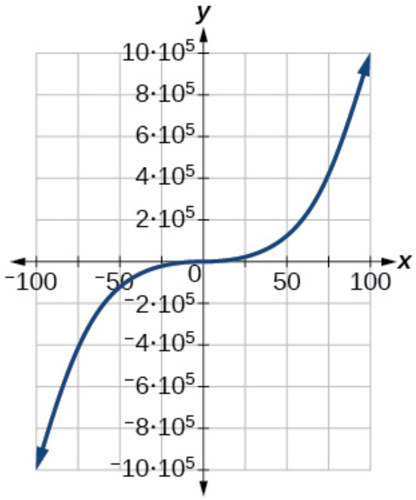
Problem: $[-10,\ 10]$

Exercise:

Problem:$[-100,\ 100]$

Solution:

$[-1,000,000,\ 1{,}000{,}000]$



For the following exercises, graph $y=\sqrt{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

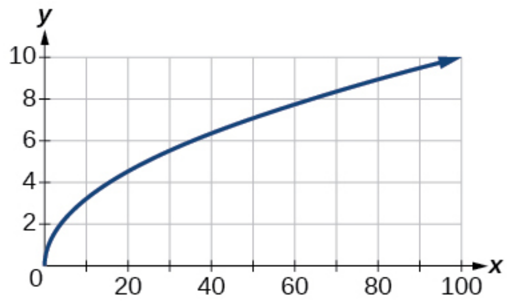
Problem: $[0,\ 0.01]$

Exercise:

Problem: $[0,\ 100]$

Solution:

$[0,\ 10]$



Exercise:

Problem: $[0,\ 10{,}000]$

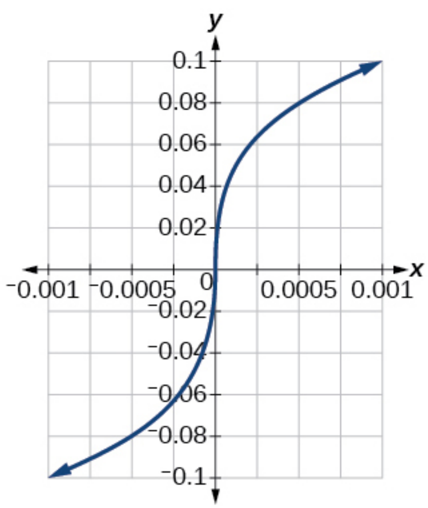
For the following exercises, graph $y=\sqrt[3]{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

Problem: $[-0.001, 0.001]$

Solution:

$[-0.1, 0.1]$



Exercise:

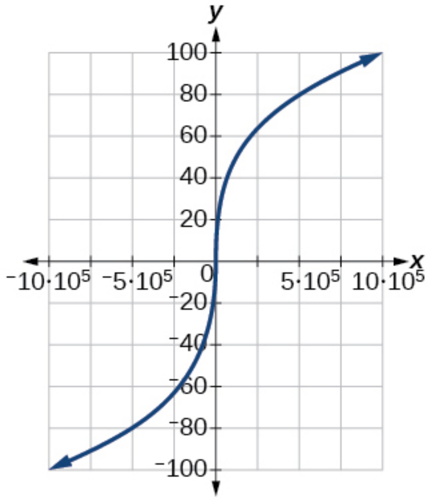
Problem: $[-1000, 1000]$

Exercise:

Problem: $[-1{,}000{,}000, 1{,}000{,}000]$

Solution:

$[-100,\ 100]$



Real-World Applications

Exercise:

Problem:

The amount of garbage, $G,$ produced by a city with population p is given by $G = f\left(p\right).$ G is measured in tons per week, and p is measured in thousands of people.

a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function $f.$
b. Explain the meaning of the statement $f\left(5\right) = 2.$

Exercise:

Problem:

The number of cubic yards of dirt, $D,$ needed to cover a garden with area a square feet is given by $D = g\left(a\right).$

a. A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function $g.$
b. Explain the meaning of the statement $g\left(100\right) = 1.$

Solution:

a. $g(5000) = 50$; b. The number of cubic yards of dirt required for a garden of 100 square feet is 1.

Exercise:

Problem:

Let $f\left(t\right)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

a. $f\left(5\right) = 30$
b. $f\left(10\right) = 40$

Exercise:

Problem:

Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:

a. $h(1) = 200$
b. $h(2) = 350$

Solution:

a. The height of a rocket above ground after 1 second is 200 ft. b. the height of a rocket above ground after 2 seconds is 350 ft.

Exercise:

Problem: Show that the function $f(x) = 3(x-5)^2 + 7$ is <u>not</u> one-to-one.

Glossary

dependent variable
: an output variable

domain
: the set of all possible input values for a relation

function
: a relation in which each input value yields a unique output value

horizontal line test
: a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
: an input variable

input
: each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
: a function for which each value of the output is associated with a unique input value

output
: each object or value in the range that is produced when an input value is entered into a function

range
: the set of output values that result from the input values in a relation

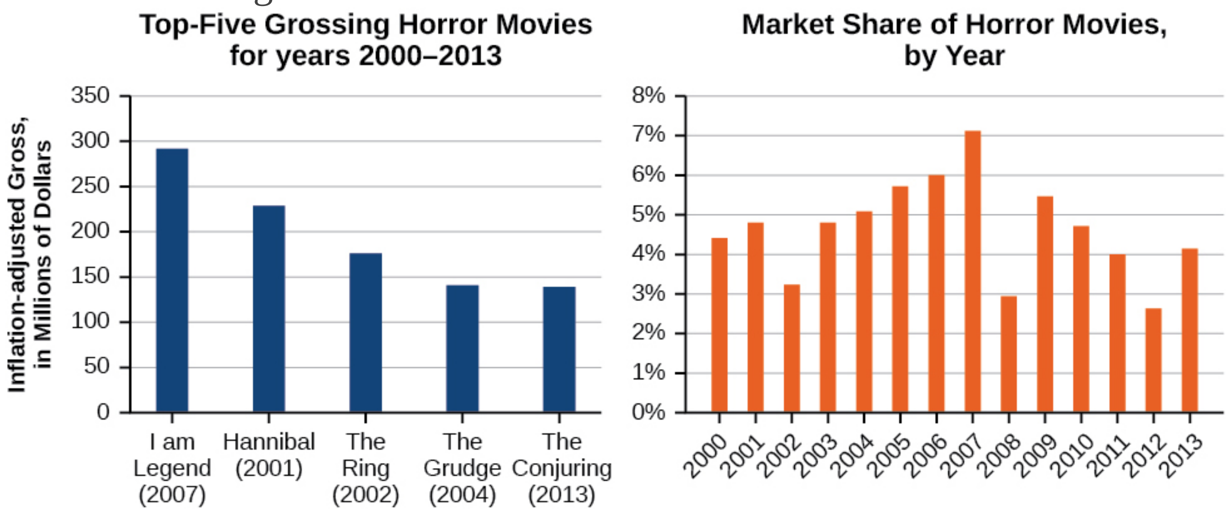
relation
: a set of ordered pairs

vertical line test
: a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

Domain and Range
In this section, you will:

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you’re in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—*I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. [link] shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

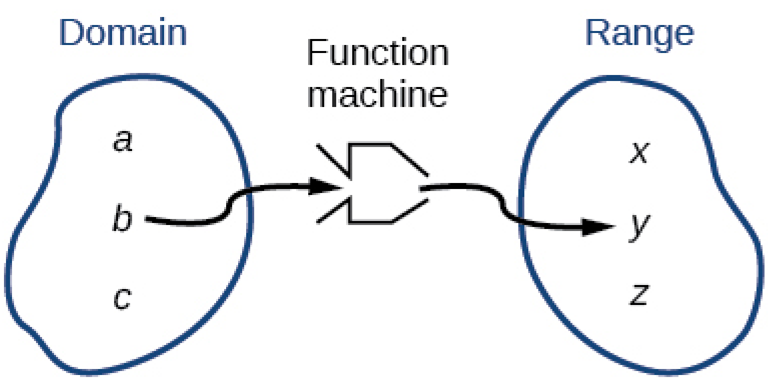


Based on data compiled by www.the-numbers.com.[footnote]
The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of domain and range. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a “holding area” that contains “raw materials” for a “function machine” and the range as another “holding area” for the machine’s products. See [link].



We can write the domain and range in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0,\ 100]$. We will discuss interval notation in greater detail later.

Let’s turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves

remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function’s equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest number from the interval is written first.
- The largest number in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint value is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint value is included, called inclusive.

See [link] for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Example:

Exercise:

Problem:

Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function:

$\{(2,\ 10),(3,\ 10),(4,\ 20),(5,\ 30),(6,\ 40)\}$.

Solution:

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

Equation:

$$\{2, 3, 4, 5, 6\}$$

Note:
Exercise:

Problem: Find the domain of the function:

$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$

Solution:

$\{-5,\ 0,\ 5,\ 10,\ 15\}$

Note:
Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example:
Exercise:

Problem:
Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution:

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Note:
Exercise:

Problem: Find the domain of the function: $f(x) = 5 - x + x^3$.

Solution:

$(-\infty, \infty)$

Note:
Given a function written in an equation form that includes a fraction, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.

3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example:
Exercise:

Problem:
Finding the Domain of a Function Involving a Denominator

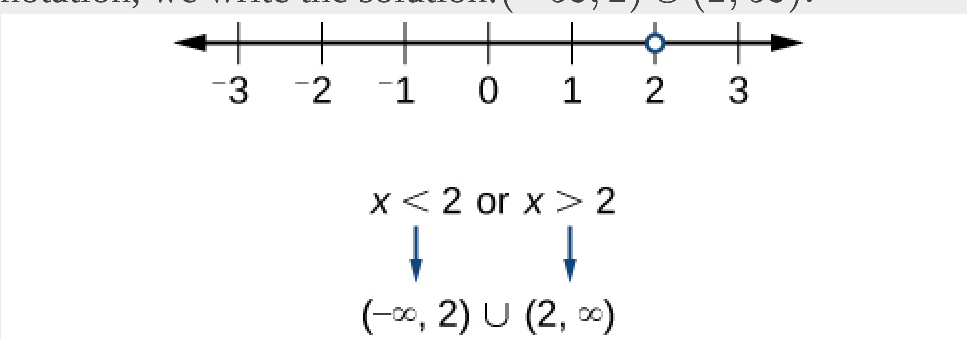
Find the domain of the function $f(x)=\frac{x+1}{2-x}$.

Solution:

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.
Equation:

$$\begin{array}{rcl}2-x&=&0\\-x&=&-2\\x&=&2\end{array}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x<2$ or $x>2$ as shown in [link]. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty,2)\cup(2,\infty)$.



Note:
Exercise:

Problem: Find the domain of the function: $f(x) = \frac{1+4x}{2x-1}$.

Solution:

$\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

Note:
Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example:
Exercise:

Problem:
Finding the Domain of a Function with an Even Root

Find the domain of the function $f(x) = \sqrt{7-x}$.

Solution:

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.
Equation:

$$\begin{aligned} 7-x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to $7,$ or $(-\infty, 7]$.

Note:
Exercise:

Problem: Find the domain of the function $f(x)=\sqrt{5+2x}$.

Solution:

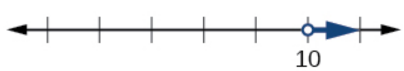
$\left[-\frac{5}{2}, \infty\right)$

Note:
Can there be functions in which the domain and range do not intersect at all?
Yes. For example, the function $f(x)=-\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in **set-builder notation**. For example, $\{x|10 \le x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar | is read as "such that," so we would read $\{x|10 \le x < 30\}$ as "the set of *x*-values such that 10 is less than or equal to $x,$ and x is less than 30."

[link] compares inequality notation, set-builder notation, and interval notation.

	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \le 10$	$\{h \mid 5 < h \le 10\}$	$(5, 10]$
	$5 \le h < 10$	$\{h \mid 5 \le h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \ge 10$	$\{h \mid h \ge 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets$\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

Equation:

$$\{x| \ \ |x| \geq 3\} = (-\infty, -3] \cup [3, \infty)$$

Note:

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form$\{x| \text{statement about } x\}$ which is read as, "the set of all x such that the statement about x is true." For example,

Equation:

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

Equation:

$$(4, 12]$$

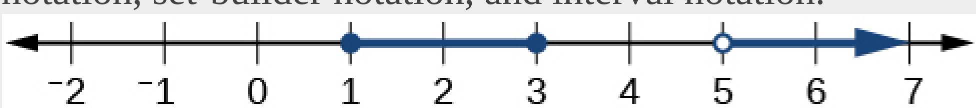
Note:
Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

Example:
Exercise:

Problem:
Describing Sets on the Real-Number Line

Describe the intervals of values shown in [link] using inequality notation, set-builder notation, and interval notation.



Solution:

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

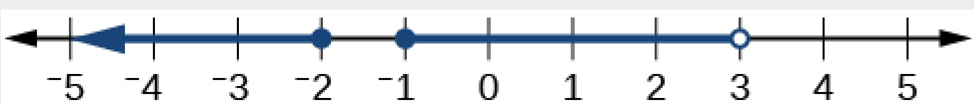
Inequality	$1 \leq x \leq 3 \text{ or } x > 5$
Set-builder notation	$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$
Interval notation	$[1, 3] \cup (5, \infty)$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Note:
Exercise:

Problem: Given [link], specify the graphed set in

a. words
b. set-builder notation
c. interval notation

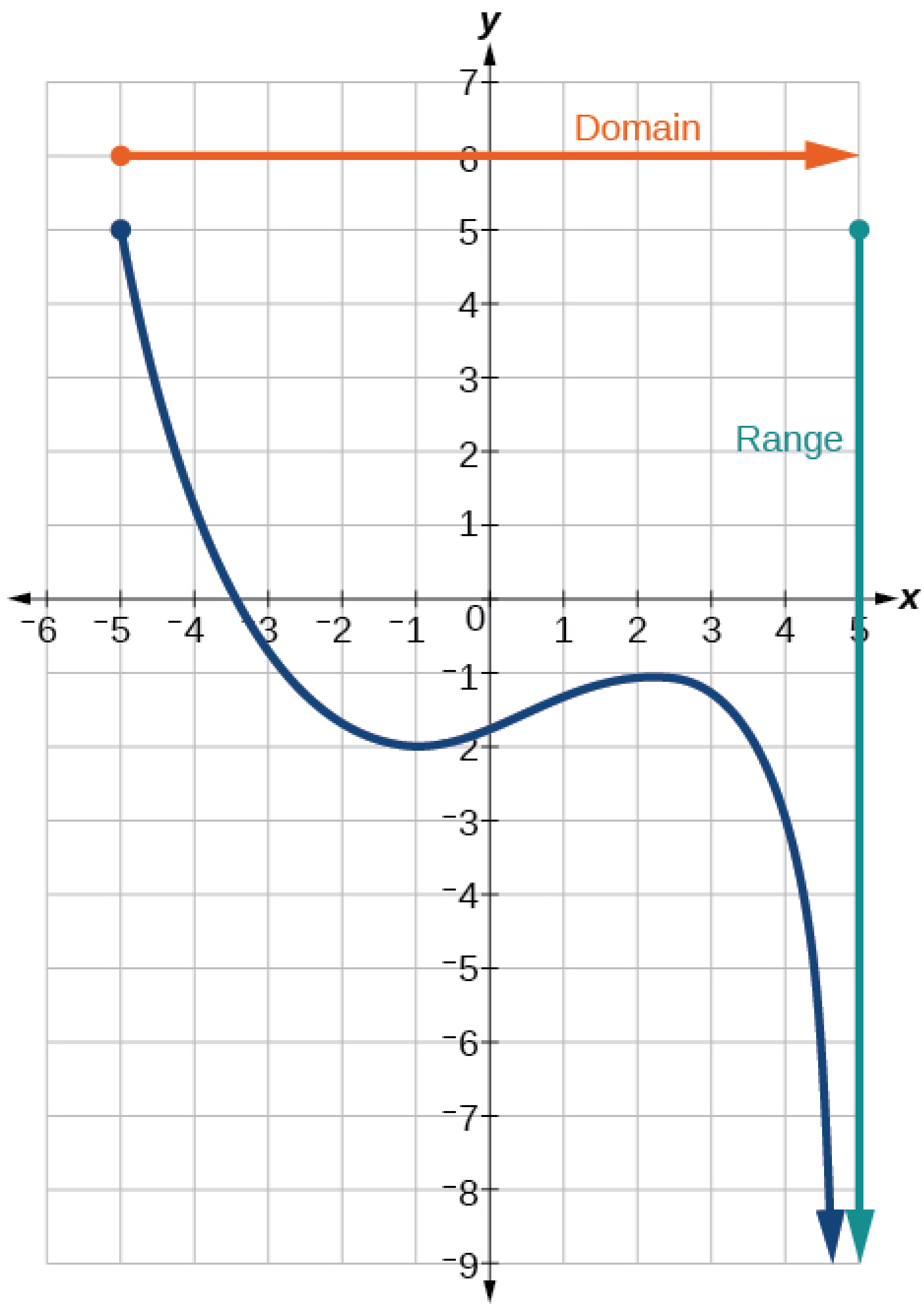


Solution:

a. values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;
b. $\{x \mid x \leq -2 \text{ or } -1 \leq x < 3\}$;
c. $(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See [link].

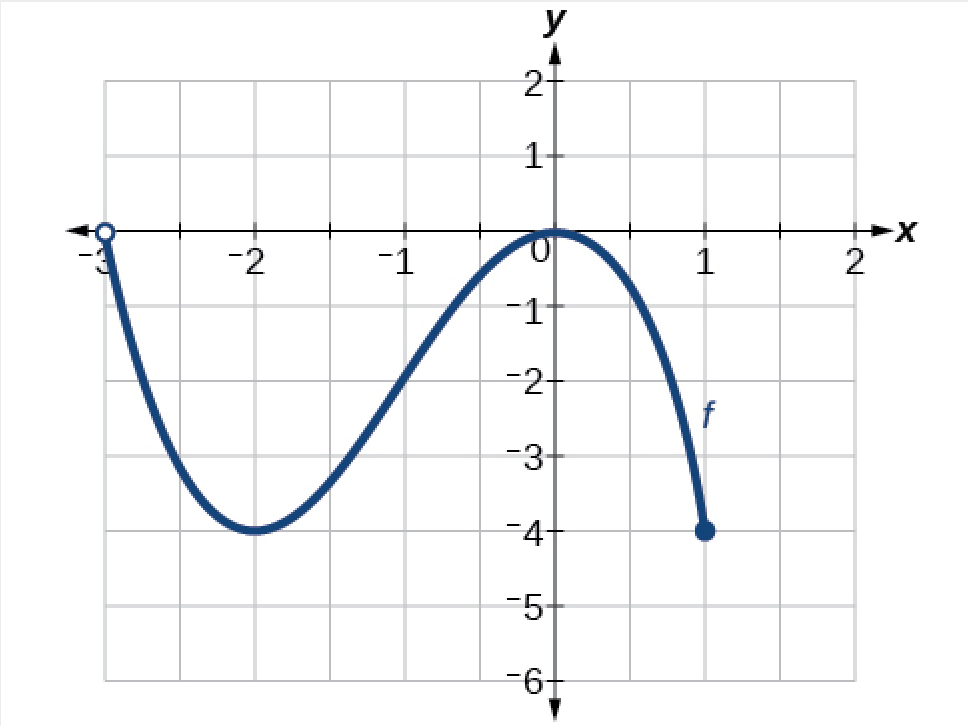


We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example:
Exercise:

Problem:
Finding Domain and Range from a Graph

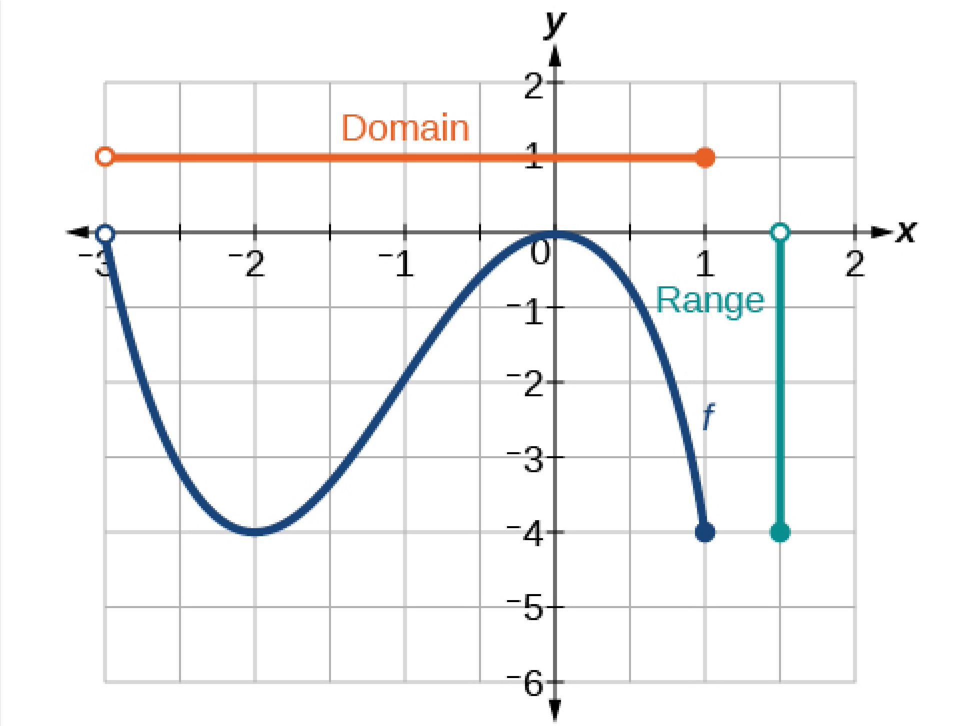
Find the domain and range of the function f whose graph is shown in [link].



Solution:

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

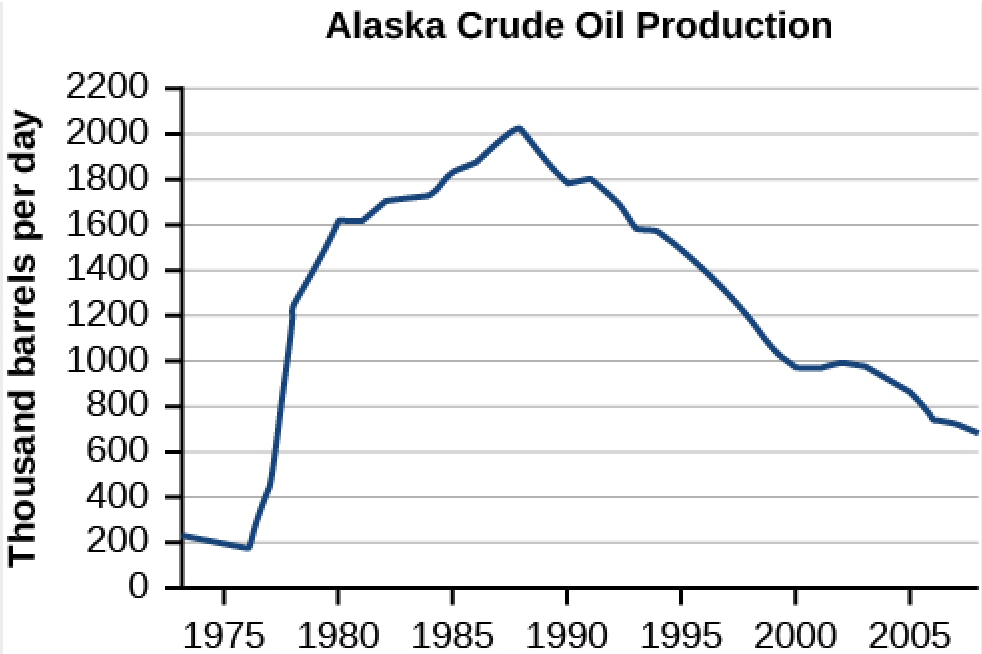
The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See [link].



Example:
Exercise:

Problem:
Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in [link].



(credit: modification of work by the U.S. Energy Information Administration)[footnote] http://www.eia.gov/dnav/pet/hist/LeafHandler.ashx?n=PET&s=MCRFPAK2&f=A.

Solution:

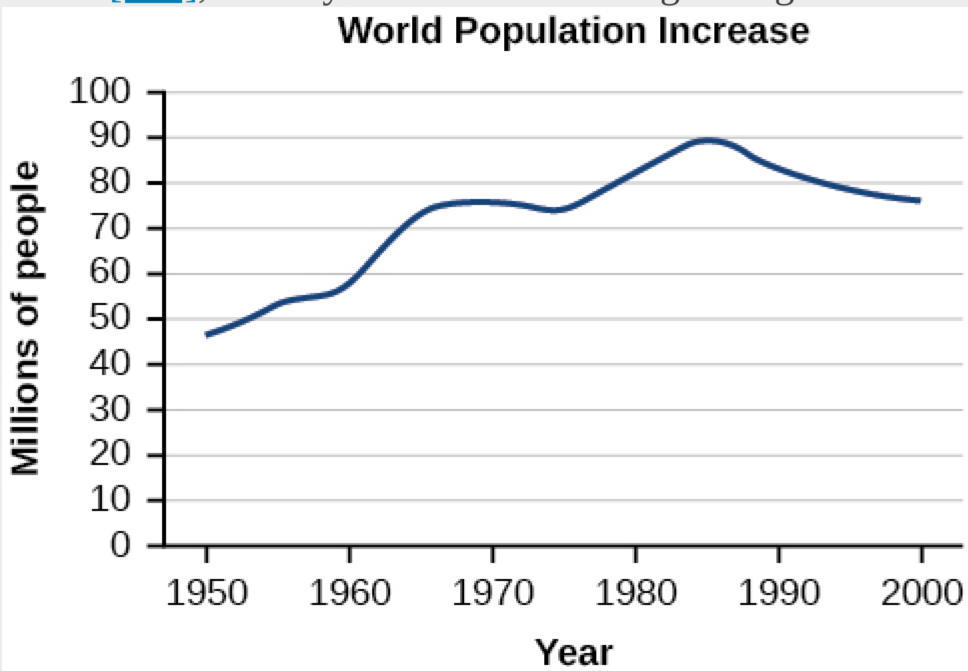
The input quantity along the horizontal axis is "years," which we represent with the variable t for time. The output quantity is "thousands of barrels of oil per day," which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is [1973, 2008], and the range is about [180, 2010]. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Note:
Exercise:

Problem:

Given [link], identify the domain and range using interval notation.



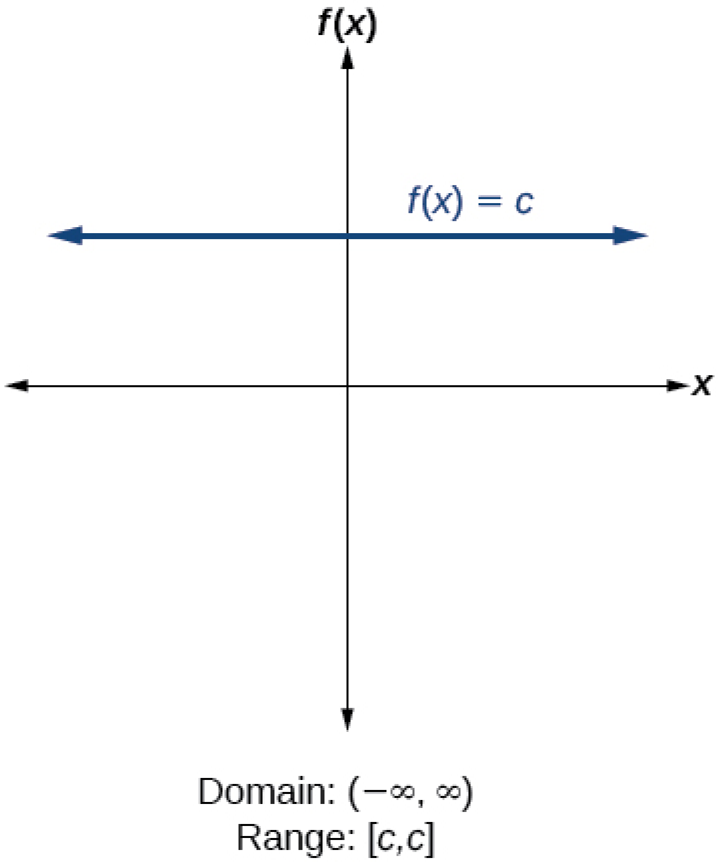
Solution:

domain =[1950,2002] range = [47,000,000,89,000,000]

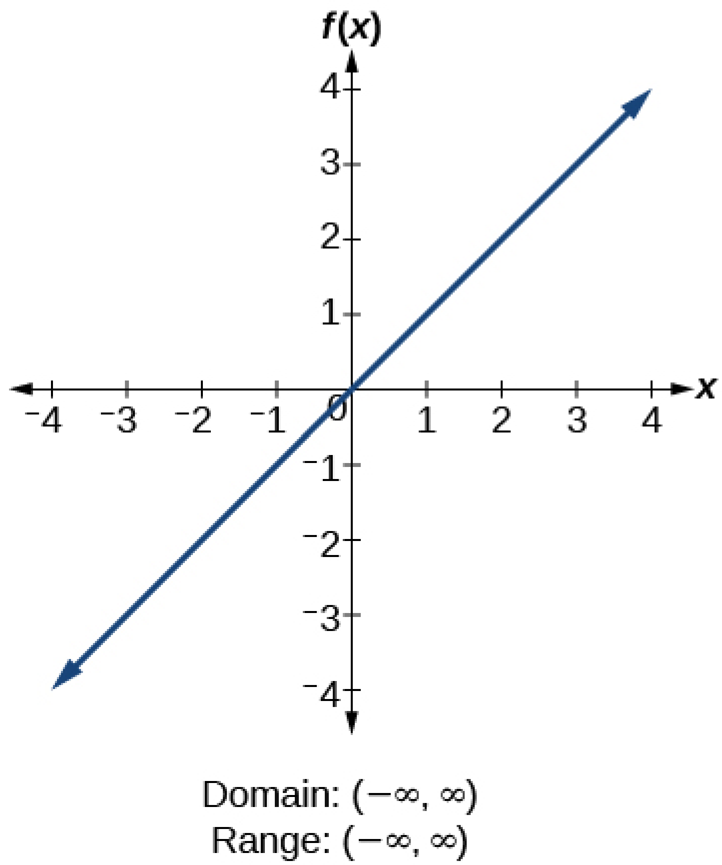
Note:
Can a function's domain and range be the same?
Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

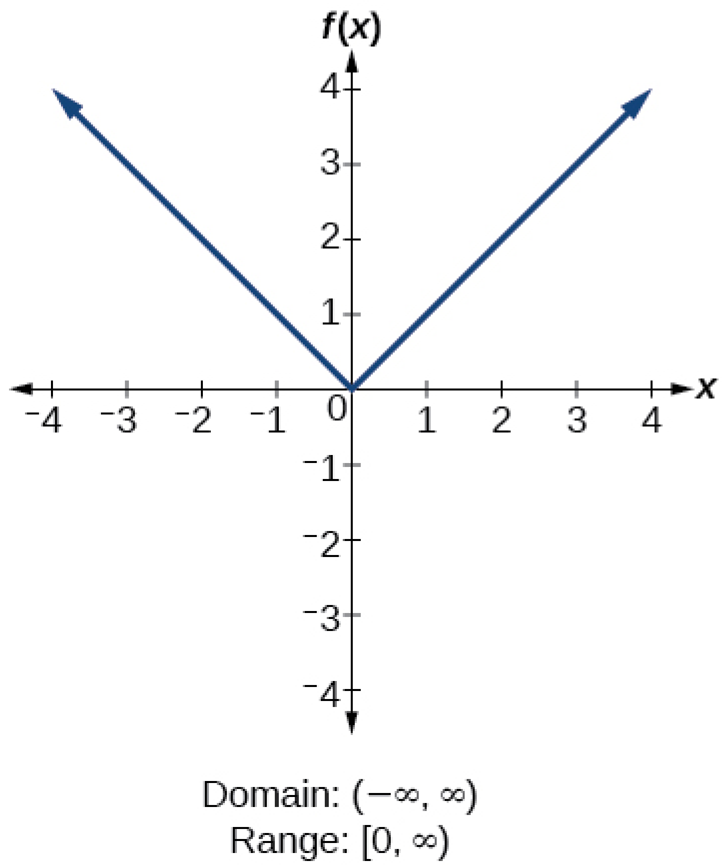
We will now return to our set of toolkit functions to determine the domain and range of each.



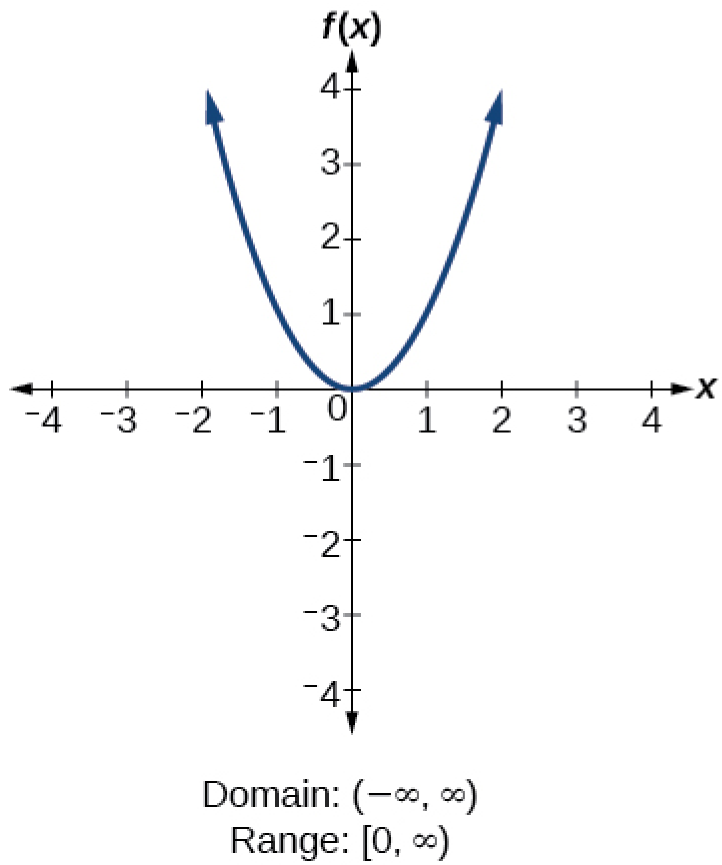
For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.



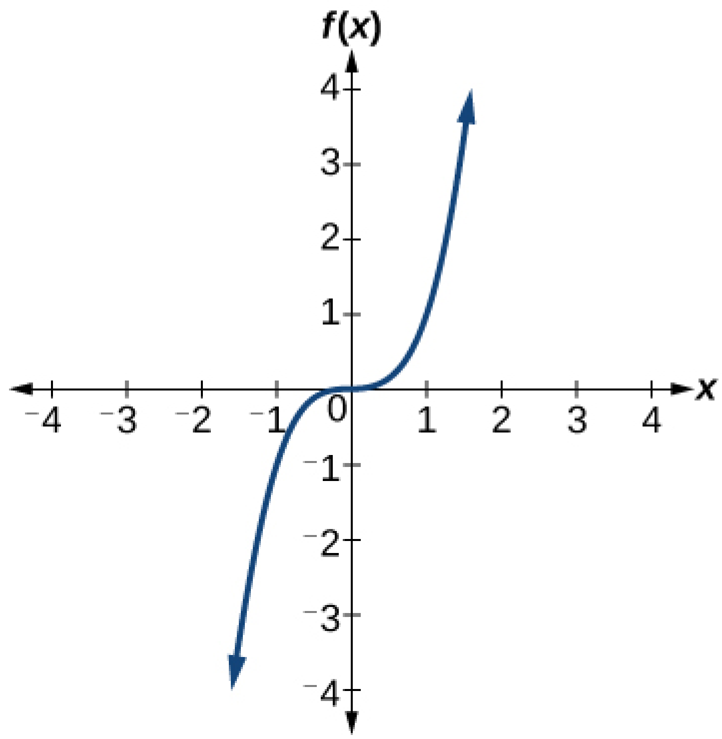
For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.



For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

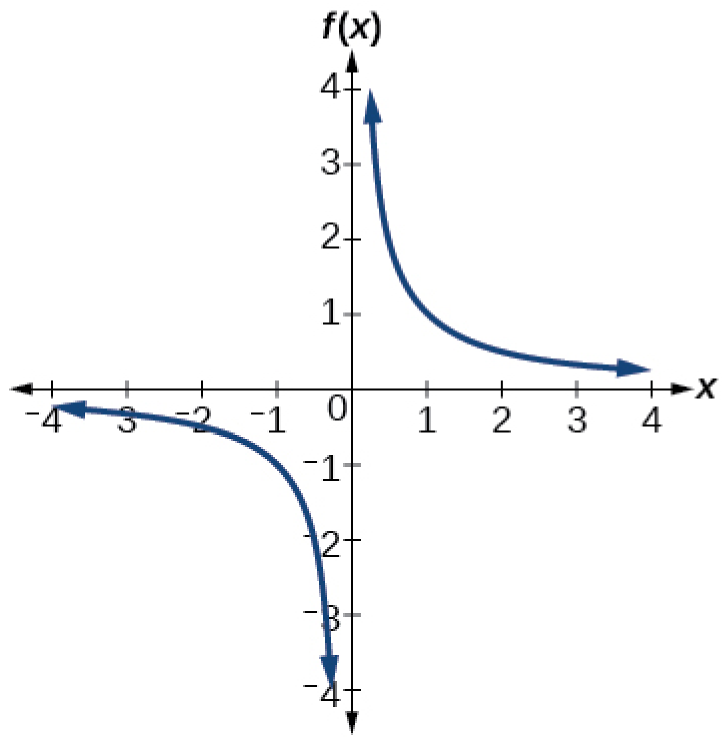


For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.



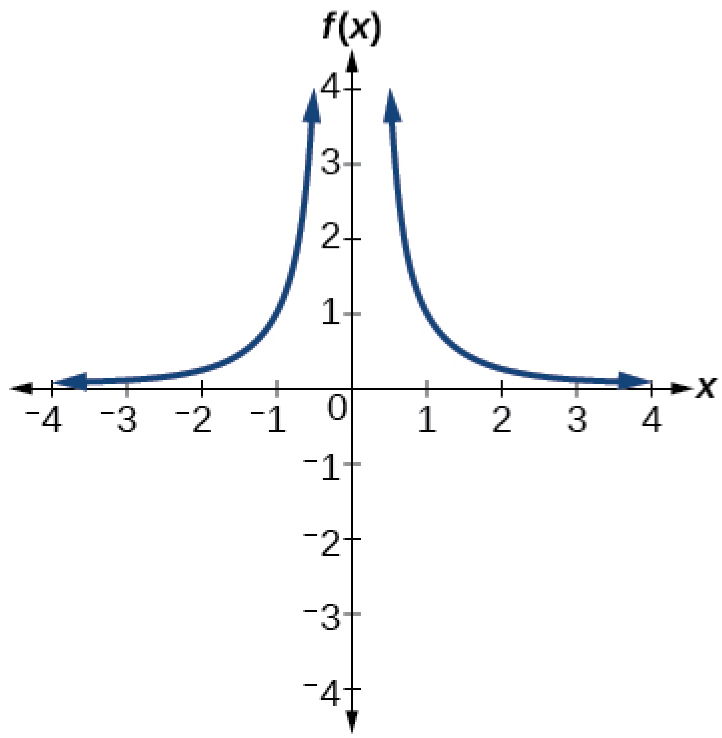
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



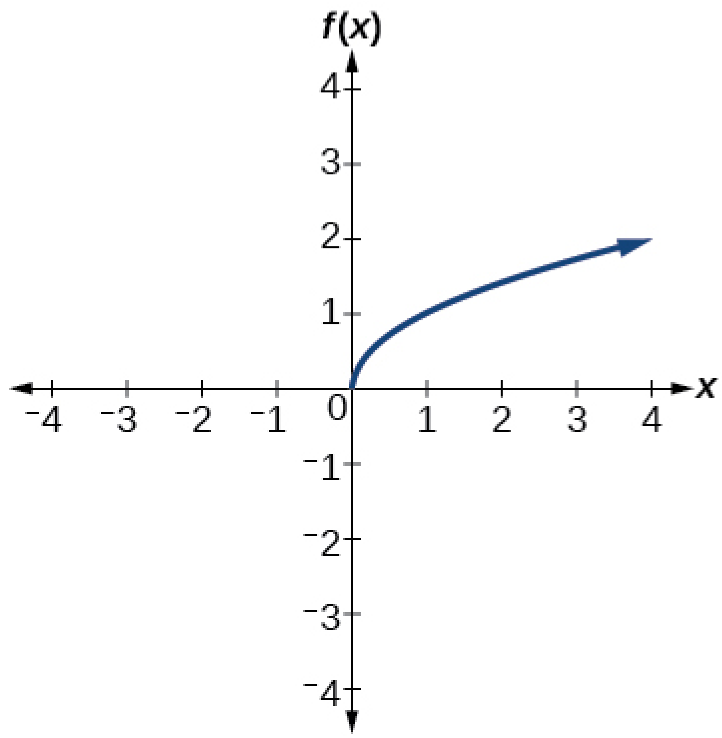
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x|\ x \neq 0\}$,the set of all real numbers that are not zero.



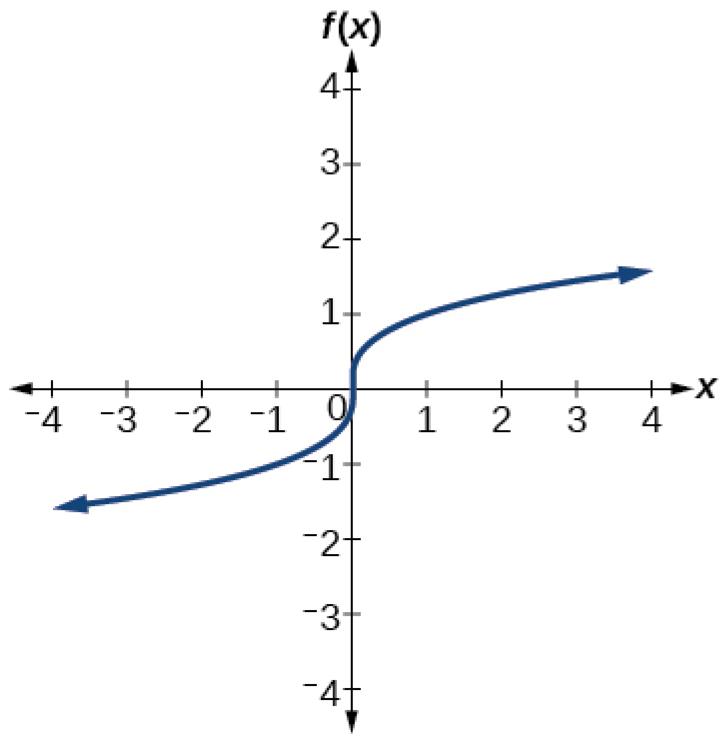
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.



Domain: [0, ∞)
Range: [0, ∞)

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.



Domain: (−∞, ∞)
Range: (−∞, ∞)

For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

Note:
Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.

3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Example:
Exercise:

Problem:
Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution:

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example:
Exercise:

Problem:
Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution:

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example:
Exercise:

Problem:
Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution:

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.
Equation:

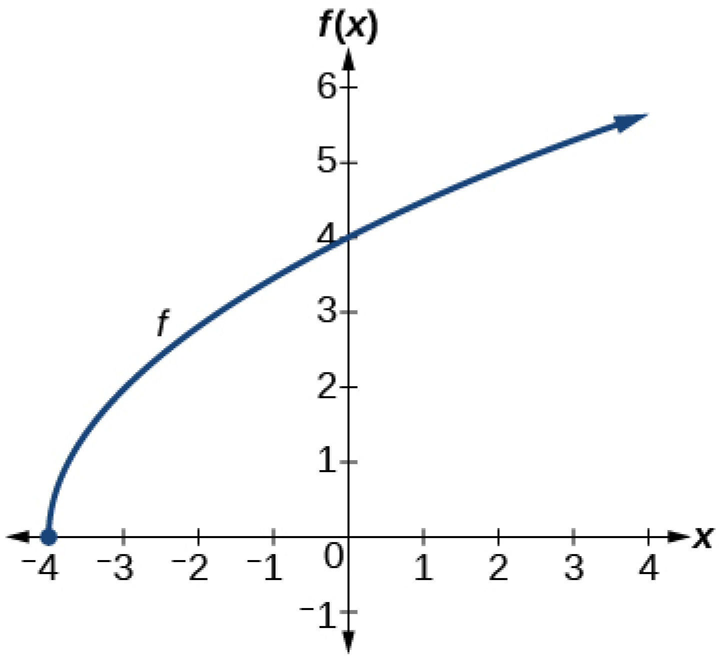
$$x + 4 \geq 0 \text{ when } x \geq -4$$

The domain of $f\left(x\right)$ is $\left[-4, \infty\right)$.

We then find the range. We know that $f\left(-4\right) = 0,$ and the function value increases as x increases without any upper limit. We conclude that the range of f is $\left[0, \infty\right)$.

Analysis

[link] represents the function f.



Note:
Exercise:

Problem: Find the domain and range of $f\left(x\right)=-\sqrt{2-x}.$

Solution:

domain: $\left(-\infty,2\right];$ range: $\left(-\infty,0\right]$

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x)=|x|.$ With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the magnitude, or modulus, of a real number value

regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.
Equation:

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.
Equation:

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a piecewise function. A **piecewise function** is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain “boundaries.” For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10{,}000$ and $\$1000 + 0.2(S - \$10{,}000)$ if $S > \$10{,}000$.

Note:
Piecewise Function
A **piecewise function** is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

Equation:

$$f(x) = \begin{array}{ll} \text{formula 1} & \text{if } x \text{ is in domain 1} \\ \text{formula 2} & \text{if } x \text{ is in domain 2} \\ \text{formula 3} & \text{if } x \text{ is in domain 3} \end{array}$$

In piecewise notation, the absolute value function is

Equation:

$$|x| = \begin{cases} x & \text{if } \ x \geq 0 \\ -x & \text{if } \ x < 0 \end{cases}$$

Note:

Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example:

Exercise:

Problem:

Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a function relating the number of people, $n,$ to the cost, $C.$

Solution:

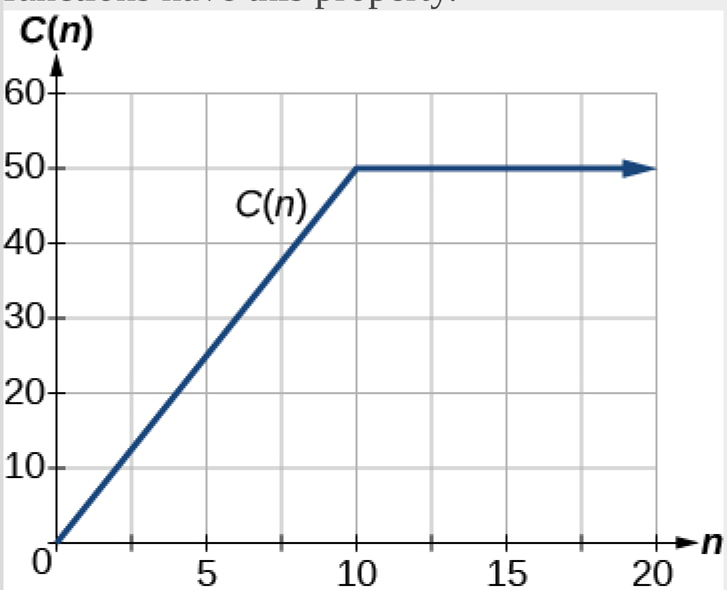
Two different formulas will be needed. For n-values under 10, $C=5n.$ For values of n that are 10 or greater, $C=50.$

Equation:

$$C(n)=\begin{cases}5n & \text{if} \quad 0<n<10\\ 50 & \text{if} \quad n\geq 10\end{cases}$$

Analysis

The function is represented in [link]. The graph is a diagonal line from $n=0$ to $n=10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n=10,$ but not all piecewise functions have this property.



Example:
Exercise:

Problem:
Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, $C,$ in dollars for g gigabytes of data transfer.

Equation:

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g-2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Solution:

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

Equation:

$$C(1.5) = \$25$$

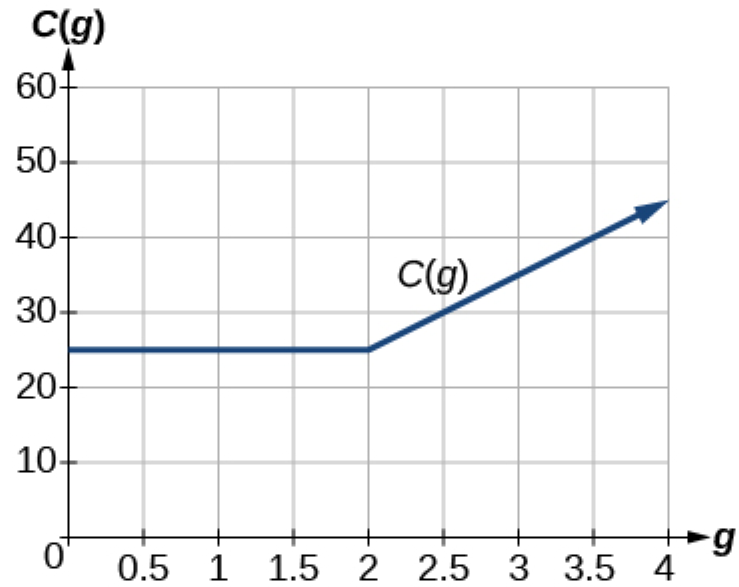
To find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we use the second formula.

Equation:

$$C(4) = 25 + 10(4-2) = \$45$$

Analysis

The function is represented in [link]. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.



Note:
Given a piecewise function, sketch a graph.

1. Indicate on the *x*-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example:
Exercise:

Problem:
Graphing a Piecewise Function

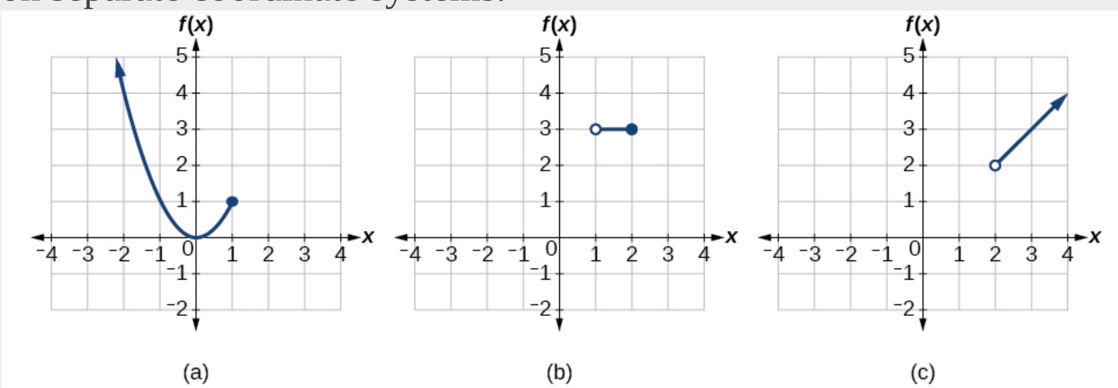
Sketch a graph of the function.
Equation:

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \le 1 \\ 3 & \text{if} \quad 1 < x \le 2 \\ x & \text{if} \quad x > 2 \end{cases}$$

Solution:

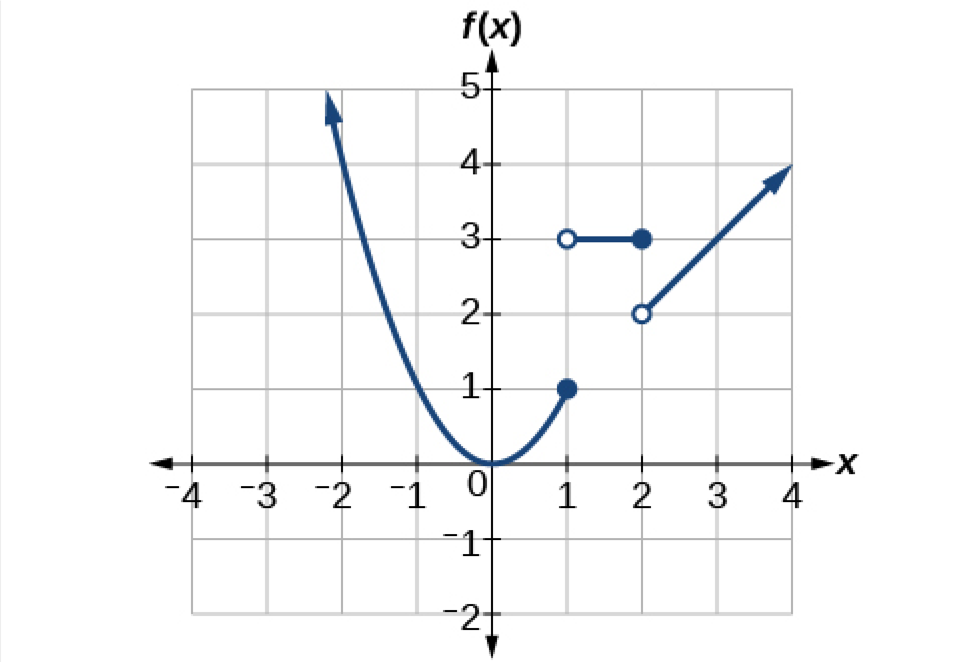
Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

[link] shows the three components of the piecewise function graphed on separate coordinate systems.



(a) $f\,(x)=x^2$ if $\ x\le 1$; (b) $f\,(x)=3$ if $1< x\le 2$; (c) $f\,(x)=x\ $ if $x>2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See [link].



Analysis

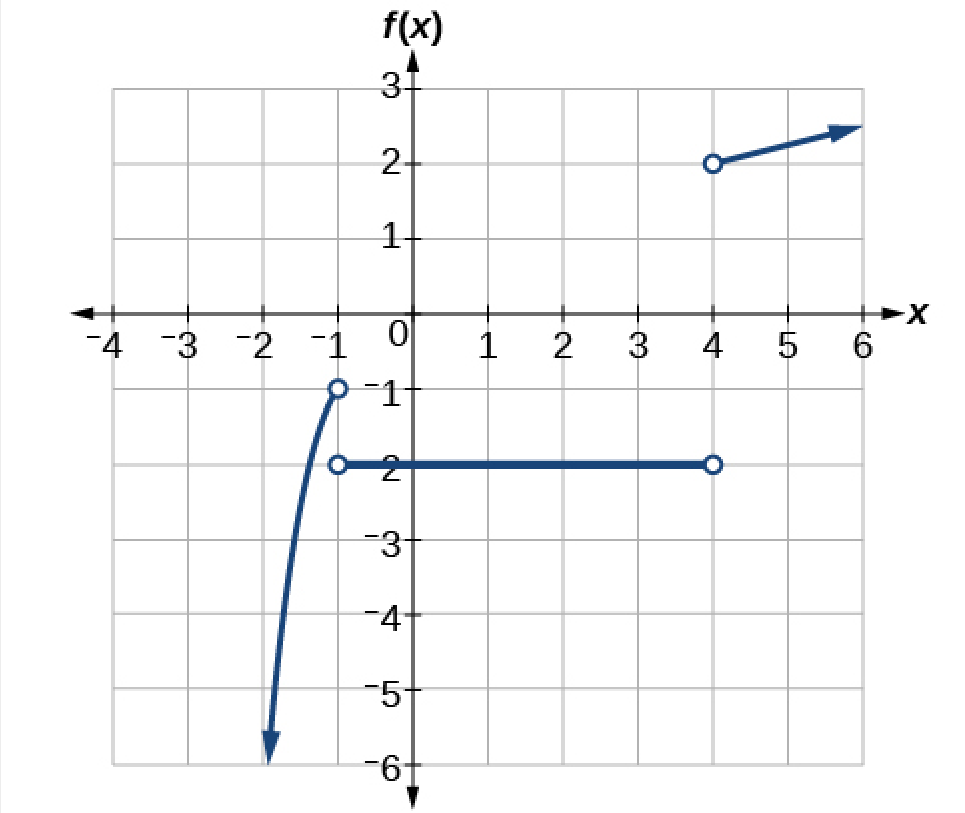
Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2,\ 3)$ are.

Note:
Exercise:

Problem: Graph the following piecewise function.
Equation:

$$f(x) = \begin{array}{ccc} x^3 & \text{if} & x < -1 \\ -2 & \text{if} & -1 < x < 4 \\ \sqrt{x} & \text{if} & x > 4 \end{array}$$

Solution:



Note:
Can more than one formula from a piecewise function be applied to a value in the domain?
No. Each value corresponds to one equation in a piecewise formula.

Note:
Access these online resources for additional instruction and practice with domain and range.

- Domain and Range of Square Root Functions
- Determining Domain and Range
- Find Domain and Range Given the Graph
- Find Domain and Range Given a Table

- Find Domain and Range Given Points on a Coordinate Plane

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs. See [link].
- The domain of a function can also be determined by identifying the input values of a function written as an equation. See [link], [link], and [link].
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation. See [link].
- For many functions, the domain and range can be determined from a graph. See [link] and [link].
- An understanding of toolkit functions can be used to find the domain and range of related functions. See [link], [link], and [link].
- A piecewise function is described by more than one formula. See [link] and [link].
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain. See [link].

Section Exercises

Verbal

Exercise:

Problem: Why does the domain differ for different functions?

Solution:

The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

Exercise:

Problem:

How do we determine the domain of a function defined by an equation?

Exercise:

Problem:

Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.

Solution:

There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$.

Exercise:

Problem:

When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

Exercise:

Problem: How do you graph a piecewise function?

Solution:

Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints

with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function.

Algebraic

For the following exercises, find the domain of each function using interval notation.

Exercise:

Problem: $f(x) = -2x(x-1)(x-2)$

Exercise:

Problem: $f(x) = 5 - 2x^2$

Solution:

$(-\infty, \infty)$

Exercise:

Problem: $f\left(x\right) = 3\sqrt{x-2}$

Exercise:

Problem: $f\left(x\right) = 3 - \sqrt{6-2x}$

Solution:

$(-\infty, 3]$

Exercise:

Problem: $f(x) = \sqrt{4-3x}$

Exercise:

Problem: $f(x) = \sqrt{x^2 + 4}$

Solution:

$(-\infty, \infty)$

Exercise:

Problem: $f(x) = \sqrt[3]{1 - 2x}$

Exercise:

Problem: $f(x) = \sqrt[3]{x - 1}$

Solution:

$(-\infty, \infty)$

Exercise:

Problem: $f(x) = \frac{9}{x-6}$

Exercise:

Problem: $f\left(x\right) = \frac{3x+1}{4x+2}$

Solution:

$(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

Exercise:

Problem: $f\left(x\right) = \frac{\sqrt{x+4}}{x-4}$

Exercise:

Problem: $f(x) = \frac{x-3}{x^2+9x-22}$

Solution:

$(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$

Exercise:

Problem: $f(x) = \frac{1}{x^2-x-6}$

Exercise:

Problem: $f(x) = \frac{2x^3-250}{x^2-2x-15}$

Solution:

$(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

Exercise:

Problem: $\frac{5}{\sqrt{x-3}}$

Exercise:

Problem: $\frac{2x+1}{\sqrt{5-x}}$

Solution:

$(-\infty, 5)$

Exercise:

Problem: $f(x) = \frac{\sqrt{x-4}}{\sqrt{x-6}}$

Exercise:

Problem: $f(x) = \frac{\sqrt{x-6}}{\sqrt{x-4}}$

Solution:

$[6, \infty)$

Exercise:

Problem: $f(x) = \frac{x}{x}$

Exercise:

Problem: $f(x) = \frac{x^2-9x}{x^2-81}$

Solution:

$(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$

Exercise:

Problem: Find the domain of the function $f(x) = \sqrt{2x^3 - 50x}$ by:

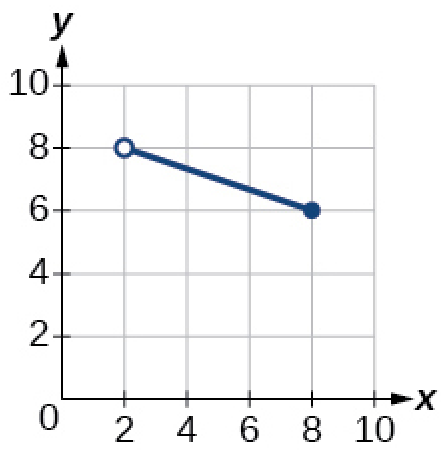
a. using algebra.
b. graphing the function in the radicand and determining intervals on the *x*-axis for which the radicand is nonnegative.

Graphical

For the following exercises, write the domain and range of each function using interval notation.

Exercise:

Problem:

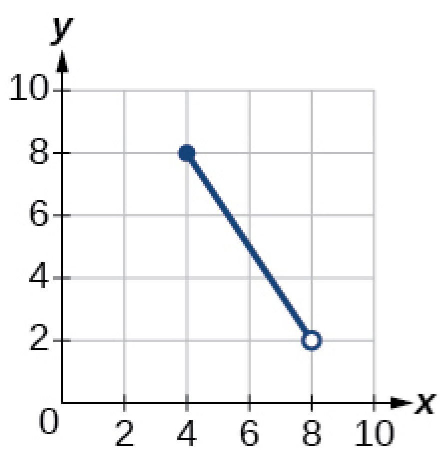


Solution:

domain: $(2,8]$, range $[6,8)$

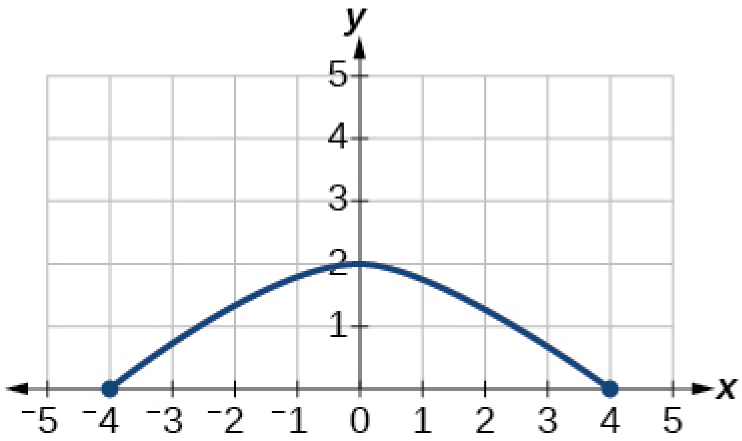
Exercise:

Problem:



Exercise:

Problem:

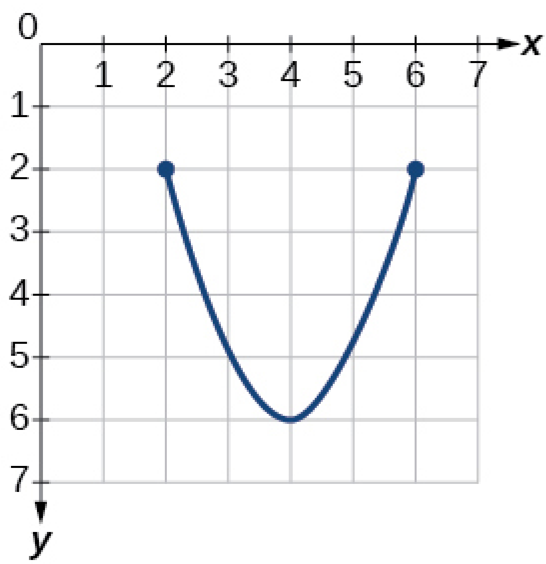


Solution:

domain: $[-4,\ 4]$, range: $[0,\ 2]$

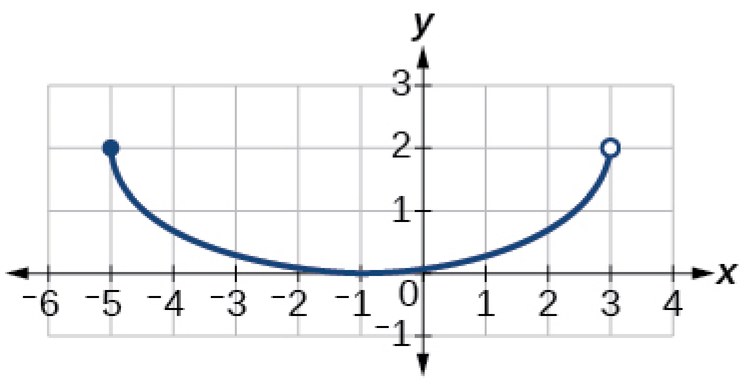
Exercise:

Problem:



Exercise:

Problem:

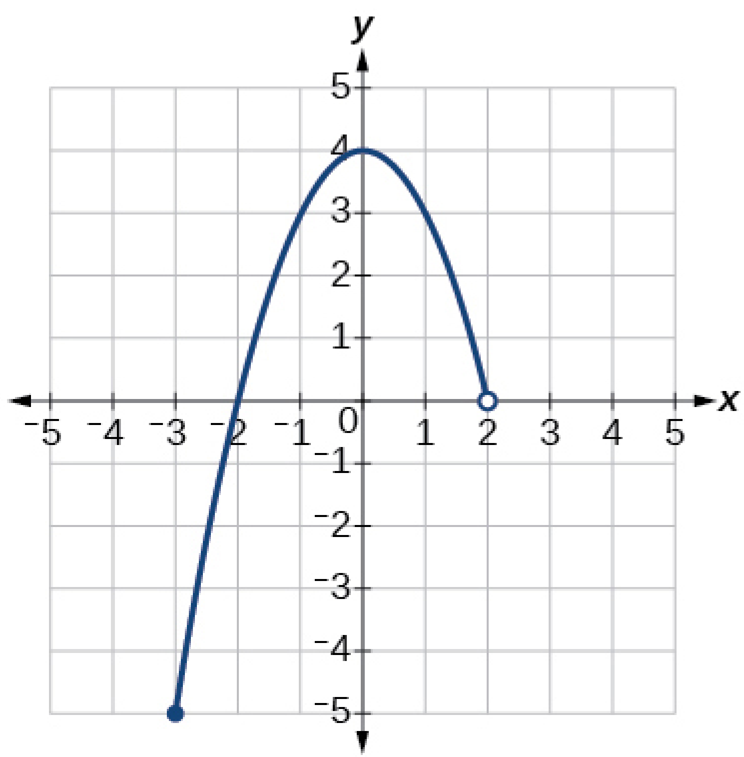


Solution:

domain: $[-5,\ 3)$, range: $[0, 2]$

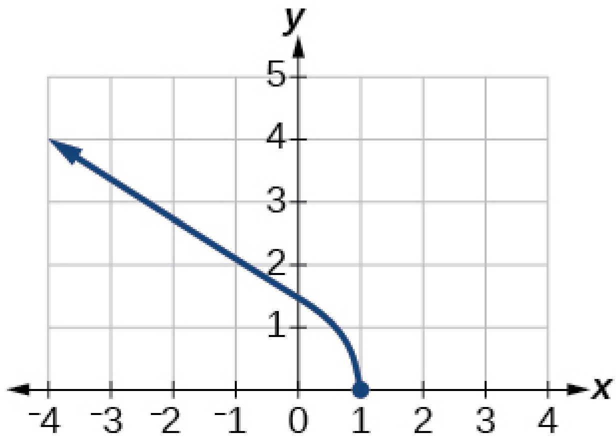
Exercise:

Problem:



Exercise:

Problem:

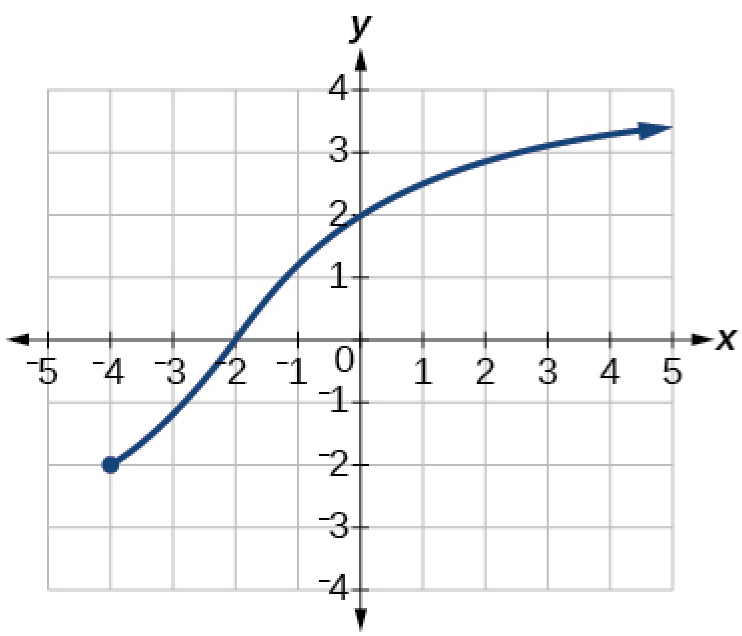


Solution:

domain: $(-\infty, 1]$, range: $[0, \infty)$

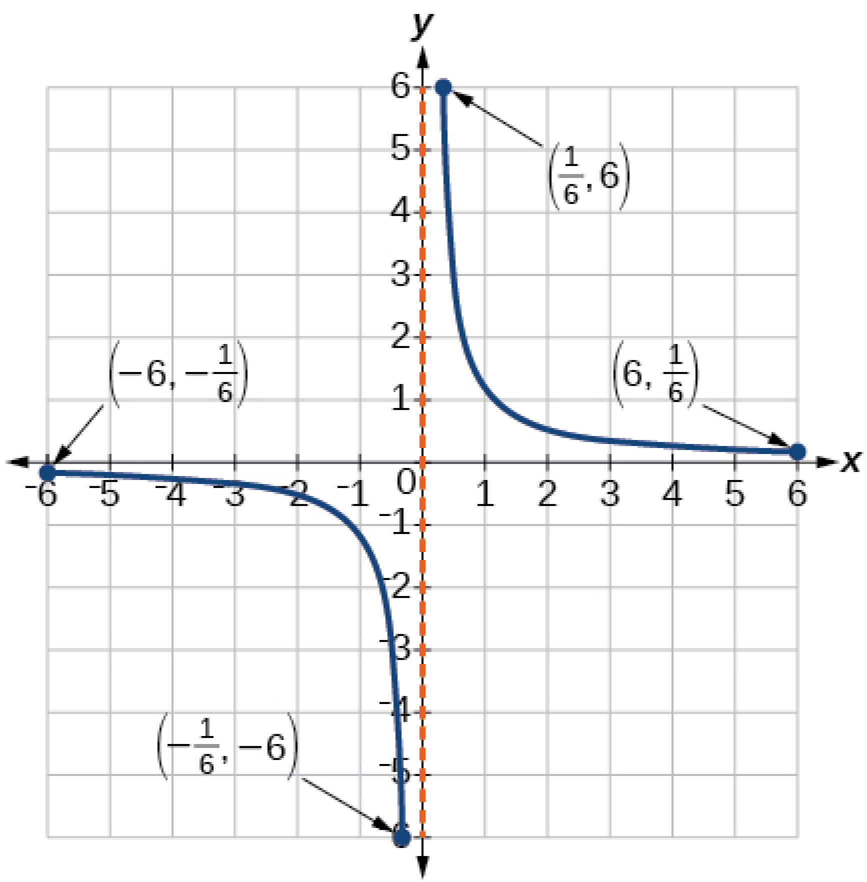
Exercise:

Problem:



Exercise:

Problem:

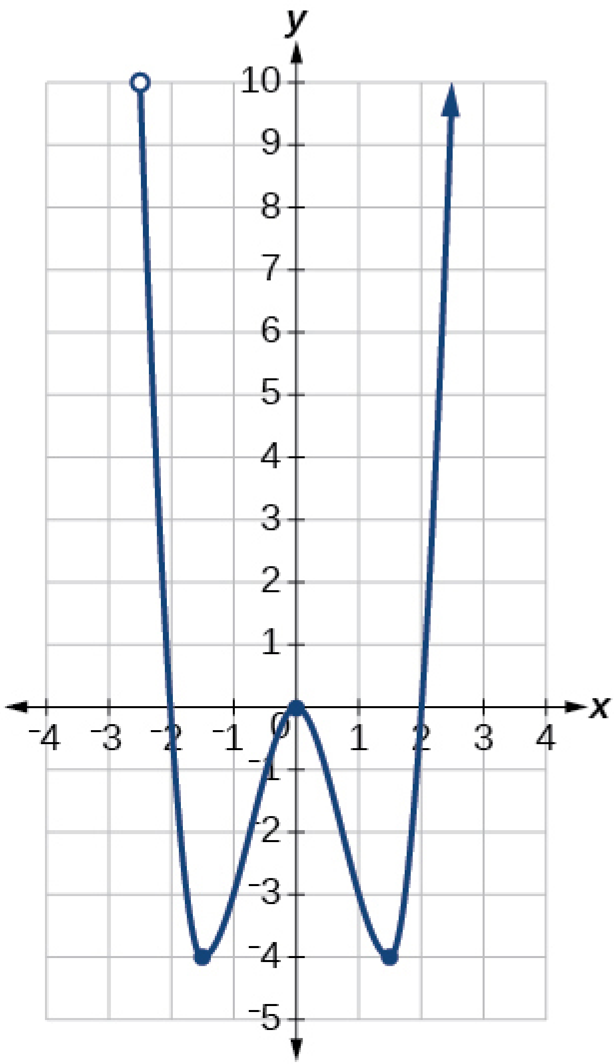


Solution:

domain: $\left[-6, -\frac{1}{6}\right] \cup \left[\frac{1}{6}, 6\right]$; range: $\left[-6, -\frac{1}{6}\right] \cup \left[\frac{1}{6}, 6\right]$

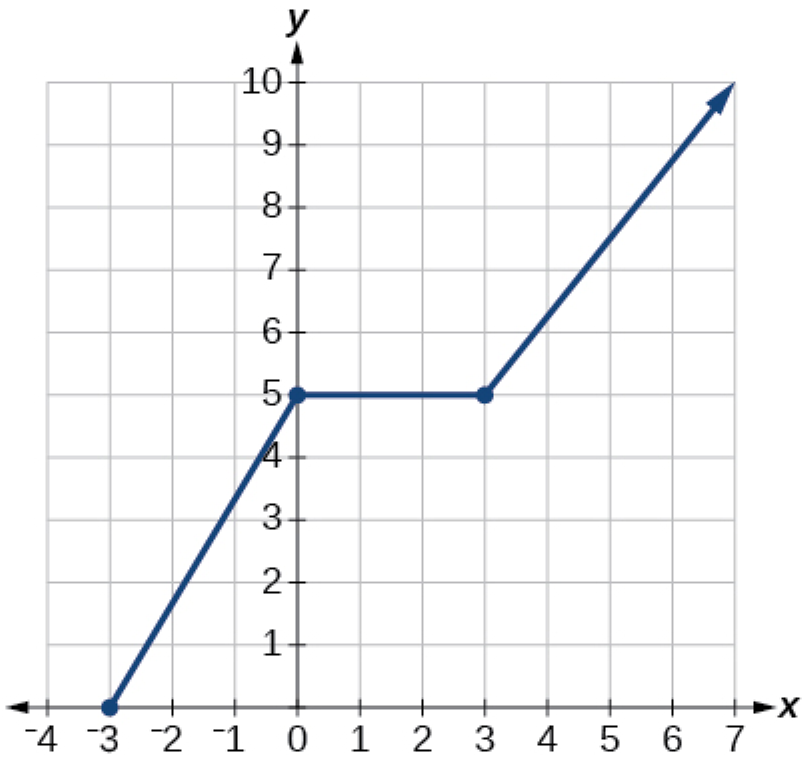
Exercise:

Problem:



Exercise:

Problem:



Solution:

domain: $[-3,\ \infty)$; range: $[0, \infty)$

For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

Exercise:

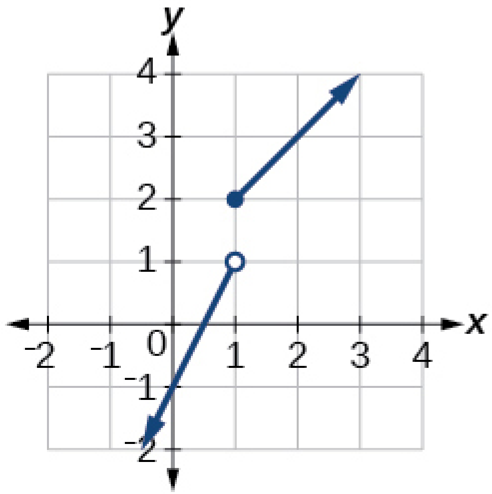
Problem: $f(x) = \begin{cases} x+1 & \text{if} \quad x < -2 \\ -2x-3 & \text{if} \quad x \geq -2 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} 2x-1 & \text{if} \quad x < 1 \\ 1+x & \text{if} \quad x \geq 1 \end{cases}$

Solution:

domain: $(-\infty, \infty)$



Exercise:

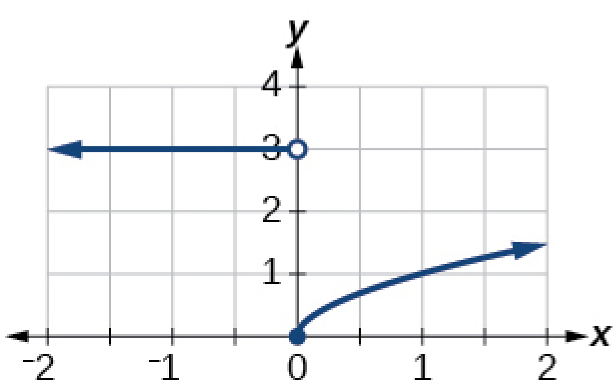
Problem: $f(x) = \begin{cases} x+1 \text{ if } x < 0 \\ x-1 \text{ if } x > 0 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} 3 & \text{if} & x < 0 \\ \sqrt{x} & \text{if} & x \geq 0 \end{cases}$

Solution:

domain: $(-\infty, \infty)$



Exercise:

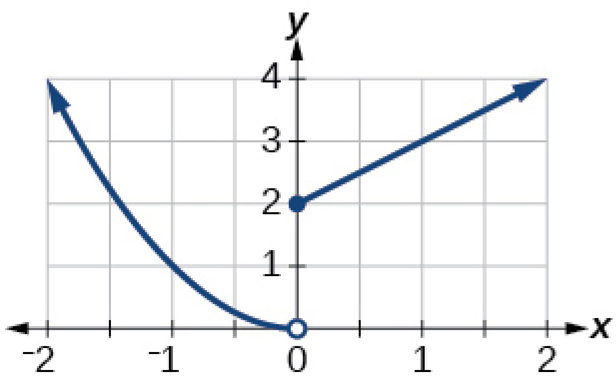
Problem: $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 - x & \text{if } x > 0 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} x^2 \text{ if } & x < 0 \\ x + 2 \text{ if } & x \geq 0 \end{cases}$

Solution:

domain: $(-\infty, \infty)$



Exercise:

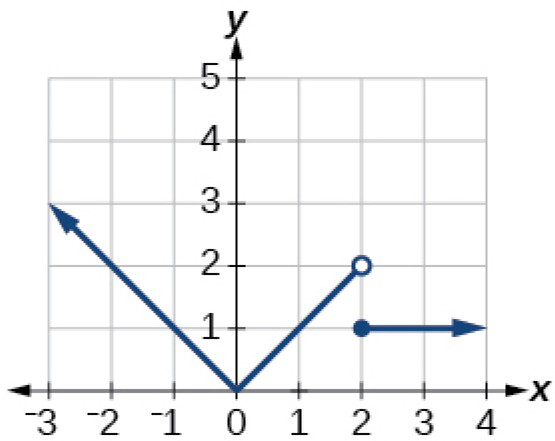
Problem: $f(x) = \begin{cases} x + 1 & \text{if} & x < 1 \\ x^3 & \text{if} & x \geq 1 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} |x| & \text{if } x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$

Solution:

domain: $(-\infty, \infty)$



Numeric

For the following exercises, given each function f,evaluate $f(-3)$, $f(-2)$, $f(-1)$, and $f(0)$.

Exercise:

Problem: $f(x) = \begin{cases} x+1 & \text{if} \quad x < -2 \\ -2x-3 & \text{if} \quad x \geq -2 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} 1 & \text{if } x \leq -3 \\ 0 & \text{if } x > -3 \end{cases}$

Solution:

$f(-3) = 1; \quad f(-2) = 0; \quad f(-1) = 0; \quad f(0) = 0$

Exercise:

Problem: $f(x) = \begin{cases} -2x^2 + 3 & \text{if } x \leq -1 \\ 5x - 7 & \text{if } x > -1 \end{cases}$

For the following exercises, given each function f, evaluate $f(-1),\ f(0),\ f(2)$, and $f(4)$.

Exercise:

Problem: $f(x) = \begin{cases} 7x + 3 & \text{if} \quad x < 0 \\ 7x + 6 & \text{if} \quad x \geq 0 \end{cases}$

Solution:

$$f(-1) = -4; \quad f(0) = 6; \quad f(2) = 20; \quad f(4) = 34$$

Exercise:

Problem: $f\left(x\right) = \begin{cases} x^2 - 2 & \text{if} \quad x < 2 \\ 4 + |x - 5| & \text{if} \quad x \geq 2 \end{cases}$

Exercise:

Problem: $f\left(x\right) = \begin{matrix} 5x & \text{if} & x < 0 \\ 3 & \text{if} & 0 \leq x \leq 3 \\ x^2 & \text{if} & x > 3 \end{matrix}$

Solution:

$$f(-1) = -5; \quad f(0) = 3; \quad f(2) = 3; \quad f(4) = 16$$

For the following exercises, write the domain for the piecewise function in interval notation.

Exercise:

Problem: $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} x^2-2 & \text{if } x < 1 \\ -x^2+2 & \text{if } x > 1 \end{cases}$

Solution:

domain: $(-\infty, 1) \cup (1, \infty)$

Exercise:

Problem: $f(x) = \begin{cases} 2x-3 & \text{if } \ x < 0 \\ -3x^2 & \text{if } \ x \geq 2 \end{cases}$

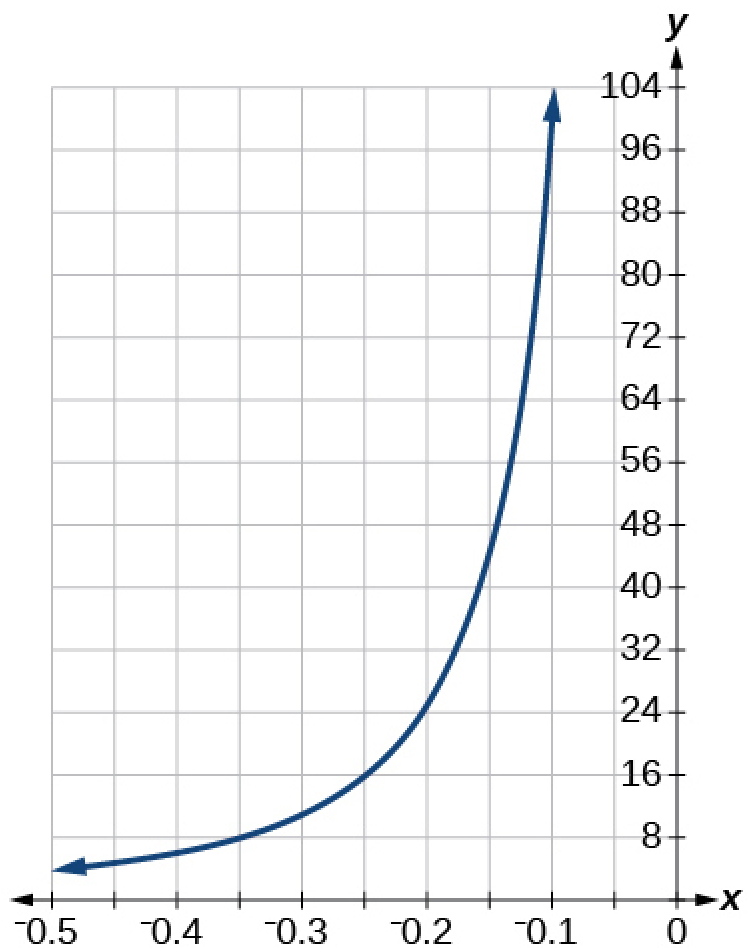
Technology

Exercise:

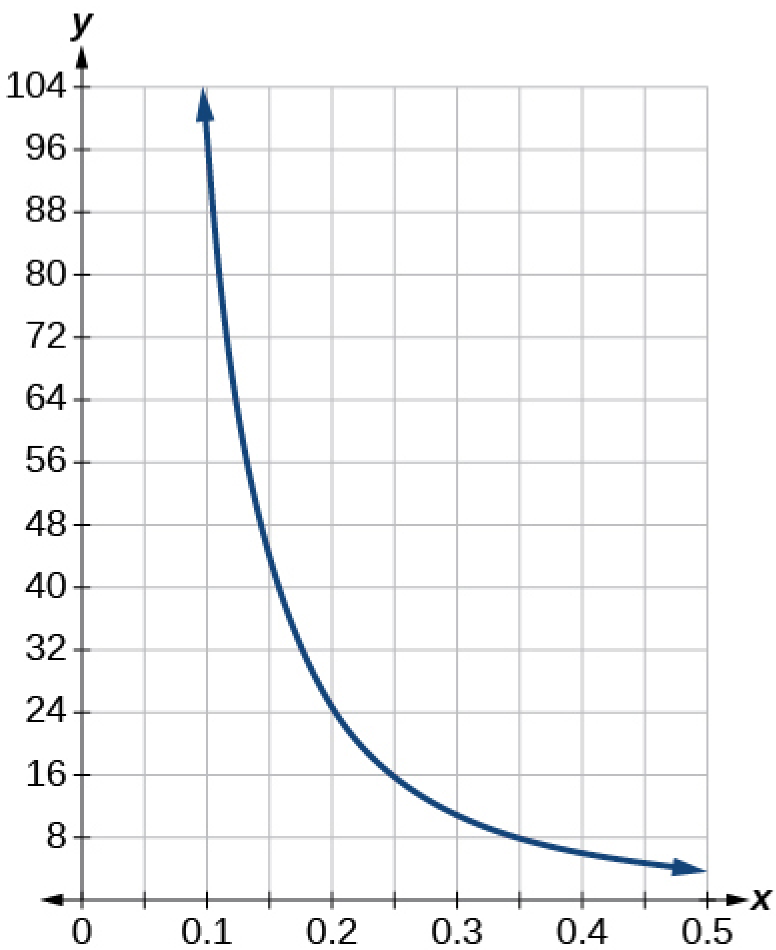
Problem:

Graph $y = \frac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Solution:



window: $[-0.5, -0.1]$; range: $[4,\ 100]$



window: $[0.1,\ 0.5]$; range: $[4,\ 100]$

Exercise:

Problem:

Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1,\ 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Extension

Exercise:

Problem:

Suppose the range of a function f is $[-5,\ 8]$. What is the range of $|f(x)|$?

Solution:

$[0,\ 8]$

Exercise:

Problem:

Create a function in which the range is all nonnegative real numbers.

Exercise:

Problem: Create a function in which the domain is $x > 2$.

Solution:

Many answers. One function is $f(x) = \frac{1}{\sqrt{x-2}}$.

Real-World Applications

Exercise:

Problem:

The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

Solution:

The domain is $[0,\ 6];$ it takes 6 seconds for the projectile to leave the ground and return to the ground

Exercise:

Problem:

The cost in dollars of making x items is given by the function $C(x) = 10x + 500.$

a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.
b. What is the cost of making 25 items?
c. Suppose the maximum cost allowed is $1500. What are the domain and range of the cost function, $C(x)$?

Glossary

interval notation
: a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function
: a function in which more than one formula is used to define the output

set-builder notation
: a method of describing a set by a rule that all of its members obey; it takes the form $\{x|\ \text{statement about } x\}$

Rates of Change and Behavior of Graphs
In this section, you will:

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

Gasoline costs have experienced some wild fluctuations over the last several decades. [link] [footnote] lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.
http://www.eia.gov/totalenergy/data/annual/showtext.cfm?t=ptb0524. Accessed 3/5/2014.

y	2005	2006	2007	2008	2009	2010	2011	2012
$C(y)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from $2.31 to $3.68, an increase of $1.37. While this is interesting, it might be more useful to look at how much the price changed *per year*. In this section, we will investigate changes such as these.

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in [link] did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.
Equation:

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}}\\ &= \frac{\Delta y}{\Delta x}\\ &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as “delta-*y* over delta-*x*” or “the change in y divided by the change in x.” Occasionally we write Δf instead of Δy, which still represents the change in the function’s output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

Equation:

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7 \text{ years}} \approx 0.196 \text{ dollars per year}$$

On average, the price of gas increased by about 19.6¢ each year.

Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

Note:

Rate of Change

A rate of change describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are “output units per input units.”

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

Equation:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Note:

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.

3. Find the ratio $\frac{\Delta y}{\Delta x}$.

Example:
Exercise:

Problem:
Computing an Average Rate of Change

Using the data in [link], find the average rate of change of the price of gasoline between 2007 and 2009.

Solution:

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

Equation:

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year} \end{aligned}$$

Analysis

Note that a decrease is expressed by a negative change or “negative increase.” A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Note:
Exercise:

Problem:

Using the data in [link], find the average rate of change between 2005 and 2010.

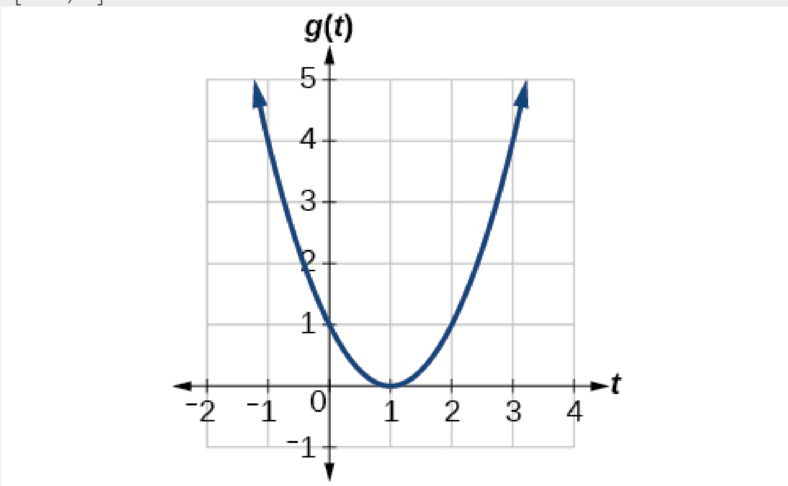
Solution:

$\frac{\$2.84 - \$2.31}{5 \text{ years}} = \frac{\$0.53}{5 \text{ years}} = \$0.106$ per year.

Example:
Exercise:

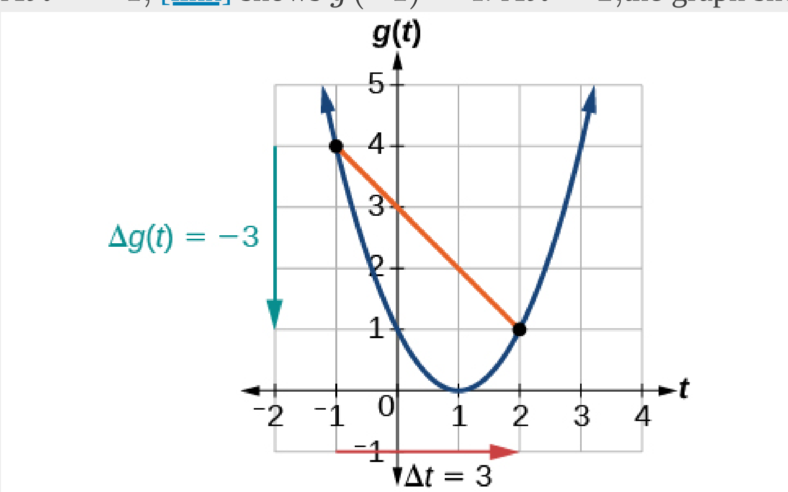
Problem:
Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in [link], find the average rate of change on the interval $[-1, 2]$.



Solution:

At $t=-1$, [link] shows $g(-1)=4$. At $t=2$,the graph shows $g(2)=1$.



The horizontal change $\Delta t=3$ is shown by the red arrow, and the vertical change $\Delta g(t)=-3$ is shown by the turquoise arrow. The average rate of change is shown by the slope of the orange line segment. The output changes by –3 while the input changes by 3, giving an average rate of change of
Equation:

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1$$

Analysis

Note that the order we choose is very important. If, for example, we use $\frac{y_2-y_1}{x_1-x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1, y_1) and (x_2, y_2).

Example:
Exercise:

Problem:
Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away and leaving on a trip, Anna records her distance from home over time. The values are shown in [link]. Find her average speed over the first 6 hours.

***t* (hours)**	0	1	2	3	4	5	6	7
***D*(*t*) (miles)**	10	55	90	153	214	240	292	300

Solution:

Here, the average speed is the average rate of change. She traveled 282 miles in 6 hours.

Equation:

$$\begin{aligned} \frac{292-10}{6-0} &= \frac{282}{6} \\ &= 47 \end{aligned}$$

The average speed is 47 miles per hour.

Analysis

Because the speed is not constant, the average speed depends on the interval chosen. For the interval [2,3], the average speed is 63 miles per hour.

Example:
Exercise:

Problem:
Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f\left(x\right)=x^2-\frac{1}{x}$ on the interval $\left[2,4\right]$.

Solution:

We can start by computing the function values at each endpoint of the interval.
Equation:

$$\begin{array}{ccc}f(2)&=&2^2-\frac{1}{2}\\&=&4-\frac{1}{2}\\&=&\frac{7}{2}\end{array}\qquad\begin{array}{ccc}f(4)&=&4^2-\frac{1}{4}\\&=&16-\frac{1}{4}\\&=&\frac{63}{4}\end{array}$$

Now we compute the average rate of change.
Equation:

$$\begin{array}{ccc}\text{Average rate of change}&=&\frac{f(4)-f(2)}{4-2}\\&=&\frac{\frac{63}{4}-\frac{7}{2}}{4-2}\\&=&\frac{\frac{49}{4}}{2}\\&=&\frac{49}{8}\end{array}$$

Note:
Exercise:

Problem: Find the average rate of change of $f\left(x\right)=x-2\sqrt{x}$ on the interval $\left[1,\ 9\right]$.

Solution:

$\frac{1}{2}$

Example:
Exercise:

Problem:
Finding the Average Rate of Change of a Force

The electrostatic force F,measured in newtons, between two charged particles can be related to the distance between the particles d,in centimeters, by the formula $F\left(d\right)=\frac{2}{d^2}$.Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution:

We are computing the average rate of change of $F\left(d\right)=\frac{2}{d^2}$ on the interval $\left[2,6\right]$.
Equation:

$$\begin{array}{lll}\text{Average rate of change} & = & \frac{F(6)-F(2)}{6-2} \\ & = & \frac{\frac{2}{6^2}-\frac{2}{2^2}}{6-2} \qquad \text{Simplify.} \\ & = & \frac{\frac{2}{36}-\frac{2}{4}}{4} \\ & = & \frac{-\frac{16}{36}}{4} \qquad \text{Combine numerator terms.} \\ & = & -\frac{1}{9} \qquad \text{Simplify}\end{array}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

Example:
Exercise:

Problem:
Finding an Average Rate of Change as an Expression

Find the average rate of change of $g\left(t\right)=t^2+3t+1$ on the interval $\left[0,\ a\right]$. The answer will be an expression involving a in simplest form.

Solution:

We use the average rate of change formula.
Equation:

$$\begin{aligned}\text{Average rate of change} &= \frac{g(a)-g(0)}{a-0} && \text{Evaluate.}\\ &= \frac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0} && \text{Simplify.}\\ &= \frac{a^2+3a+1-1}{a} && \text{Simplify and factor.}\\ &= \frac{a(a+3)}{a} && \text{Divide by the common factor } a.\\ &= a+3 \end{aligned}$$

This result tells us the average rate of change in terms of a between $t=0$ and any other point $t=a$. For example, on the interval $[0,5]$, the average rate of change would be $5+3=8$.

Note:
Exercise:

Problem:

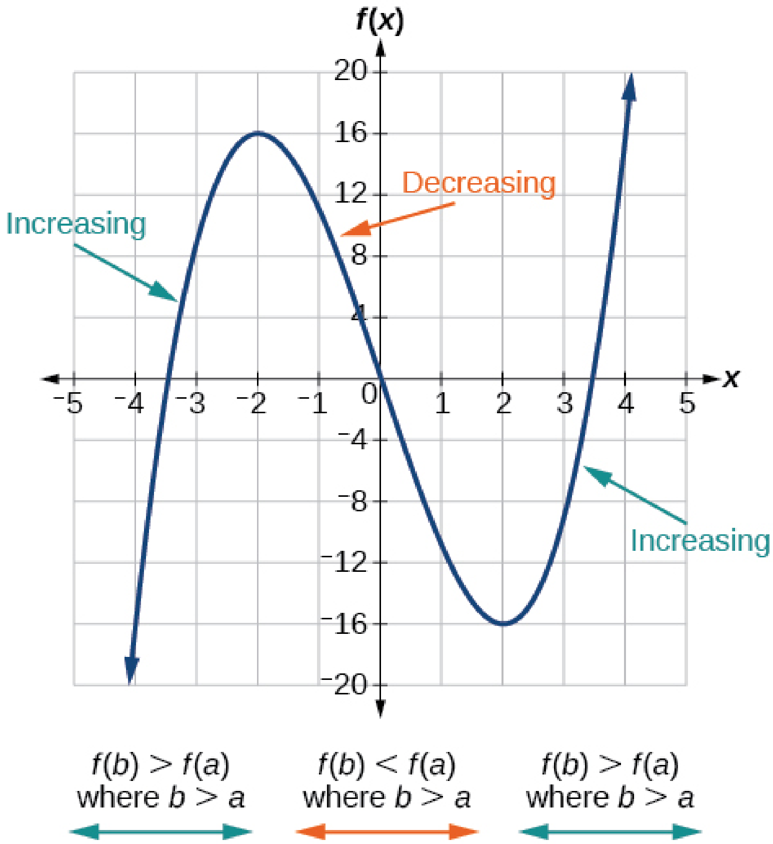
Find the average rate of change of $f(x)=x^2+2x-8$ on the interval $[5,a]$ in simplest forms in terms of a.

Solution:

$a+7$

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. [link] shows examples of increasing and decreasing intervals on a function.

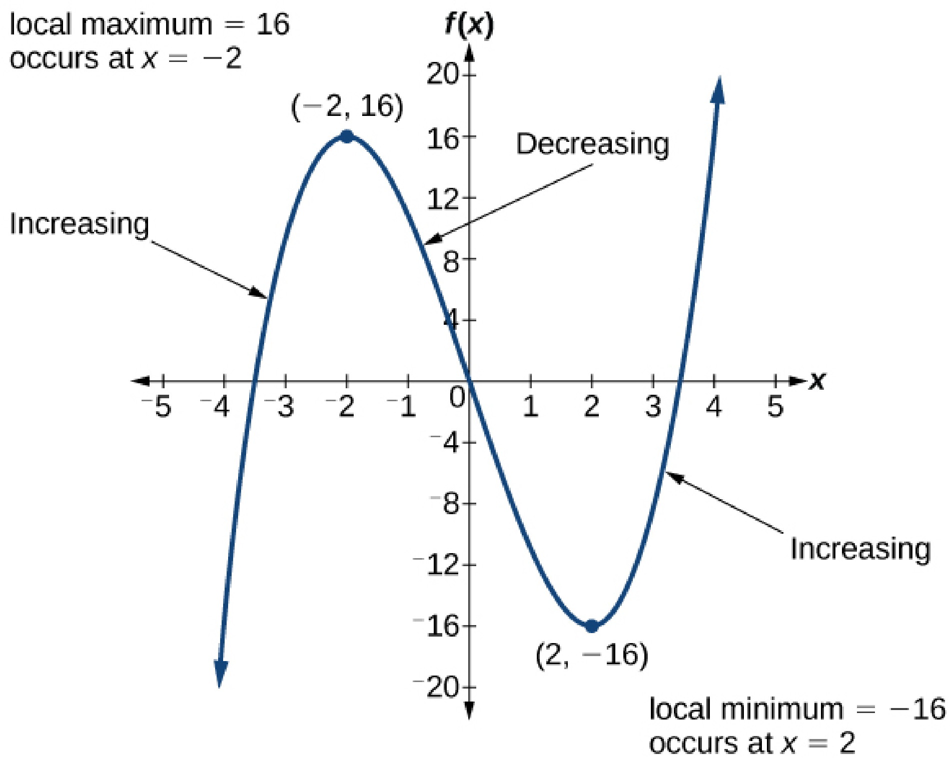


The function $f\left(x\right)={x}^{3}-12x$ is increasing on $\left(-\infty ,\text{ }-2\right)\cup \text{ }\left(2,\text{ }\infty \right)$ and is decreasing on $\left(-2,2\right)$.

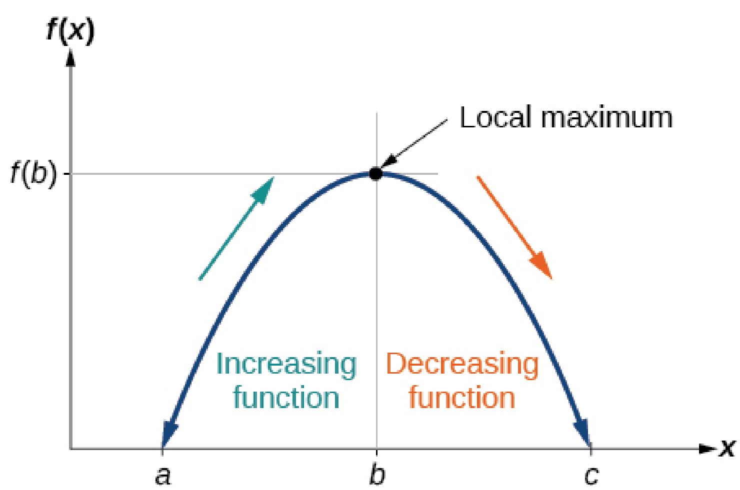
While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is “local minima.” Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is “extremum.”) Often, the term *local* is replaced by the term *relative*. In this text, we will use the term *local*.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of *local* extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function’s entire domain.

For the function whose graph is shown in [link], the local maximum is 16, and it occurs at $x=-2$. The local minimum is -16 and it occurs at $x=2$.



To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. [link] illustrates these ideas for a local maximum.



Definition of a local maximum

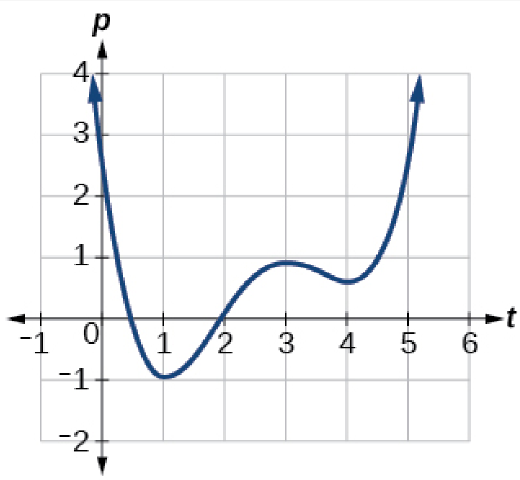
These observations lead us to a formal definition of local extrema.

Note:
Local Minima and Local Maxima
A function f is an **increasing function** on an open interval if $f(b) > f(a)$ for any two input values a and b in the given interval where $b > a$.
A function f is a **decreasing function** on an open interval if $f(b) < f(a)$ for any two input values a and b in the given interval where $b > a$.
A function f has a local maximum at $x = b$ if there exists an interval (a, c) with $a < b < c$ such that, for any x in the interval (a, c), $f(x) \leq f(b)$. Likewise, f has a local minimum at $x = b$ if there exists an interval (a, c) with $a < b < c$ such that, for any x in the interval (a, c), $f(x) \geq f(b)$.

Example:
Exercise:

Problem:
Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in [link], identify the intervals on which the function appears to be increasing.



Solution:

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval (1,3) and the interval $(4, \infty)$.

Analysis

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1, t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

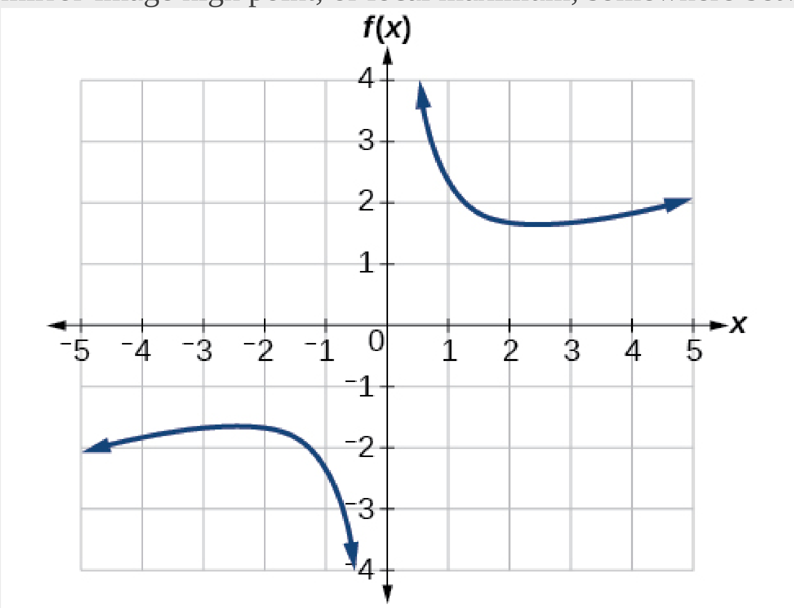
Example:
Exercise:

Problem:
Finding Local Extrema from a Graph

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

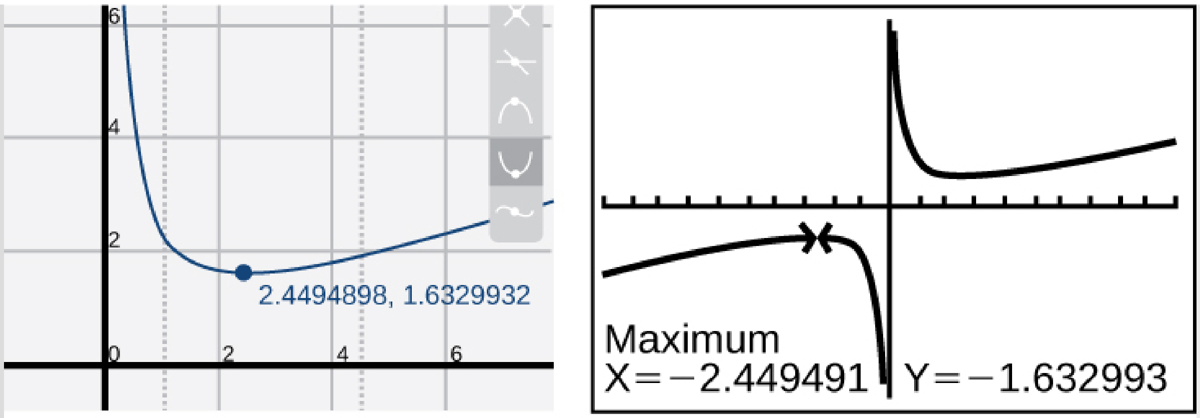
Solution:

Using technology, we find that the graph of the function looks like that in [link]. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$.



Analysis

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. [link] provides screen images from two different technologies, showing the estimate for the local maximum and minimum.



(a) (b)

Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449,\infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

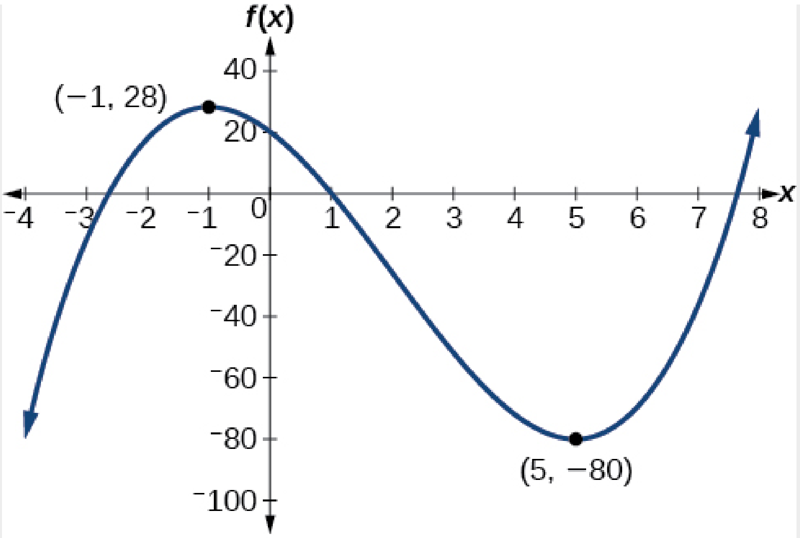
Note:
Exercise:

Problem:

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Solution:

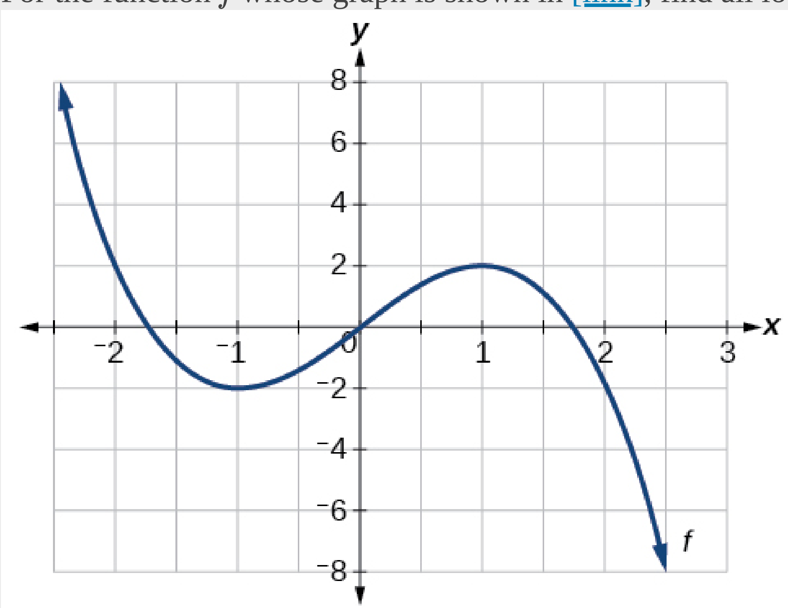
The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



Example:
Exercise:

Problem:
Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in [link], find all local maxima and minima.



Solution:

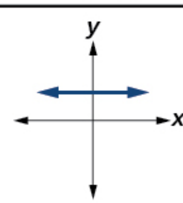
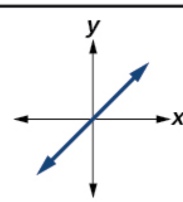
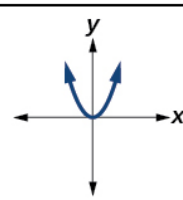
Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate

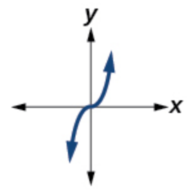
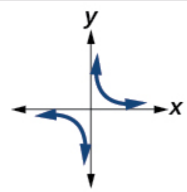
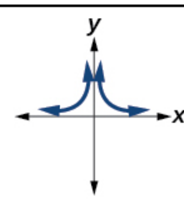
at $x = 1,$ which is $2.$

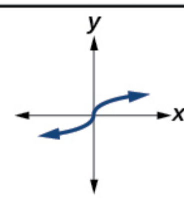
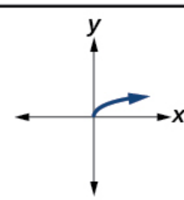
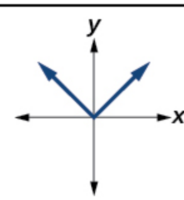
The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1.$ The local minimum is the y-coordinate at $x = -1,$ which is $-2.$

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in [link], [link], and [link].

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

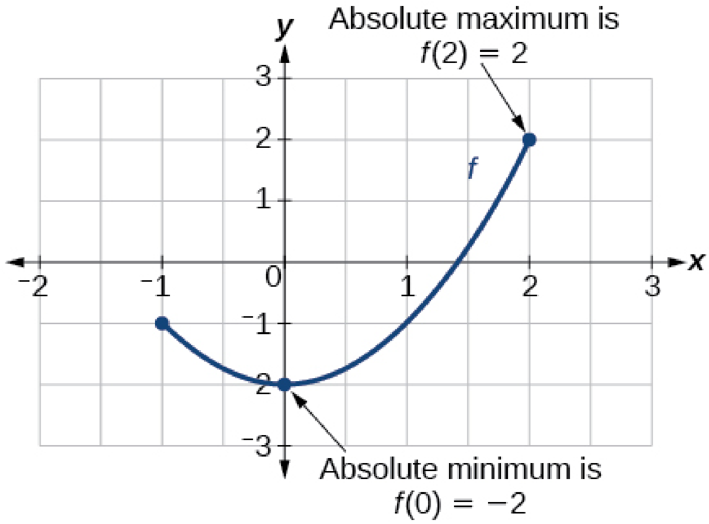
Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

Function	Increasing/Decreasing	Example
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \|x\|$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$	

Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively.

To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function. See [link].



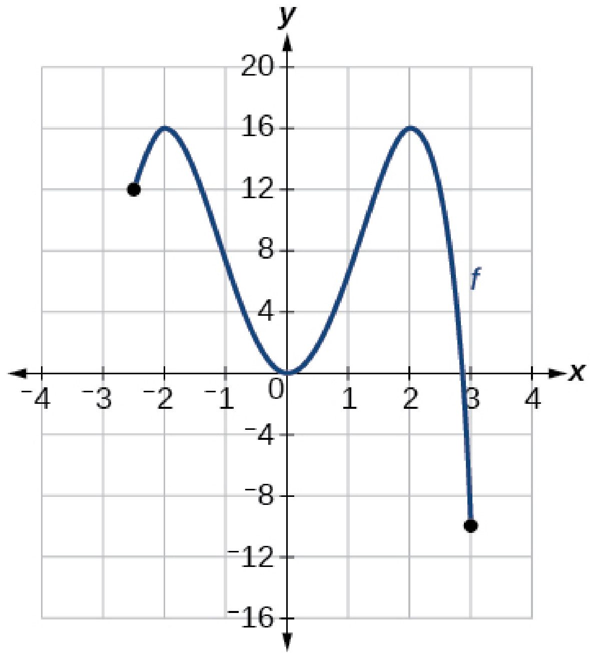
Not every function has an absolute maximum or minimum value. The toolkit function $f\left(x\right)=x^3$ is one such function.

Note:
Absolute Maxima and Minima
The **absolute maximum** of f at $x=c$ is $f\left(c\right)$ where $f\left(c\right)\ge f\left(x\right)$ for all x in the domain of $f.$
The **absolute minimum** of f at $x=d$ is $f\left(d\right)$ where $f\left(d\right)\le f\left(x\right)$ for all x in the domain of $f.$

Example:
Exercise:

Problem:
Finding Absolute Maxima and Minima from a Graph

For the function f shown in [link], find all absolute maxima and minima.



Solution:

Observe the graph of f. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the *y*-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at $x = 3$, because it is the lowest point on the domain of the function's graph. The absolute minimum is the *y*-coordinate at $x = 3$, which is -10.

Note:
Access this online resource for additional instruction and practice with rates of change.

- Average Rate of Change

Key Equations

Average rate of change	$\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See [link].
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See [link].
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See [link].
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See [link] and [link].
- The average rate of change can sometimes be determined as an expression. See [link].
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See [link].
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See [link] and [link].
- The highest and lowest points on a graph indicate the maxima and minima. See [link].

Section Exercises

Verbal

Exercise:

Problem: Can the average rate of change of a function be constant?

Solution:

Yes, the average rate of change of all linear functions is constant.

Exercise:

Problem:

If a function f is increasing on (a, b) and decreasing on (b, c), then what can be said about the local extremum of f on (a, c)?

Exercise:

Problem:

How are the absolute maximum and minimum similar to and different from the local extrema?

Solution:

The absolute maximum and minimum relate to the entire graph, whereas the local extrema relate only to a specific region around an open interval.

Exercise:

Problem:

How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?

Algebraic

For the following exercises, find the average rate of change of each function on the interval specified for real numbers b or h in simplest form.

Exercise:

Problem: $f(x) = 4x^2 - 7$ on $[1,\ b]$

Solution:

$4(b + 1)$

Exercise:

Problem: $g(x) = 2x^2 - 9$ on $[4,\ b]$

Exercise:

Problem: $p(x) = 3x + 4$ on $[2,\ 2 + h]$

Solution:

3

Exercise:

Problem: $k(x) = 4x - 2$ on $[3,\ 3 + h]$

Exercise:

Problem: $f\left(x\right)=2x^2+1$ on $\left[x,x+h\right]$

Solution:

$4x+2h$

Exercise:

Problem: $g\left(x\right)=3x^2-2$ on $\left[x,x+h\right]$

Exercise:

Problem: $a\left(t\right)=\frac{1}{t+4}$ on $\left[9,9+h\right]$

Solution:

$\frac{-1}{13(13+h)}$

Exercise:

Problem: $b\left(x\right)=\frac{1}{x+3}$ on $\left[1,1+h\right]$

Exercise:

Problem: $j\left(x\right)=3x^3$ on $\left[1,1+h\right]$

Solution:

$3h^2+9h+9$

Exercise:

Problem: $r\left(t\right)=4t^3$ on $\left[2,2+h\right]$

Exercise:

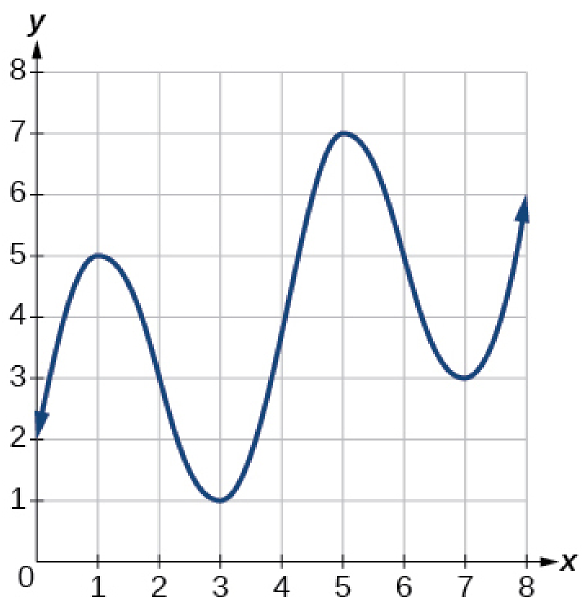
Problem: $\frac{f(x+h)-f(x)}{h}$ given $f\left(x\right)=2x^2-3x$ on $\left[x,x+h\right]$

Solution:

$4x+2h-3$

Graphical

For the following exercises, consider the graph of f shown in [link].



Exercise:

Problem: Estimate the average rate of change from $x = 1$ to $x = 4$.

Exercise:

Problem: Estimate the average rate of change from $x = 2$ to $x = 5$.

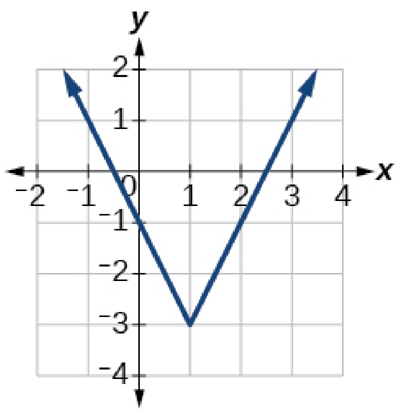
Solution:

$\frac{4}{3}$

For the following exercises, use the graph of each function to estimate the intervals on which the function is increasing or decreasing.

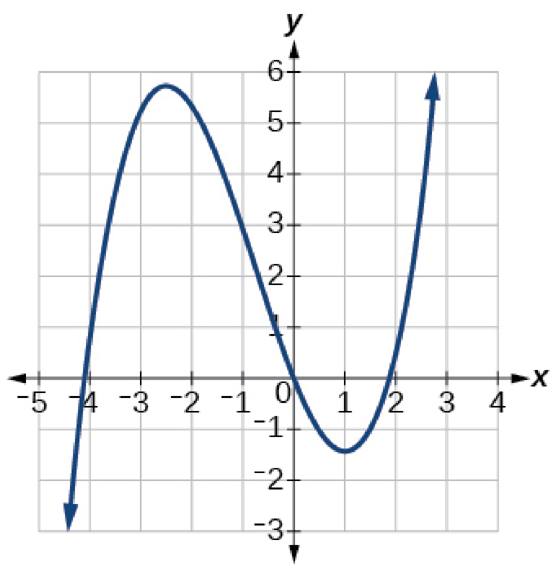
Exercise:

Problem:



Exercise:

Problem:

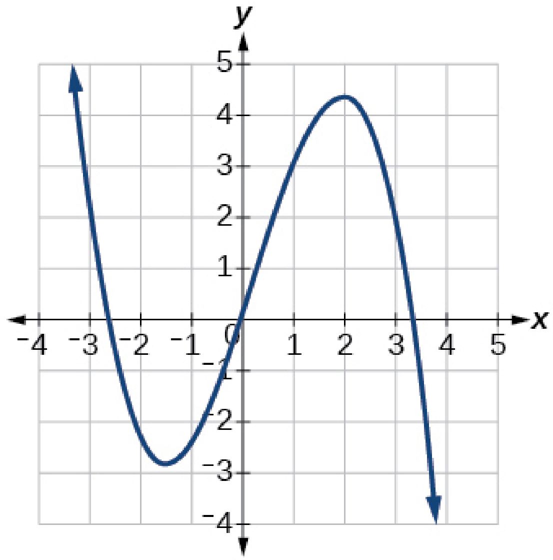


Solution:

increasing on $(-\infty, -2.5) \cup (1, \infty)$, decreasing on $(-2.5,\ 1)$

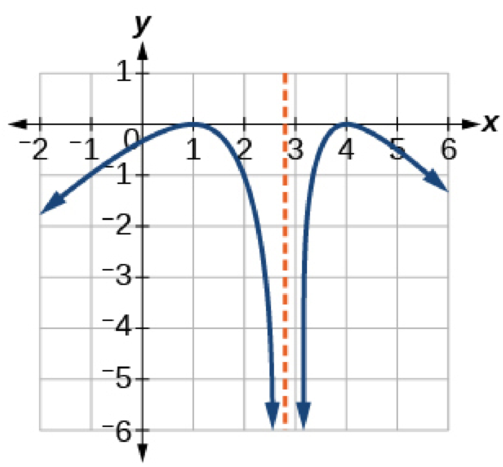
Exercise:

Problem:



Exercise:

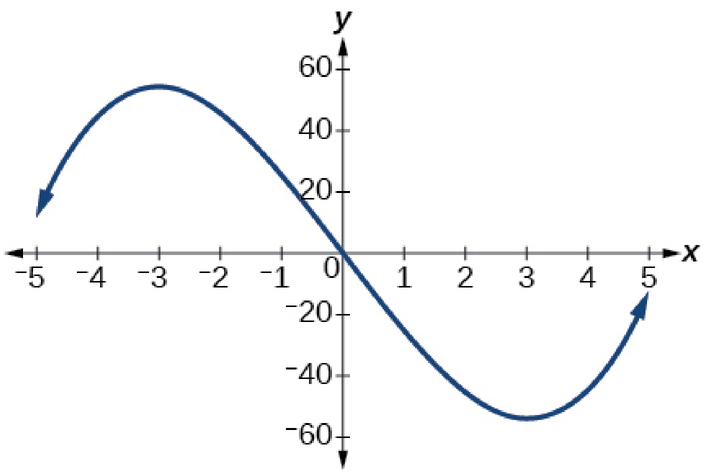
Problem:



Solution:

increasing on $(-\infty, 1) \cup (3, 4)$, decreasing on $(1, 3) \cup (4, \infty)$

For the following exercises, consider the graph shown in [link].



Exercise:

Problem: Estimate the intervals where the function is increasing or decreasing.

Exercise:

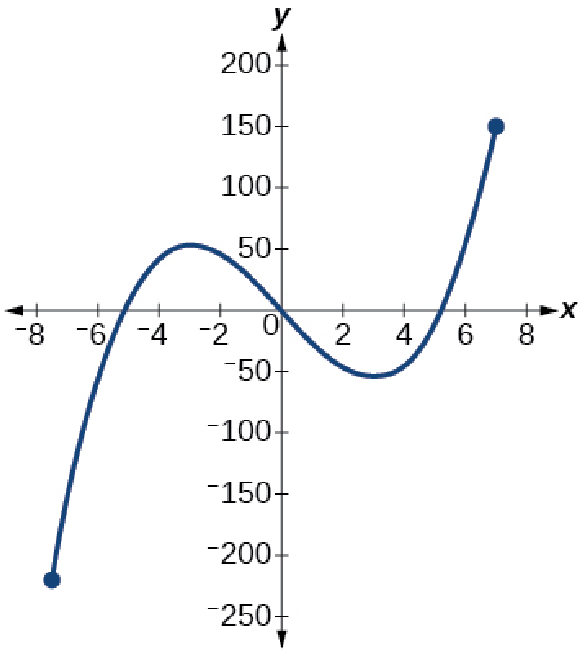
Problem:

Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

Solution:

local maximum: $(-3,\ 60)$, local minimum: $(3,\ -60)$

For the following exercises, consider the graph in [link].



Exercise:

Problem:

If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.

Exercise:

Problem:

If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

Solution:

absolute maximum at approximately $(7,\ 150)$, absolute minimum at approximately $(-7.5,\ -220)$

Numeric

Exercise:

Problem:

[link] gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

Year	Sales (millions of dollars)
1998	201
1999	219
2000	233
2001	243
2002	249
2003	251
2004	249
2005	243
2006	233

Exercise:

Problem:

[link] gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (a) between 2002 and 2004, and (b) between 2002 and 2006?

Year	Population (thousands)
2000	87
2001	84
2002	83
2003	80

Year	Population (thousands)
2004	77
2005	76
2006	78
2007	81
2008	85

Solution:

a. –3000; b. –1250

For the following exercises, find the average rate of change of each function on the interval specified.

Exercise:

Problem: $f\left(x\right)=x^{2}$ on $\left[1,\ 5\right]$

Exercise:

Problem: $h\left(x\right)=5-2x^{2}$ on $\left[-2,4\right]$

Solution:

-4

Exercise:

Problem: $q\left(x\right)=x^{3}$ on $\left[-4,2\right]$

Exercise:

Problem: $g\left(x\right)=3x^{3}-1$ on $\left[-3,3\right]$

Solution:

27

Exercise:

Problem: $y = \frac{1}{x}$ on $[1,\ 3]$

Exercise:

Problem: $p\left(t\right) = \frac{\left(t^{2}-4\right)\left(t+1\right)}{t^{2}+3}$ on $[-3, 1]$

Solution:

–0.167

Exercise:

Problem: $k\left(t\right) = 6t^{2} + \frac{4}{t^{3}}$ on $[-1, 3]$

Technology

For the following exercises, use a graphing utility to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.

Exercise:

Problem: $f\left(x\right) = x^{4} - 4x^{3} + 5$

Solution:

Local minimum at $(3, -22),$ decreasing on $(-\infty,\ 3),$ increasing on $(3,\ \infty)$

Exercise:

Problem: $h\left(x\right) = x^{5} + 5x^{4} + 10x^{3} + 10x^{2} - 1$

Exercise:

Problem: $g\left(t\right) = t\sqrt{t+3}$

Solution:

Local minimum at $(-2, -2),$ decreasing on $(-3, -2),$ increasing on $(-2,\ \infty)$

Exercise:

Problem: $k\left(t\right) = 3t^{\frac{2}{3}} - t$

Exercise:

Problem: $m\left(x\right)=x^4+2x^3-12x^2-10x+4$

Solution:

Local maximum at $\left(-0.5,\ 6\right),$ local minima at $\left(-3.25,-47\right)$ and $\left(2.1,-32\right),$ decreasing on $\left(-\infty,-3.25\right)$ and $\left(-0.5,\ 2.1\right),$ increasing on $\left(-3.25,\ \ -0.5\right)$ and $\left(2.1,\ \infty\right)$

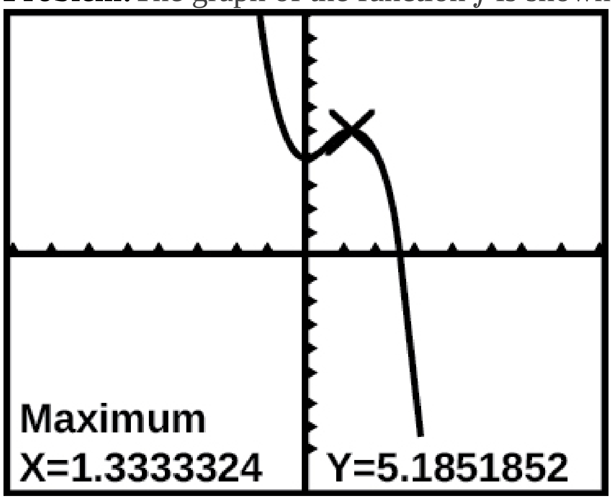
Exercise:

Problem: $n\left(x\right)=x^4-8x^3+18x^2-6x+2$

Extension

Exercise:

Problem: The graph of the function *f* is shown in [link].



Based on the calculator screen shot, the point $\left(1.333,\ 5.185\right)$ is which of the following?

A. a relative (local) maximum of the function
B. the vertex of the function
C. the absolute maximum of the function
D. a zero of the function

Solution:

A

Exercise:

Problem:

Let $f(x) = \frac{1}{x}$. Find a number c such that the average rate of change of the function f on the interval $(1, c)$ is $-\frac{1}{4}$.

Exercise:

Problem:

Let $f\left(x\right) = \frac{1}{x}$. Find the number b such that the average rate of change of f on the interval $(2, b)$ is $-\frac{1}{10}$.

Solution:

$b = 5$

Real-World Applications

Exercise:

Problem:

At the start of a trip, the odometer on a car read 21,395. At the end of the trip, 13.5 hours later, the odometer read 22,125. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?

Exercise:

Problem:

A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped 10.7 gallons. What is the average rate of flow of the gasoline into the gas tank?

Solution:

2.7 gallons per minute

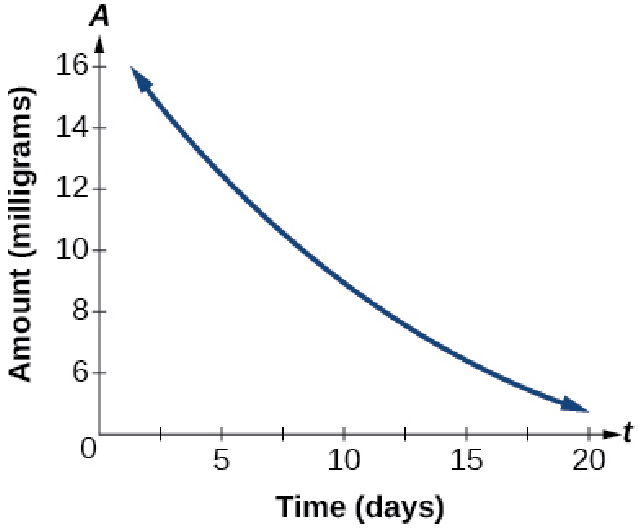
Exercise:

Problem:

Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where t is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.

Exercise:

Problem: The graph in [link] illustrates the decay of a radioactive substance over t days.



Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

Solution:

approximately –0.6 milligrams per day

Glossary

absolute maximum
: the greatest value of a function over an interval

absolute minimum
: the lowest value of a function over an interval

average rate of change
: the difference in the output values of a function found for two values of the input divided by the difference between the inputs

decreasing function

a function is decreasing in some open interval if $f(b) < f(a)$ for any two input values a and b in the given interval where $b > a$

increasing function
: a function is increasing in some open interval if $f(b) > f(a)$ for any two input values a and b in the given interval where $b > a$

local extrema
: collectively, all of a function's local maxima and minima

local maximum
: a value of the input where a function changes from increasing to decreasing as the input value increases.

local minimum
: a value of the input where a function changes from decreasing to increasing as the input value increases.

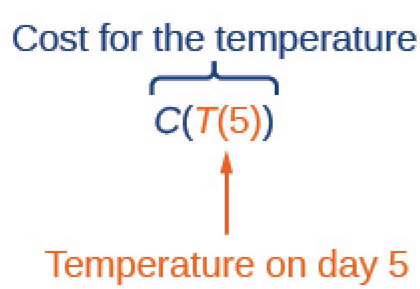
rate of change
: the change of an output quantity relative to the change of the input quantity

Composition of Functions
In this section, you will:

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C\left(T\right)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T\left(d\right)$ gives the average daily temperature on day d of the year. For any given day, $\text{Cost}=C\left(T\left(d\right)\right)$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T\left(d\right).$ For example, we could evaluate $T\left(5\right)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write $C\left(T\left(5\right)\right).$



By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

Equation:

$$T\left(y\right)=h\left(y\right)+w\left(y\right)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

Equation:

$$T=h+w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f\left(x\right)$ and $g\left(x\right)$ with real number outputs, we define new functions $f+g,$ $f-g,$ $fg,$ and $\frac{f}{g}$ by the relations

Equation:

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x) \\
(f-g)(x) &= f(x)-g(x) \\
(fg)(x) &= f(x)g(x) \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \qquad \text{where } g(x) \neq 0
\end{aligned}$$

Example:
Exercise:

Problem:
Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

Solution:

Begin by writing the general form, and then substitute the given functions.

Equation:

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ &= x^2-x\\ &= x(x-1)\end{aligned}$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)}\\ \left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ &= \frac{(x+1)(x-1)}{x-1} && \text{where } x\neq 1\\ &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x\neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

Note:
Exercise:

Problem: Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.
Equation:

$$f(x)=x-1 \quad \text{and} \quad g(x)=x^2-1$$

Are they the same function?

Solution:

$$(fg)(x) = f(x)g(x) = (x-1)\left(x^2-1\right) = x^3 - x^2 - x + 1$$
$$(f-g)(x) = f(x) - g(x) = (x-1) - \left(x^2-1\right) = x - x^2$$

No, the functions are not the same.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a composition of functions**.** The resulting function is known as a **composite function**. We represent this combination by the following notation:

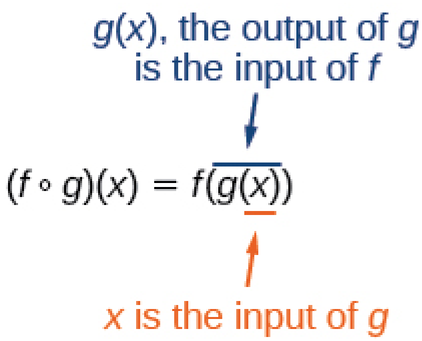
Equation:

$$(f \circ g)(x) = f(g(x))$$

We read the left-hand side as“f composed with g at x ,” and the right-hand side as“f of g of x.”The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and

yields an output $g\left(x\right).$ Then the function f takes $g\left(x\right)$ as an input and yields an output $f\left(g\left(x\right)\right).$



In general, $f\circ g$ and $g\circ f$ are different functions. In other words, in many cases $f\left(g\left(x\right)\right)\ne g\left(f\left(x\right)\right)$ for all $x.$ We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f\left(x\right)={x}^{2}$ and $g\left(x\right)=x+2$,then

Equation:

$$\begin{array}{ccc}f(g(x))& =& f(x+2)\\ & =& (x+2)^{2}\\ & =& x^{2}+4x+4\end{array}$$

but

Equation:

$$\begin{array}{ccc}g(f(x))& =& g\left(x^{2}\right)\\ & =& x^{2}+2\end{array}$$

These expressions are not equal for all values of $x,$ so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x=-\frac{1}{2}.$

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the

composition has to make sense in terms of inputs and outputs.

Note:
Composition of Functions
When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that
Equation:

$$(f \circ g)(x) = f(g(x))$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.
It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example:
Exercise:

Problem:
Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is commutative.
Equation:

$$f(x) = 2x + 1 \qquad g(x) = 3 - x$$

Solution:

Let’s begin by substituting $g(x)$ into $f(x)$.
Equation:

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.
Equation:

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= -2x+2 \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Example:
Exercise:

Problem:
Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution:

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t = 3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in

3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example:
Exercise:

Problem:
Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f\left(g(y)\right)$ or $g\left(f(x)\right)$?

Solution:

The function $y = f\left(x\right)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.
Equation:

$$\text{number of miles } = f\left(\text{number of hours}\right)$$

The function $g\left(y\right)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:
Equation:

$$\text{number of gallons } = g\left(\text{number of miles}\right)$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f\left(g(y)\right)$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as

the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g\left(f(x)\right)$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Note:
Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?
Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

Note:
Exercise:

Problem:

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Solution:

A gravitational force is still a force, so $a\left(G(r)\right)$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G\left(a(F)\right)$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example:
Exercise:

Problem:
Using a Table to Evaluate a Composite Function

Using [link], evaluate $f(g(3))$ and $g(f(3))$.

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2

x	$f(x)$	$g(x)$
4	1	7

Solution:

To evaluate $f(g(3)\ ),$ we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function $g: \ g(3) = 2.$ We can then use that result as the input to the function $f,$ so $g(3)$ is replaced by 2 and we get $f(2).$ Then, using the table that defines the function $f,$ we find that $f(2) = 8.$

Equation:

$$\begin{aligned} g(3) &= 2 \\ f(g(3)) &= f(2) = 8 \end{aligned}$$

To evaluate $g(f(3)\ ),$ we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3.$ Then, using the table for $g,$ we can evaluate

Equation:

$$g(f(3)) = g(3) = 2$$

[link] shows the composite functions $f \circ g$ and $g \circ f$ as tables.

x	$g\left(x\right)$	$f\left(g\left(x\right)\right)$	$f\left(x\right)$	$g\left(f\left(x\right)\right)$
3	2	8	3	2

Note:
Exercise:

Problem: Using [link], evaluate $f(g(1))$ and $g(f(4))$.

Solution:

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

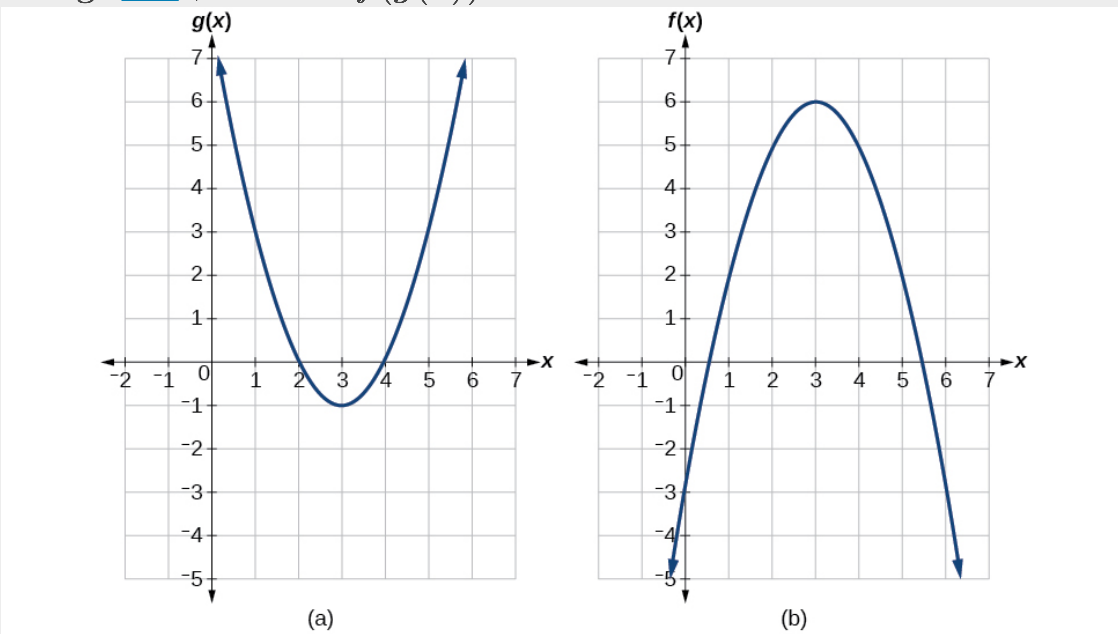
Note:
Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example:
Exercise:

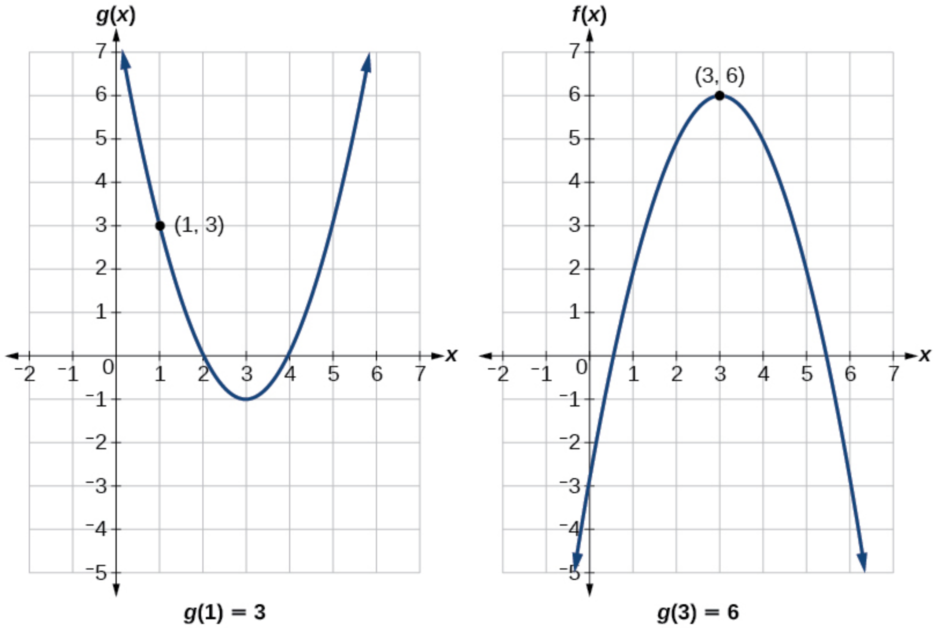
Problem:
Using a Graph to Evaluate a Composite Function

Using [link], evaluate $f(g(1))$.



Solution:

To evaluate $f(g(1))$, we start with the inside evaluation. See [link].



We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

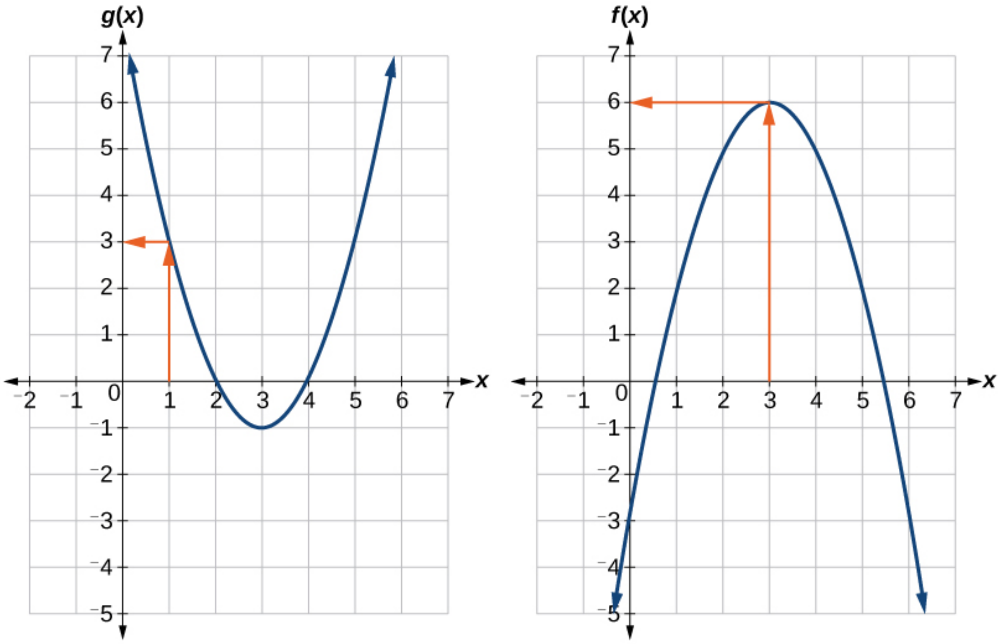
Equation:

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

[link] shows how we can mark the graphs with arrows to trace the path from the input value to the output value.



Note:
Exercise:

Problem: Using [link], evaluate $g(f(2))$.

Solution:

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f\left(g\left(x\right)\right).$ To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t)=t^2-t,$ we substitute the value inside the parentheses into the formula wherever we see the input variable.

Note:
Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example:
Exercise:

Problem:
Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t)=t^2-t$ and $h(x)=3x+2,$ evaluate $f(h(1)).$

Solution:

Because the inside expression is $h(1),$ we start by evaluating $h(x)$ at 1.

Equation:

$$\begin{array}{ccc}h(1) & = & 3(1)+2\\ h(1) & = & 5\end{array}$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.
Equation:

$$\begin{aligned} f(h(1)) &= f(5) \\ f(h(1)) &= 5^2 - 5 \\ f(h(1)) &= 20 \end{aligned}$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Note:
Exercise:

Problem: Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Solution:

a. 8; b. 20

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite

function for an input x as $f\left(g\left(x\right)\right),$ we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g\left(x\right)$ must be a member of the domain of $f,$ otherwise the second function evaluation in $f\left(g\left(x\right)\right)$ cannot be completed, and the expression is still undefined. Thus the domain of $f\circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of $f.$ Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g\left(x\right)$ is in the domain of $f.$

Note:
Domain of a Composite Function
The domain of a composite function $f\left(g\left(x\right)\right)$ is the set of those inputs x in the domain of g for which $g\left(x\right)$ is in the domain of $f.$

Note:
Given a function composition $f(g(x))$,determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g\left(x\right)$ is in the domain of $f.$ That is, exclude those inputs x from the domain of g for which $g\left(x\right)$ is not in the domain of $f.$ The resulting set is the domain of $f\circ g$.

Example:
Exercise:

Problem:
Finding the Domain of a Composite Function

Find the domain of
Equation:

$$(f \circ g)(x) \quad \text{where} \quad f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

Solution:

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.
Equation:

$$\begin{aligned} \tfrac{4}{3x-2} &= 1 \\ 4 &= 3x - 2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that
Equation:

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as
Equation:

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

Example:
Exercise:

Problem:
Finding the Domain of a Composite Function Involving Radicals

Find the domain of
Equation:

$$(f \circ g)(x) \;\text{ where }\; f(x) = \sqrt{x+2} \;\text{ and }\; g(x) = \sqrt{3-x}$$

Solution:

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function
Equation:

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, $\sqrt{3-x}+2 \geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $\sqrt{3-x} \geq 0$, or, $3-x \geq 0$, which gives a domain of (-$\infty, 3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Note:

Exercise:

Problem: Find the domain of
Equation:

$$(f \circ g)(x) \text{ where } f(x) = \frac{1}{x-2} \text{ and } g(x) = \sqrt{x+4}$$

Solution:

$[-4, 0) \cup (0, \infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

Example:
Exercise:

Problem:
Decomposing a Function

Write $f(x) = \sqrt{5 - x^2}$ as the composition of two functions.

Solution:

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$.

As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as

Equation:

$$h(x) = 5 - x^2 \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.

Equation:

$$g(h(x)) = g\left(5 - x^2\right) = \sqrt{5 - x^2}$$

Note:

Exercise:

Problem: Write $f(x) = \frac{4}{3 - \sqrt{4 + x^2}}$ as the composition of two functions.

Solution:

Possible answer:

$g\left(x\right) = \sqrt{4 + x^2}$
$h\left(x\right) = \frac{4}{3 - x}$
$f = h \circ g$

Note:

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions

- Composite Function Notation Application
- Composite Functions Using Graphs
- Decompose Functions
- Composite Function Values

Key Equation

Composite function	$(f \circ g)(x) = f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See [link].
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See [link] and [link].
- The order of function composition must be considered when interpreting the meaning of composite functions. See [link].
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See [link].
- A composite function can be evaluated from a graph. See [link].
- A composite function can be evaluated from a formula. See [link].
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See [link] and [link].

- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See [link].

Section Exercises

Verbal

Exercise:

Problem:

How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

Solution:

Find the numbers that make the function in the denominator g equal to zero, and check for any other domain restrictions on f and g, such as an even-indexed root or zeros in the denominator.

Exercise:

Problem: What is the composition of two functions, $f \circ g$?

Exercise:

Problem:

If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

Solution:

Yes. Sample answer: Let $f(x) = x + 1$ and $g(x) = x - 1$. Then $f(g(x)) = f(x - 1) = (x - 1) + 1 = x$ and $g(f(x)) = g(x + 1) = (x + 1) - 1 = x$. So $f \circ g = g \circ f$.

Exercise:

Problem:

How do you find the domain for the composition of two functions, $f \circ g$?

Algebraic

For the following exercises, determine the domain for each function in interval notation.

Exercise:

Problem:

Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$,find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$.

Solution:

$(f+g)(x) = 2x + 6,$ domain: $(-\infty, \infty)$

$(f-g)(x) = 2x^2 + 2x - 6,$ domain: $(-\infty, \infty)$

$(fg)(x) = -x^4 - 2x^3 + 6x^2 + 12x,$ domain: $(-\infty, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{x^2+2x}{6-x^2},$ domain: $(-\infty, -\sqrt{6}) \cup (-\sqrt{6}, \sqrt{6}) \cup (\sqrt{6}, \infty)$

Exercise:

Problem:

Given $f(x) = -3x^2 + x$ and $g(x) = 5,$ find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$.

Exercise:

Problem:

Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$,find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$.

Solution:

$(f + g)\,(x) = \frac{4x^3+8x^2+1}{2x}$, domain: $(-\infty, 0) \cup (0, \infty)$

$(f - g)\,(x) = \frac{4x^3+8x^2-1}{2x}$, domain: $(-\infty, 0) \cup (0, \infty)$

$(fg)\,(x) = x + 2,$ domain: $(-\infty, 0) \cup (0, \infty)$

$\left(\frac{f}{g}\right)(x) = 4x^3 + 8x^2,$ domain: $(-\infty, 0) \cup (0, \infty)$

Exercise:

Problem:

Given $f(x) = \frac{1}{x-4}$ and $g(x) = \frac{1}{6-x}$, find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$.

Exercise:

Problem:

Given $f(x) = 3x^2$ and $g(x) = \sqrt{x - 5}$, find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$.

Solution:

$(f + g)(x) = 3x^2 + \sqrt{x - 5},$ domain: $[5, \infty)$

$(f - g)(x) = 3x^2 - \sqrt{x - 5},$ domain: $[5, \infty)$

$(fg)(x) = 3x^2\sqrt{x - 5},$ domain: $[5, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{3x^2}{\sqrt{x-5}}$, domain: $(5, \infty)$

Exercise:

Problem: Given $f(x) = \sqrt{x}$ and $g(x) = |x - 3|$, find $\frac{g}{f}$.

Exercise:

Problem:

For the following exercise, find the indicated function given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$.

a. $f(g(2))$
b. $f(g(x))$
c. $g(f(x))$
d. $(g \circ g)(x)$
e. $(f \circ f)(-2)$

Solution:

a. 3; b. $f(g(x)) = 2(3x - 5)^2 + 1$; c. $f(g(x)) = 6x^2 - 2$; d. $(g \circ g)(x) = 3(3x - 5) - 5 = 9x - 20$; e. $(f \circ f)(-2) = 163$

For the following exercises, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.

Exercise:

Problem: $f(x) = x^2 + 1,\ g(x) = \sqrt{x + 2}$

Exercise:

Problem: $f(x) = \sqrt{x} + 2,\ g(x) = x^2 + 3$

Solution:

$$f(g(x)) = \sqrt{x^2+3}+2,\ g(f(x)) = x+4\sqrt{x}+7$$

Exercise:

Problem: $f(x) = |x|,\ g(x) = 5x+1$

Exercise:

Problem: $f(x) = \sqrt[3]{x},\ g(x) = \frac{x+1}{x^3}$

Solution:

$$f(g(x)) = \sqrt[3]{\frac{x+1}{x^3}} = \frac{\sqrt[3]{x+1}}{x},\ g(f(x)) = \frac{\sqrt[3]{x}+1}{x}$$

Exercise:

Problem: $f(x) = \frac{1}{x-6},\ g(x) = \frac{7}{x}+6$

Exercise:

Problem: $f(x) = \frac{1}{x-4},\ g(x) = \frac{2}{x}+4$

Solution:

$$(f \circ g)(x) = \frac{1}{\frac{2}{x}+4-4} = \frac{x}{2},\ \ (g \circ f)(x) = 2x-4$$

For the following exercises, use each set of functions to find $f\left(g\left(h(x)\right)\right)$. Simplify your answers.

Exercise:

Problem: $f(x) = x^4+6,\ g(x) = x-6,$ and $h(x) = \sqrt{x}$

Exercise:

Problem: $f(x) = x^2+1,\ g(x) = \frac{1}{x},$ and $h(x) = x+3$

Solution:

$f(g(h(x))) = \left(\frac{1}{x+3}\right)^2 + 1$

Exercise:

Problem: Given $f(x) = \frac{1}{x}$ and $g(x) = x - 3,$ find the following:

a. $(f \circ g)(x)$
b. the domain of $(f \circ g)(x)$ in interval notation
c. $(g \circ f)(x)$
d. the domain of $(g \circ f)(x)$
e. $\left(\frac{f}{g}\right)x$

Exercise:

Problem:

Given $f(x) = \sqrt{2 - 4x}$ and $g(x) = -\frac{3}{x},$ find the following:

a. $(g \circ f)(x)$
b. the domain of $(g \circ f)(x)$ in interval notation

Solution:

a. $(g \circ f)(x) = -\frac{3}{\sqrt{2-4x}}$; b. $\left(-\infty, \frac{1}{2}\right)$

Exercise:

Problem:

Given the functions $f(x) = \frac{1-x}{x}$ and $g(x) = \frac{1}{1+x^2}$,find the following:

a. $(g \circ f)(x)$
b. $(g \circ f)(2)$

Exercise:

Problem:

Given functions $p(x) = \frac{1}{\sqrt{x}}$ and $m(x) = x^2 - 4,$ state the domain of each of the following functions using interval notation:

a. $\frac{p(x)}{m(x)}$
b. $p(m(x))$
c. $m(p(x))$

Solution:

a. $(0, 2) \cup (2, \infty);$ b. $(-\infty, -2) \cup (2, \infty);$ c. $(0, \infty)$

Exercise:

Problem:

Given functions $q(x) = \frac{1}{\sqrt{x}}$ and $h(x) = x^2 - 9,$ state the domain of each of the following functions using interval notation.

a. $\frac{q(x)}{h(x)}$
b. $q\left(h(x)\right)$
c. $h\left(q(x)\right)$

Exercise:

Problem:

For $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x - 1},$ write the domain of $(f \circ g)(x)$ in interval notation.

Solution:

$(1, \infty)$

For the following exercises, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x) = f\left(g(x)\right)$.

Exercise:

Problem: $h(x) = (x+2)^2$

Exercise:

Problem: $h(x) = (x-5)^3$

Solution:

sample: $f(x) = x^3$
$g(x) = x - 5$

Exercise:

Problem: $h(x) = \frac{3}{x-5}$

Exercise:

Problem: $h(x) = \frac{4}{(x+2)^2}$

Solution:

sample: $f(x) = \frac{4}{x}$
$g(x) = (x+2)^2$

Exercise:

Problem: $h(x) = 4 + \sqrt[3]{x}$

Exercise:

Problem: $h(x) = \sqrt[3]{\frac{1}{2x-3}}$

Solution:

sample: $\begin{array}{l} f(x) = \sqrt[3]{x} \\ g(x) = \frac{1}{2x-3} \end{array}$

Exercise:

Problem: $h(x) = \frac{1}{(3x^2-4)^{-3}}$

Exercise:

Problem: $h(x) = \sqrt[4]{\frac{3x-2}{x+5}}$

Solution:

sample: $\begin{array}{l} f(x) = \sqrt[4]{x} \\ g(x) = \frac{3x-2}{x+5} \end{array}$

Exercise:

Problem: $h(x) = \left(\frac{8+x^3}{8-x^3}\right)^4$

Exercise:

Problem: $h(x) = \sqrt{2x+6}$

Solution:

sample: $\begin{array}{l} f(x) = \sqrt{x} \\ g(x) = 2x+6 \end{array}$

Exercise:

Problem: $h(x) = (5x-1)^3$

Exercise:

Problem: $h(x) = \sqrt[3]{x-1}$

Solution:

sample: $\begin{array}{l} f(x) = \sqrt[3]{x} \\ g(x) = (x-1) \end{array}$

Exercise:

Problem: $h(x) = \left|x^2+7\right|$

Exercise:

Problem: $h(x) = \frac{1}{(x-2)^3}$

Solution:

sample: $\begin{array}{l} f(x) = x^3 \\ g(x) = \frac{1}{x-2} \end{array}$

Exercise:

Problem: $h(x) = \left(\frac{1}{2x-3}\right)^2$

Exercise:

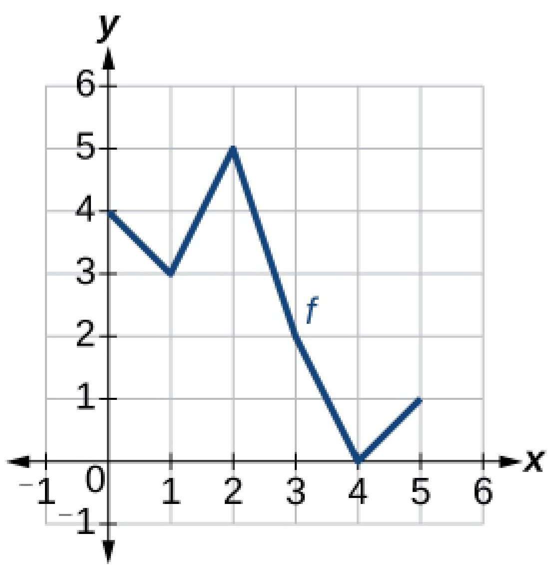
Problem: $h(x) = \sqrt{\frac{2x-1}{3x+4}}$

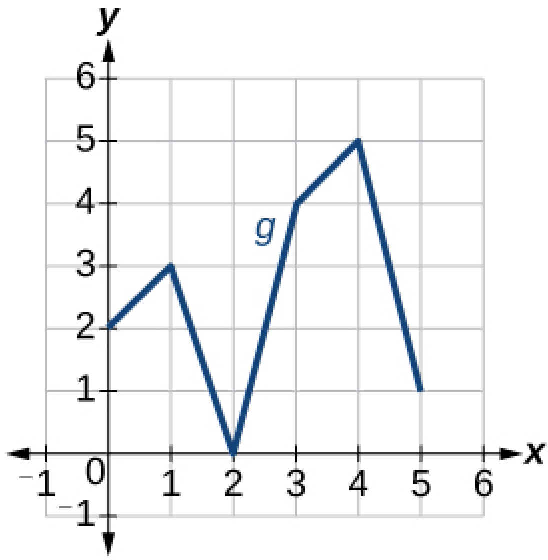
Solution:

sample: $f(x) = \sqrt{x}$
$g(x) = \frac{2x-1}{3x+4}$

Graphical

For the following exercises, use the graphs of f,shown in [link], and g, shown in [link], to evaluate the expressions.





Exercise:

Problem: $f\left(g(3)\right)$

Exercise:

Problem: $f\left(g(1)\right)$

Solution:

2

Exercise:

Problem: $g\left(f(1)\right)$

Exercise:

Problem: $g\left(f(0)\right)$

Solution:

5

Exercise:

Problem: $f\left(f(5)\right)$

Exercise:

Problem: $f\left(f(4)\right)$

Solution:

4

Exercise:

Problem: $g\left(g(2)\right)$

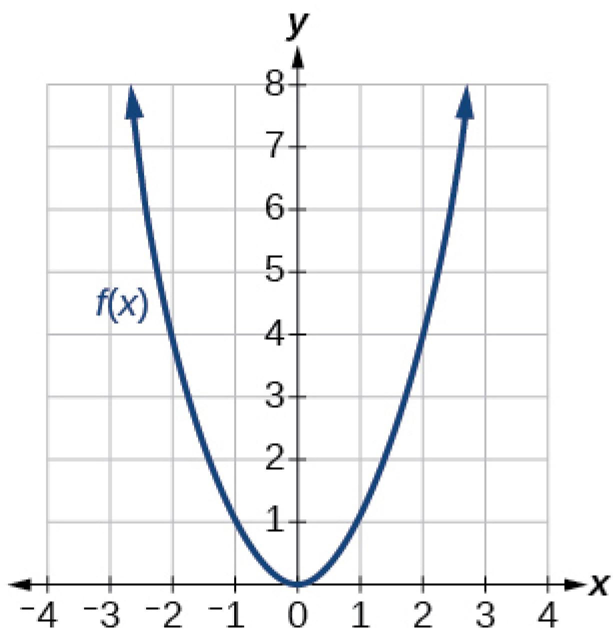
Exercise:

Problem: $g\left(g(0)\right)$

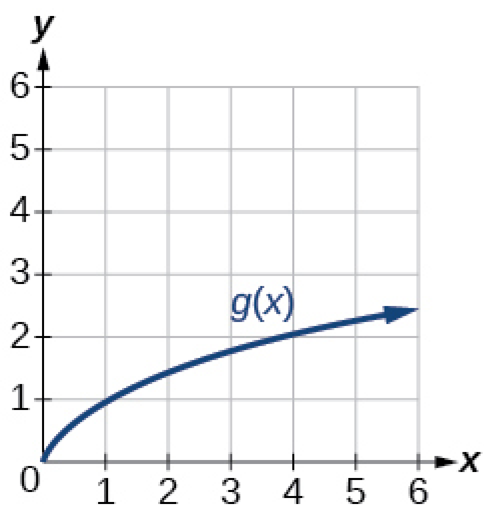
Solution:

0

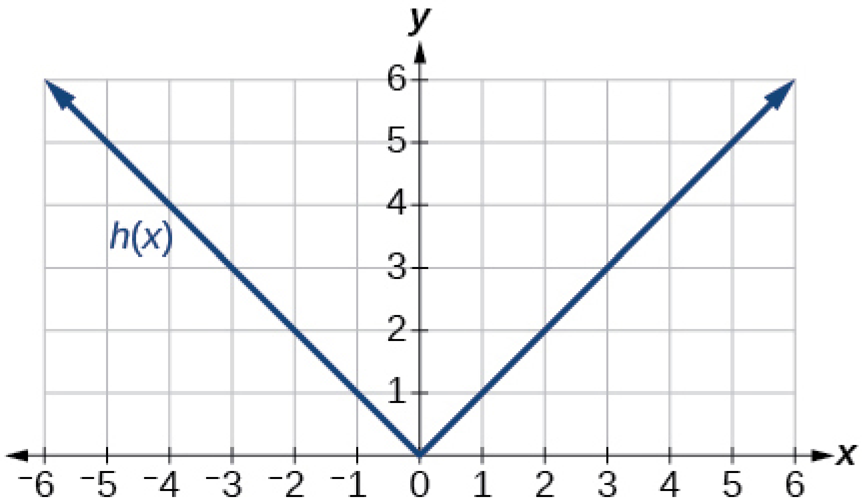
For the following exercises, use graphs of $f(x)$,shown in [link], $g(x)$,shown in [link], and $h(x)$,shown in [link], to evaluate the expressions.



y
x
8
7
6
5
4
3
2
1
−4
−3
−2
−1
0
1
2
3
4
f(x)



y
x
6
5
4
3
2
1
0
1
2
3
4
5
6
g(x)



Exercise:

Problem: $g\left(f\left(1\right)\right)$

Exercise:

Problem: $g\left(f\left(2\right)\right)$

Solution:

2

Exercise:

Problem: $f\left(g\left(4\right)\right)$

Exercise:

Problem: $f\left(g\left(1\right)\right)$

Solution:

1

Exercise:

Problem: $f\left(h\left(2\right)\right)$

Exercise:

Problem: $h\left(f\left(2\right)\right)$

Solution:

4

Exercise:

Problem: $f\left(g\left(h\left(4\right)\right)\right)$

Exercise:

Problem: $f\left(g\left(f\left(-2\right)\right)\right)$

Solution:

4

Numeric

For the following exercises, use the function values for f and g shown in [link] to evaluate each expression.

x	$f(x)$	$g(x)$
0	7	9
1	6	5

x	$f(x)$	$g(x)$
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

Exercise:

Problem: $f\left(g\left(8\right)\right)$

Exercise:

Problem: $f\left(g\left(5\right)\right)$

Solution:

9

Exercise:

Problem: $g\left(f\left(5\right)\right)$

Exercise:

Problem: $g\left(f\left(3\right)\right)$

Solution:

4

Exercise:

Problem: $f\left(f\left(4\right)\right)$

Exercise:

Problem: $f\left(f\left(1\right)\right)$

Solution:

2

Exercise:

Problem: $g\left(g\left(2\right)\right)$

Exercise:

Problem: $g\left(g\left(6\right)\right)$

Solution:

3

For the following exercises, use the function values for f and g shown in [link] to evaluate the expressions.

x	$f(x)$	$g(x)$
-3	11	-8
-2	9	-3
-1	7	0
0	5	1
1	3	0
2	1	-3
3	-1	-8

Exercise:

Problem: $(f \circ g)(1)$

Exercise:

Problem: $(f \circ g)(2)$

Solution:

11

Exercise:

Problem: $(g \circ f)(2)$

Exercise:

Problem: $(g \circ f)(3)$

Solution:

0

Exercise:

Problem: $(g \circ g)(1)$

Exercise:

Problem: $(f \circ f)(3)$

Solution:

7

For the following exercises, use each pair of functions to find $f\left(g\left(0\right)\right)$ and $g\left(f(0)\right)$.

Exercise:

Problem: $f(x) = 4x + 8,\ g(x) = 7 - x^2$

Exercise:

Problem: $f(x) = 5x + 7,\ g(x) = 4 - 2x^2$

Solution:

$f(g(0)) = 27,\ g\left(f(0)\right) = -94$

Exercise:

Problem: $f(x) = \sqrt{x + 4},\ g(x) = 12 - x^3$

Exercise:

Problem: $f(x) = \frac{1}{x+2},\ g(x) = 4x + 3$

Solution:

$f(g(0)) = \frac{1}{5}$, $g(f(0)) = 5$

For the following exercises, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.

Exercise:

Problem: $f\left(g(2)\right)$

Exercise:

Problem: $f\left(g(x)\right)$

Solution:

$18x^2 + 60x + 51$

Exercise:

Problem: $g\left(f(-3)\right)$

Exercise:

Problem: $(g \circ g)(x)$

Solution:

$g \circ g(x) = 9x + 20$

Extensions

For the following exercises, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x - 1}$.

Exercise:

Problem: Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.

Exercise:

Problem: Find $(f \circ g)(2)$ and $(g \circ f)(2)$.

Solution:

2

Exercise:

Problem: What is the domain of $(g \circ f)(x)$?

Exercise:

Problem: What is the domain of $(f \circ g)(x)$?

Solution:

$(-\infty, \infty)$

Exercise:

Problem: Let $f(x) = \frac{1}{x}$.

a. Find $(f \circ f)(x)$.
b. Is $(f \circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x+1$.

Exercise:

Problem: True or False: $(g \circ f)(x) = F(x)$.

Solution:

False

Exercise:

Problem: True or False: $(f \circ g)(x) = F(x)$.

For the following exercises, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.

Exercise:

Problem: $(f \circ g)(6)$; $(g \circ f)(6)$

Solution:

$(f \circ g)(6) = 6$; $(g \circ f)(6) = 6$

Exercise:

Problem: $(g \circ f)(a)$; $(f \circ g)(a)$

Exercise:

Problem: $(f \circ g)(11)$; $(g \circ f)(11)$

Solution:

$(f \circ g)(11) = 11$, $(g \circ f)(11) = 11$

Real-World Applications

Exercise:

Problem:

The function $D(p)$ gives the number of items that will be demanded when the price is $p.$ The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is $6, you would do which of the following?

a. Evaluate $D\left(C(6)\right).$
b. Evaluate $C\left(D(6)\right).$
c. Solve $D\left(C(x)\right)=6.$
d. Solve $C\left(D(p)\right)=6.$

Exercise:

Problem:

The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t).$ Which of the following would you do in order to determine when the patient will be at a pain level of 4?

a. Evaluate $A\left(m(4)\right).$
b. Evaluate $m\left(A(4)\right).$
c. Solve $A\left(m(t)\right)=4.$
d. Solve $m\left(A(d)\right)=4.$

Solution:

c

Exercise:

Problem:

A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price $x.$ (Hint: Use function composition to find your answer.)

Exercise:

Problem:

A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t)=25\sqrt{t+2},$ find the area of the ripple as a function of time. Find the area of the ripple at $t=2.$

Solution:

$A(t)=\pi\left(25\sqrt{t+2}\right)^2$ and $A(2)=\pi\left(25\sqrt{4}\right)^2=2500\pi$ square inches

Exercise:

Problem:

A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t)=2t+1,$ express the area burned as a function of time, t (minutes).

Exercise:

Problem:

Use the function you found in the previous exercise to find the total area burned after 5 minutes.

Solution:

$A(5) = \pi(2(5)+1)^2 = 121\pi$ square units

Exercise:

Problem:

The radius $r,$ in inches, of a spherical balloon is related to the volume, $V,$ by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}.$ Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t.$

a. Find the composite function $r\left(V(t)\right).$
b. Find the *exact* time when the radius reaches 10 inches.

Exercise:

Problem:

The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1,\ 3 < T < 33,$ where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5,$ where t is the time in hours.

a. Find the composite function $N\left(T(t)\right).$
b. Find the time (round to two decimal places) when the bacteria count reaches 6752.

Solution:

a. $N(T(t)) = 23(5t+1.5)^2 - 56(5t+1.5) + 1;$ b. 3.38 hours

Glossary

composite function
: the new function formed by function composition, when the output of one function is used as the input of another

Absolute Value Functions
In this section you will:

- Graph an absolute value function.
- Solve an absolute value equation.

Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will continue our investigation of absolute value functions.

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the absolute value function is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign. Knowing this, we can use absolute value functions to solve some kinds of real-world problems.

Note:
Absolute Value Function
The absolute value function can be defined as a piecewise function
Equation:

$$f(x) = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Example:

Exercise:

Problem:
Using Absolute Value to Determine Resistance

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often $\pm 1\%,\ \pm 5\%,$ or $\pm 10\%.$

Suppose we have a resistor rated at 680 ohms, $\pm 5\,\%.$ Use the absolute value function to express the range of possible values of the actual resistance.

Solution:

We can find that 5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,
Equation:

$$|R-680|\leq 34$$

Note:
Exercise:

Problem:

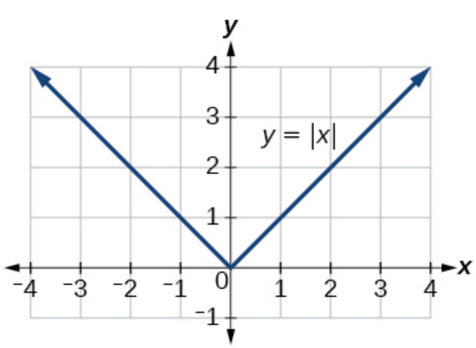
Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Solution:

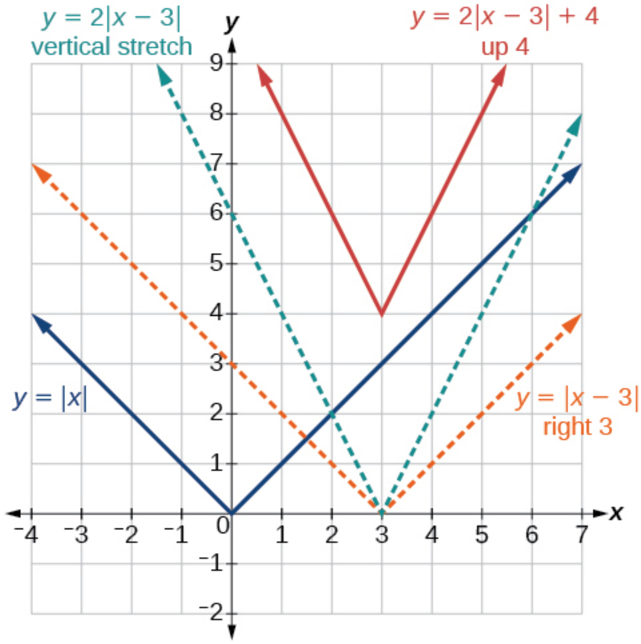
using the variable p for passing, $|p-80|\leq 20$

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in [link].



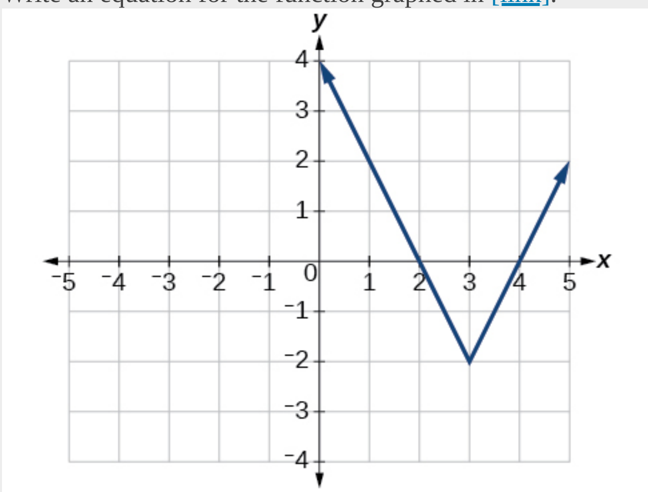
[link] shows the graph of $y = 2\left|x - 3\right| + 4$. The graph of $y = \left|x\right|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $\left(3, 4\right)$ for this transformed function.



Example:
Exercise:

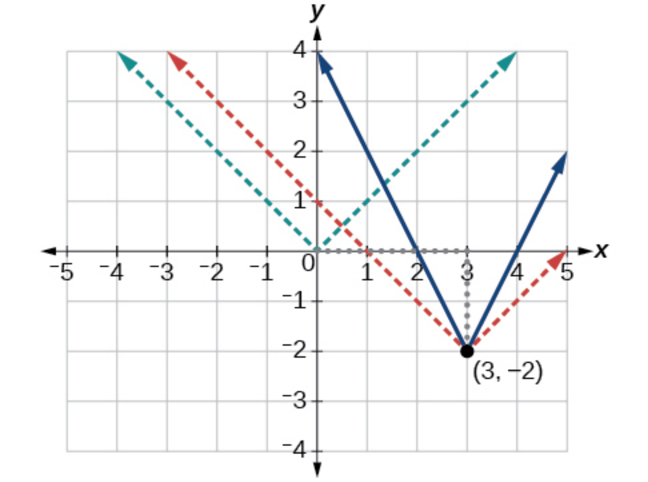
Problem:
Writing an Equation for an Absolute Value Function Given a Graph

Write an equation for the function graphed in [link].

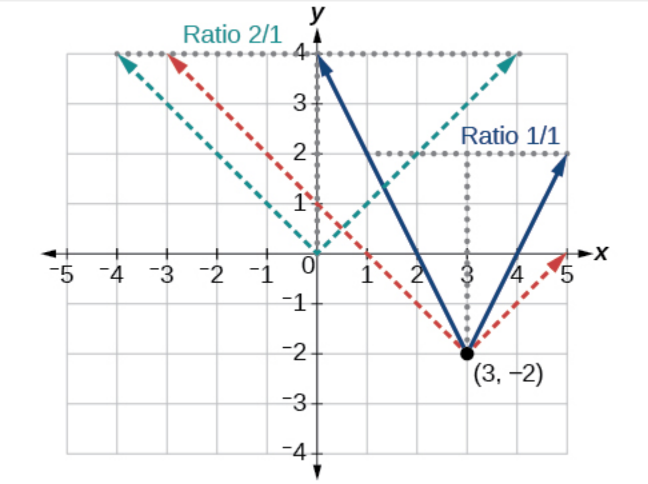


Solution:

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See [link].



We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in [link].



From this information we can write the equation
Equation:

$$\begin{array}{lll} f(x) & = & 2|x-3|-2, \qquad \text{treating the stretch as } a \text{ vertical stretch,or} \\ f(x) & = & |2(x-3)|-2, \qquad \text{treating the stretch as } a \text{ horizontal compression.} \end{array}$$

Analysis

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

Note:
If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

Equation:

$$f(x) = a|x-3| - 2$$

Now substituting in the point (1, 2)

Equation:

$$\begin{aligned} 2 &= a|1-3| - 2 \\ 4 &= 2a \\ a &= 2 \end{aligned}$$

Note:

Exercise:

Problem:

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Solution:

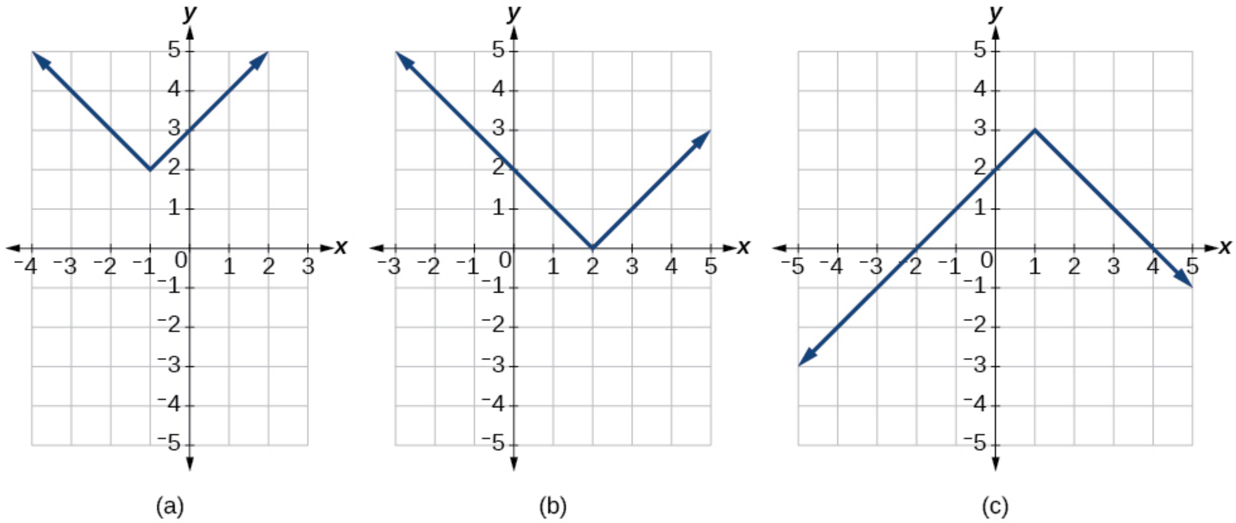
$f(x) = -|x+2| + 3$

Note:

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (see [link]).



(a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

Solving an Absolute Value Equation

In Other Type of Equations, we touched on the concepts of absolute value equations. Now that we understand a little more about their graphs, we can take another look at these types of equations. Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

Equation:

$$\begin{array}{rclcrcl} 2x-6 & = & 8 & \quad\text{or}\quad & 2x-6 & = & -8 \\ 2x & = & 14 & & 2x & = & -2 \\ x & = & 7 & & x & = & -1 \end{array}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An absolute value equation is an equation in which the unknown variable appears in absolute value bars. For example,

Equation:

$$\begin{array}{l} |x| = 4, \\ |2x-1| = 3, \text{or} \\ |5x+2| - 4 = 9 \end{array}$$

Note:
Solutions to Absolute Value Equations

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

Note:
Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-\text{A} = B$, assuming $B > 0$.
3. Solve for x.

Example:
Exercise:

Problem:
Finding the Zeros of an Absolute Value Function

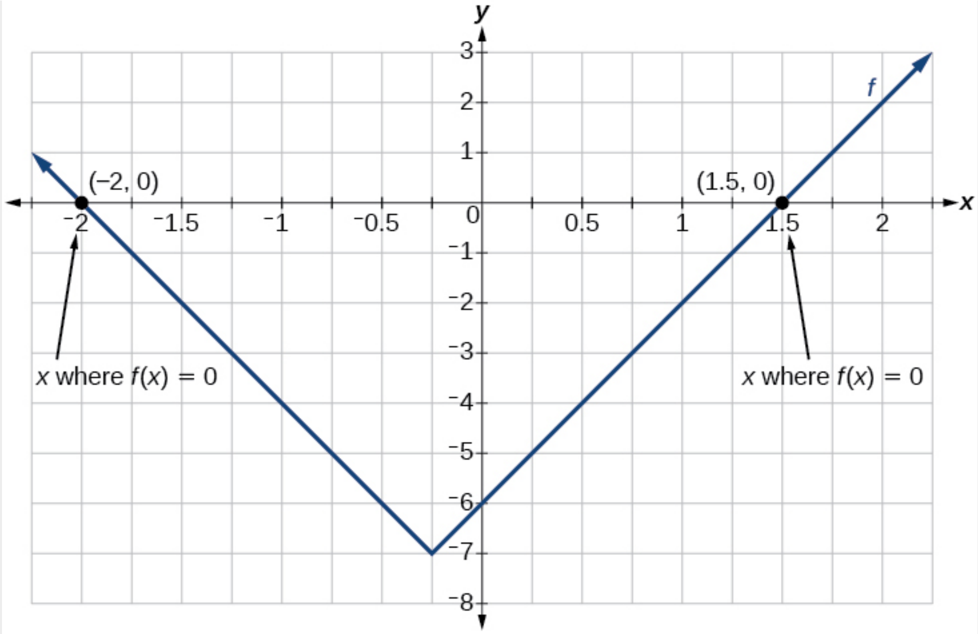
For the function $f(x) = |4x + 1| - 7$,find the values of x such that $f(x) = 0$.

Solution:
Equation:

$$\begin{array}{rcl} 0 & = & |4x+1| - 7 \\ 7 & = & |4x+1| \end{array} \qquad \begin{array}{l} \text{Substitute 0 for } f(x). \\ \text{Isolate the absolute value on one side of the equation} \end{array}$$

$$\begin{array}{rclcrcll} 7 & = & 4x+1 & \text{or} & -7 & = & 4x+1 & \text{Break into two separate equations and solve.} \\ 6 & = & 4x & & -8 & = & 4x & \\ x & = & \frac{6}{4} = 1.5 & & x & = & \frac{-8}{4} = -2 & \end{array}$$

The function outputs 0 when $x = \frac{3}{2}$ or $x = -2$. See [link].



Note:
Exercise:

Problem: For the function $f(x) = |2x - 1| - 3$, find the values of x such that $f(x) = 0$.

Solution:

$x = -1$ or $x = 2$

Note:
Should we always expect two answers when solving $|A| = B$?
No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

Note:
Access these online resources for additional instruction and practice with absolute value.

- Graphing Absolute Value Functions
- Graphing Absolute Value Functions 2

Key Concepts

- Applied problems, such as ranges of possible values, can also be solved using the absolute value function. See [link].
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction. See [link].

- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable. See [link].

Section Exercises

Verbal

Exercise:

Problem: How do you solve an absolute value equation?

Solution:

Isolate the absolute value term so that the equation is of the form $|A|=B.$ Form one equation by setting the expression inside the absolute value symbol, $A,$ equal to the expression on the other side of the equation, $B.$ Form a second equation by setting A equal to the opposite of the expression on the other side of the equation, $-B.$ Solve each equation for the variable.

Exercise:

Problem:

How can you tell whether an absolute value function has two *x*-intercepts without graphing the function?

Exercise:

Problem:

When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?

Solution:

The graph of the absolute value function does not cross the x-axis, so the graph is either completely above or completely below the x-axis.

Exercise:

Problem:

How can you use the graph of an absolute value function to determine the *x*-values for which the function values are negative?

Algebraic

Exercise:

Problem:

Describe all numbers x that are at a distance of 4 from the number 8. Express this set of numbers using absolute value notation.

Exercise:

Problem:

Describe all numbers x that are at a distance of $\frac{1}{2}$ from the number −4. Express this set of numbers using absolute value notation.

Solution:

$|x+4| = \frac{1}{2}$

Exercise:

Problem:

Describe the situation in which the distance that point x is from 10 is at least 15 units. Express this set of numbers using absolute value notation.

Exercise:

Problem:

Find all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units. Express this set of numbers using absolute value notation.

Solution:

$|f(x) - 8| < 0.03$

For the following exercises, find the *x*- and *y*-intercepts of the graphs of each function.

Exercise:

Problem: $f(x) = 4\,|x-3| + 4$

Exercise:

Problem: $f(x) = -3\,|x-2| - 1$

Solution:

$(0, -7);$ no x-intercepts

Exercise:

Problem: $f(x) = -2\,|x+1| + 6$

Exercise:

Problem: $f(x) = -5|x+2| + 15$

Solution:

$(0,\ 5), (1, 0), (-5, 0)$

Exercise:

Problem: $f(x) = 2|x-1| - 6$

Solution:

$(0, -4), (4, 0), (-2, 0)$

Exercise:

Problem: $f(x) = |-2x+1| - 13$

Solution:

$(0, -12), (-6, 0), (7, 0)$

Exercise:

Problem: $f(x) = -|x - 9| + 16$

Solution:

$(0, 7), (25, 0), (-7, 0)$

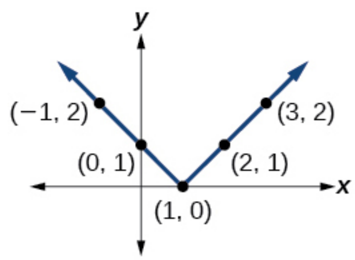
Graphical

For the following exercises, graph the absolute value function. Plot at least five points by hand for each graph.

Exercise:

Problem: $y = |x - 1|$

Solution:



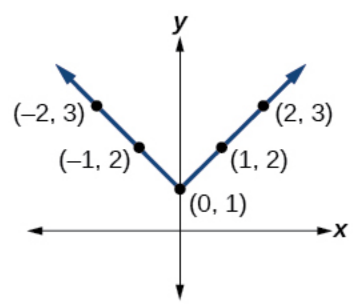
Exercise:

Problem: $y = |x + 1|$

Exercise:

Problem: $y = |x| + 1$

Solution:



For the following exercises, graph the given functions by hand.

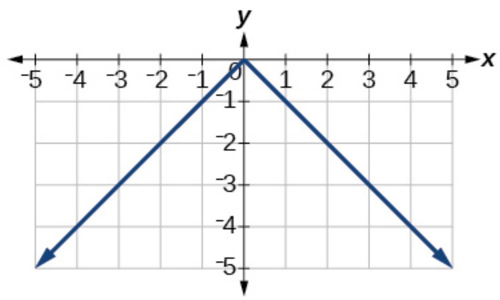
Exercise:

Problem: $y = |x| - 2$

Exercise:

Problem: $y = -|x|$

Solution:



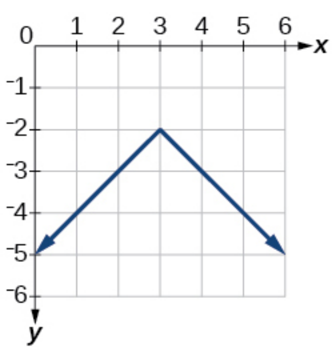
Exercise:

Problem: $y = -|x| - 2$

Exercise:

Problem: $y = -|x - 3| - 2$

Solution:



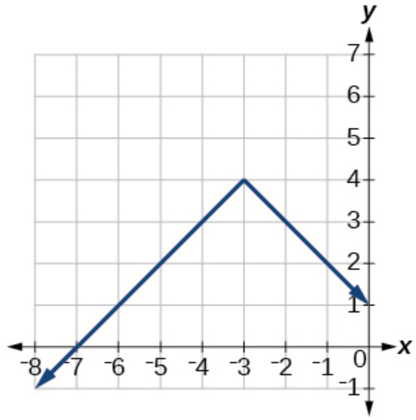
Exercise:

Problem: $f(x) = -|x - 1| - 2$

Exercise:

Problem: $f(x) = -|x + 3| + 4$

Solution:



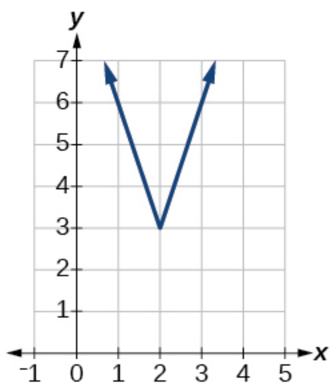
Exercise:

Problem: $f(x) = 2|x+3|+1$

Exercise:

Problem: $f(x) = 3\left|x-2\right|+3$

Solution:



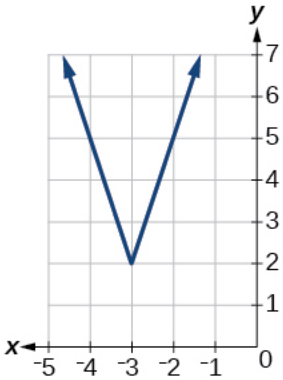
Exercise:

Problem: $f(x) = |2x-4|-3$

Exercise:

Problem: $f\left(x\right) = \left|3x+9\right|+2$

Solution:



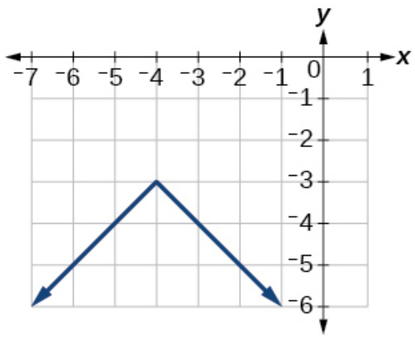
Exercise:

Problem: $f(x) = -|x-1| - 3$

Exercise:

Problem: $f(x) = -|x+4| - 3$

Solution:



Exercise:

Problem: $f(x) = \frac{1}{2}|x+4| - 3$

Technology

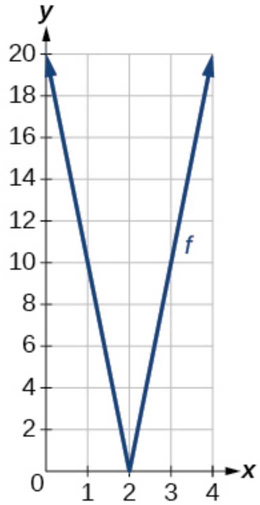
Exercise:

Problem:

Use a graphing utility to graph $f(x) = 10|x-2|$ on the viewing window $[0, 4]$. Identify the corresponding range. Show the graph.

Solution:

range: $[0, 20]$



Exercise:

Problem:

Use a graphing utility to graph $f(x) = -100|x| + 100$ on the viewing window $[-5, 5]$. Identify the corresponding range. Show the graph.

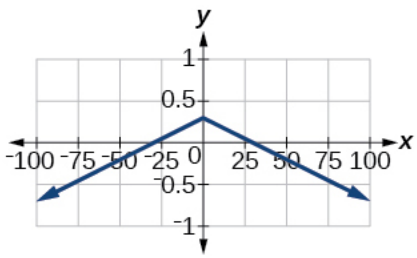
For the following exercises, graph each function using a graphing utility. Specify the viewing window.

Exercise:

Problem: $f(x) = -0.1\,|0.1(0.2 - x)| + 0.3$

Solution:

x-intercepts:



Exercise:

Problem: $f(x) = 4 \times 10^9\,\left|x - \left(5 \times 10^9\right)\right| + 2 \times 10^9$

Extensions

For the following exercises, solve the inequality.

Exercise:

Problem: If possible, find all values of a such that there are no x-intercepts for $f(x) = 2\,|x + 1| + a$.

Exercise:

Problem: If possible, find all values of a such that there are no y-intercepts for $f(x) = 2|x+1| + a$.

Solution:

There is no solution for a that will keep the function from having a y-intercept. The absolute value function always crosses the y-intercept when $x = 0$.

Real-World Applications

Exercise:

Problem:

Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least 100 miles and x represents the distance from city B to city A, express this using absolute value notation.

Exercise:

Problem:

The true proportion p of people who give a favorable rating to Congress is 8% with a margin of error of 1.5%. Describe this statement using an absolute value equation.

Solution:

$|p - 0.08| \leq 0.015$

Exercise:

Problem:

Students who score within 18 points of the number 82 will pass a particular test. Write this statement using absolute value notation and use the variable x for the score.

Exercise:

Problem:

A machinist must produce a bearing that is within 0.01 inches of the correct diameter of 5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.

Solution:

$|x - 5.0| \leq 0.01$

Exercise:

Problem:

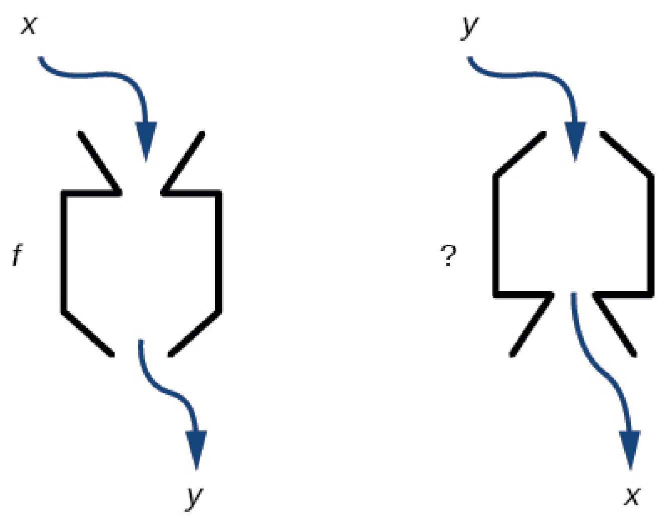
The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

Inverse Functions
In this section, you will:

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. [link] provides a visual representation of this question. In this section, we will consider the reverse nature of functions.



Can a function "machine" operate in reverse?

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of how

temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula

Equation:

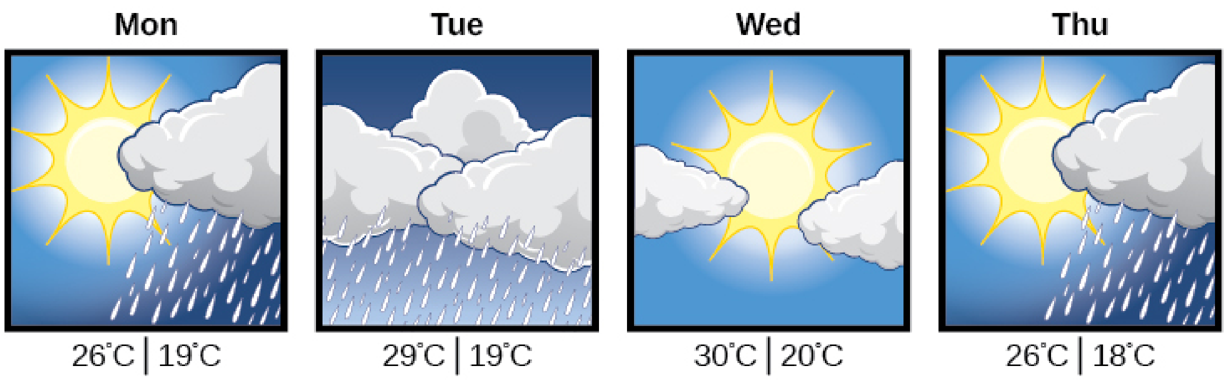
$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

Equation:

$$\frac{5}{9}(75 - 32) \approx 24^{\circ}\mathrm{C}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from [link] for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.



At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

Equation:

$$\begin{array}{rcl} 26 & = & \frac{5}{9}(F - 32) \\ 26 \cdot \frac{9}{5} & = & F - 32 \\ F & = & 26 \cdot \frac{9}{5} + 32 \approx 79 \end{array}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x),$ we represent its inverse as $f^{-1}(x),$ read as "f inverse of $x.$" The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a=1$ (1 is the identity element for multiplication) for any nonzero number $a,$ so $f^{-1}\circ f$ equals the identity function, that is,
Equation:

$$\left(f^{-1}\circ f\right)(x)=f^{-1}\left(f(x)\right)=f^{-1}\left(y\right)=x$$

This holds for all x in the domain of $f.$ Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Given a function $f(x),$ we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x))=x$ or $f(g(x))=x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y=4x$ and $y=\frac{1}{4}x$ are inverse functions.
Equation:

$$\left(f^{-1}\circ f\right)(x)=f^{-1}\left(4x\right)=\frac{1}{4}(4x)=x$$

and
Equation:

$$\left(f \circ f^{-1}\right)(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x$$

A few coordinate pairs from the graph of the function $y = 4x$ are (−2, −8), (0, 0), and (2, 8). A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are (−8, −2), (0, 0), and (8, 2). If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Note:
Inverse Function
For any one-to-one function $f(x) = y,$ a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x.$ This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of $f.$ It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of $f.$
The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of $x.$" Keep in mind that
Equation:

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

and not all functions have inverses.

Example:
Exercise:

Problem:
Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12,$ what are the corresponding input and output values for the inverse function?

Solution:

The inverse function reverses the input and output quantities, so if
Equation:

$$\begin{aligned} f(2) &= 4, \text{ then } f^{-1}(4) = 2; \\ f(5) &= 12, \text{ then f}^{-1}(12) = 5. \end{aligned}$$

Alternatively, if we want to name the inverse function $g,$ then $g(4) = 2$ and $g(12) = 5.$

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See [link].

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Note:
Exercise:

Problem:

Given that $h^{-1}(6) = 2,$ what are the corresponding input and output values of the original function h?

Solution:

$h(2) = 6$

Note:
Given two functions $f(x)$ and $g(x),$ test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
2. If either statement is true, then both are true, and $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Example:
Exercise:

Problem:
Testing Inverse Relationships Algebraically

If $f\left(x\right) = \frac{1}{x+2}$ and $g\left(x\right) = \frac{1}{x} - 2,$ is $g = f^{-1}$?

Solution:
Equation:

$$\begin{array}{rcl} g(f(x)) & = & \frac{1}{\left(\frac{1}{x+2}\right)} - 2 \\ & = & x + 2 - 2 \\ & = & x \end{array}$$

so
Equation:

$$g = f^{-1} \text{ and } f = g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.
Equation:

$$\begin{array}{rcl} f(g(x)) & = & \frac{1}{\frac{1}{x} - 2 + 2} \\ & = & \frac{1}{\frac{1}{x}} \\ & = & x \end{array}$$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

Note:
Exercise:

Problem: If $f\left(x\right)=x^{3}-4$ and $g\left(x\right)=\sqrt[3]{x+4}$, is $g=f^{-1}$?

Solution:

Yes

Example:
Exercise:

Problem:
Determining Inverse Relationships for Power Functions

If $f(x)=x^{3}$ (the cube function) and $g(x)=\frac{1}{3}x$, is $g=f^{-1}$?

Solution:
Equation:

$$f\left(g\left(x\right)\right)=\frac{x^{3}}{27}\neq x$$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x}=x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

Note:
Exercise:

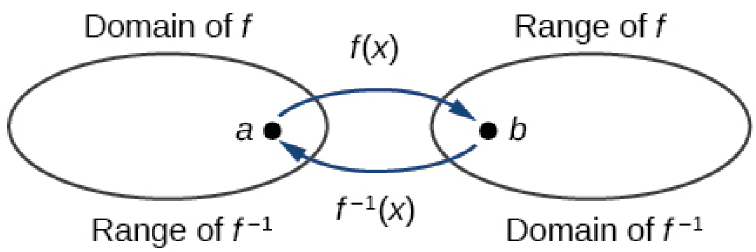
Problem: If $f\left(x\right)=\left(x-1\right)^{3}$ and $g\left(x\right)=\sqrt[3]{x}+1$, is $g=f^{-1}$?

Solution:

Yes

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in [link].



Domain and range of a function and its inverse

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $[0, \infty)$, which is a one-to-

one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of f^{-1} = $[1, \infty)$.
- The domain of f^{-1} = range of f = $[0, \infty)$.

Note:
Is it possible for a function to have more than one inverse?
No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Note:
Domain and Range of Inverse Functions
The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.
The domain of $f(x)$ is the range of $f^{-1}(x)$.

Note:
Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example:
Exercise:

Problem:

Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in [link]. We restrict the domain in such a fashion that the function assumes all *y*-values exactly once.

Constant	**Identity**	**Quadratic**	**Cubic**	**Reciprocal**
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	**Cube root**	**Square root**	**Absolute value**	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x\rvert$	

Solution:

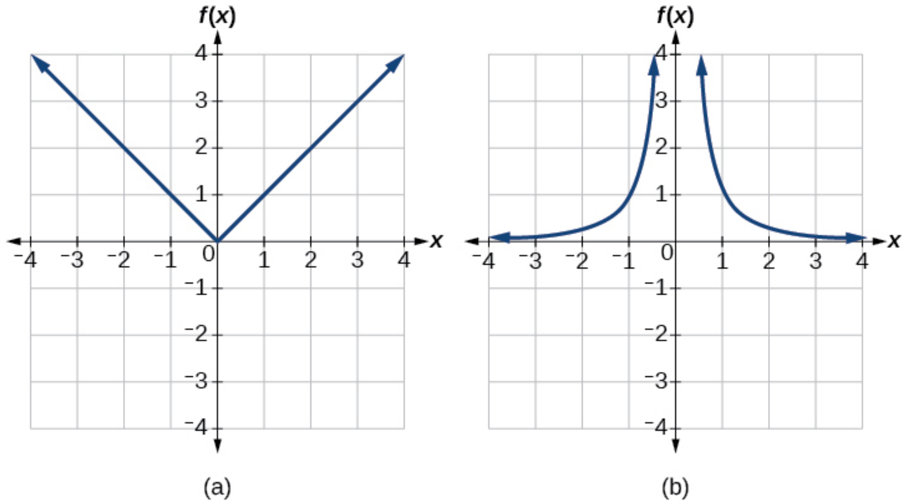
The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in [link]. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.



(a) Absolute value (b) Reciprocal square

Note:
Exercise:

Problem:

The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Solution:

The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example:
Exercise:

Problem:
Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in [link], showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Solution:

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70)=90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a)=b$, then $f^{-1}(b)=a$. By this definition, if we are given $f^{-1}(70)=a$, then we are looking for a value a so that $f(a)=70$. In this case, we are looking for a t so that $f(t)=70$, which is when $t=90$.

Note:
Exercise:

Problem: Using [link], find and interpret (a) $f(60)$, and (b) $f^{-1}(60)$.

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Solution:

a. $f(60)=50.$ In 60 minutes, 50 miles are traveled.
b. $f^{-1}(60)=70.$ To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in Functions and Function Notation that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

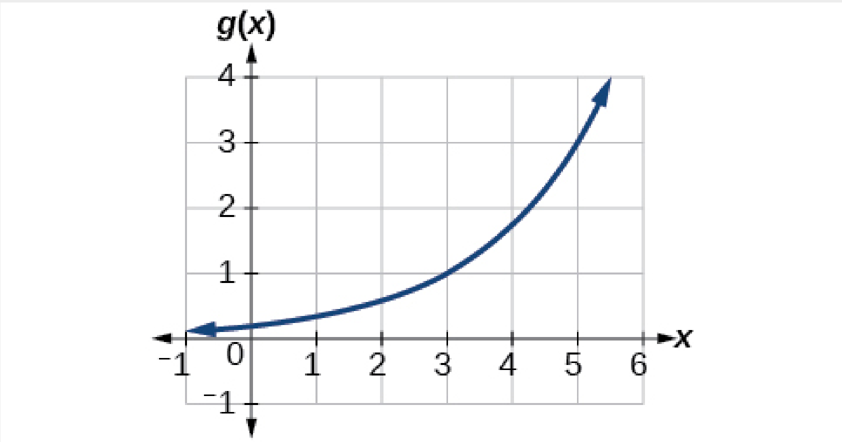
Note:
Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the *y*-axis of the given graph.
2. Read the inverse function's output from the *x*-axis of the given graph.

Example:
Exercise:

Problem:
Evaluating a Function and Its Inverse from a Graph at Specific Points

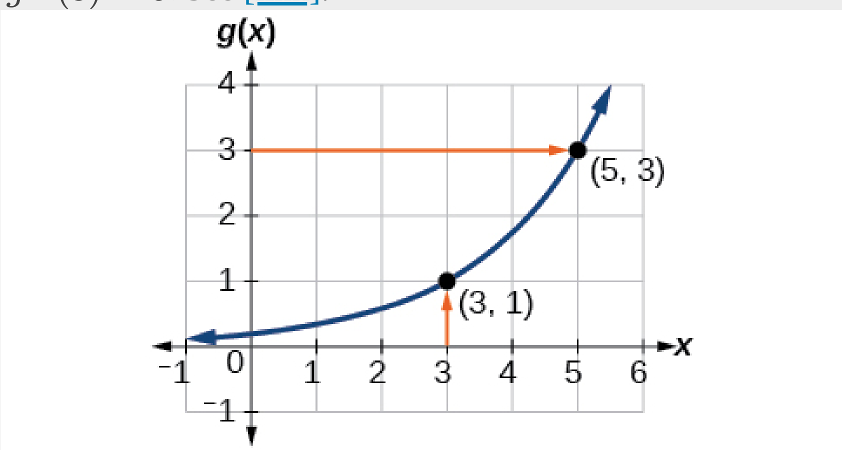
A function $g(x)$ is given in [link]. Find $g(3)$ and $g^{-1}(3)$.



Solution:

To evaluate $g(3)$, we find 3 on the *x*-axis and find the corresponding output value on the *y*-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of *x* for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See [link].



Note:
Exercise:

Problem: Using the graph in [link], (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Solution:

a. 3; b. 5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—for example, y as a function of x—we can often find the inverse function by solving to obtain x as a function of y.

Note:
Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x.
3. Interchange x and y.

Example:
Exercise:

Problem:
Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.
Equation:

$$C=\frac{5}{9}(F-32)$$

Solution:

Equation:

$$\begin{array}{rcl} C & = & \frac{5}{9}(F-32) \\ C \cdot \frac{9}{5} & = & F-32 \\ F & = & \frac{9}{5}C+32 \end{array}$$

By solving in general, we have uncovered the inverse function. If

Equation:

$$C = h(F) = \frac{5}{9}(F-32),$$

then

Equation:

$$F = h^{-1}(C) = \frac{9}{5}C + 32$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Note:
Exercise:

Problem: Solve for x in terms of y given $y = \frac{1}{3}(x-5)$.

Solution:

$x = 3y + 5$

Example:
Exercise:

Problem:
Solving to Find an Inverse Function

Find the inverse of the function $f\left(x\right)=\frac{2}{x-3}+4$.

Solution:
Equation:

$$\begin{array}{rcll} y & = & \frac{2}{x-3}+4 & \text{Set up an equation.} \\ y-4 & = & \frac{2}{x-3} & \text{Subtract 4 from both sides.} \\ x-3 & = & \frac{2}{y-4} & \text{Multiply both sides by } x-3 \text{ and divide by } y-4. \\ x & = & \frac{2}{y-4}+3 & \text{Add 3 to both sides.} \end{array}$$

So $f^{-1}\left(y\right)=\frac{2}{y-4}+3$ or $f^{-1}\left(x\right)=\frac{2}{x-4}+3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating [link].

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example:
Exercise:

Problem:
Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x)=2+\sqrt{x-4}$.

Solution:
Equation:

$$\begin{array}{rcl} y & = & 2+\sqrt{x-4} \\ (y-2)^2 & = & x-4 \\ x & = & (y-2)^2+4 \end{array}$$

So $f^{-1}(x) = (x-2)^2 + 4.$

The domain of f is $[4, \infty).$ Notice that the range of f is $[2, \infty),$ so this means that the domain of the inverse function f^{-1} is also $[2, \infty).$

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real $x.$ However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of $f.$

Note:
Exercise:

Problem:

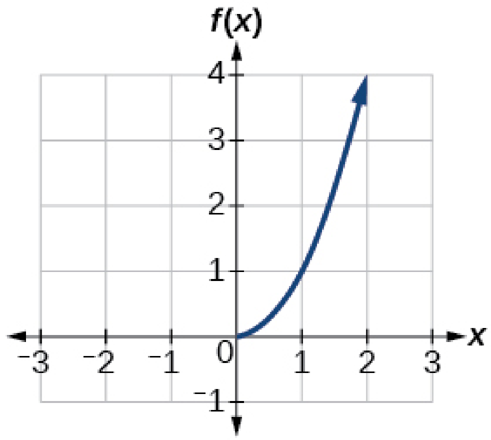
What is the inverse of the function $f(x) = 2 - \sqrt{x}$?State the domains of both the function and the inverse function.

Solution:

$f^{-1}(x) = (2-x)^2;\ \text{domain of}\ f:\ \ [0,\infty);\ \text{domain of}\ f^{-1}:\ \ (-\infty, 2]$

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in [link].

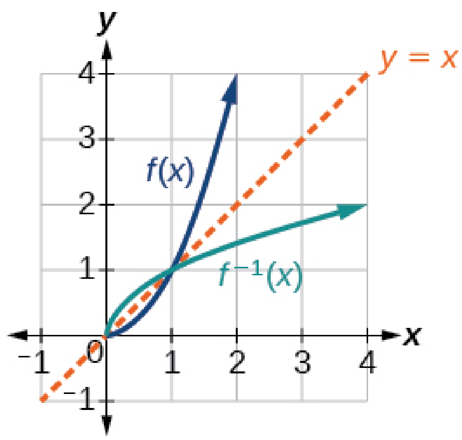


Quadratic function with domain restricted to [0, ∞).

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in [link].



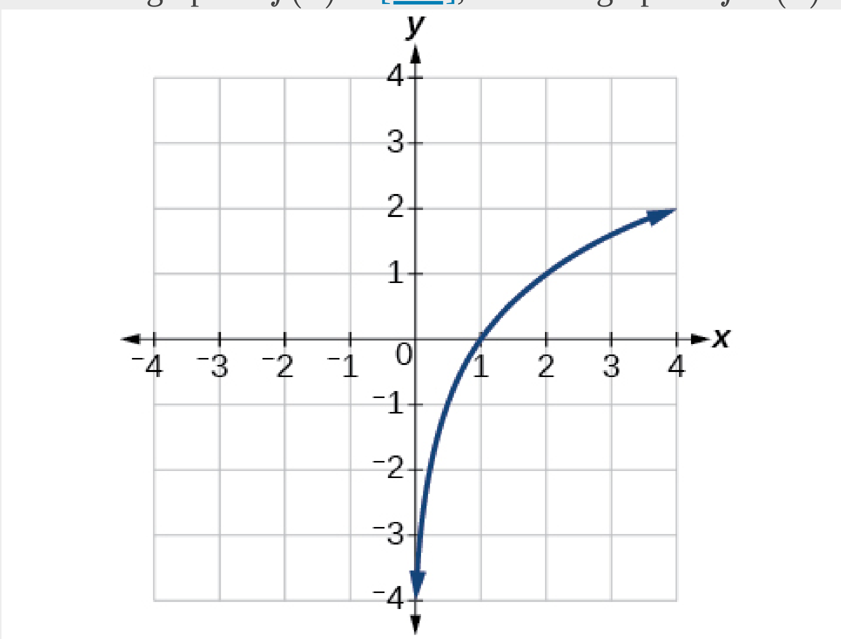
Square and square-root functions on the non-negative domain

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example:
Exercise:

Problem:
Finding the Inverse of a Function Using Reflection about the Identity Line

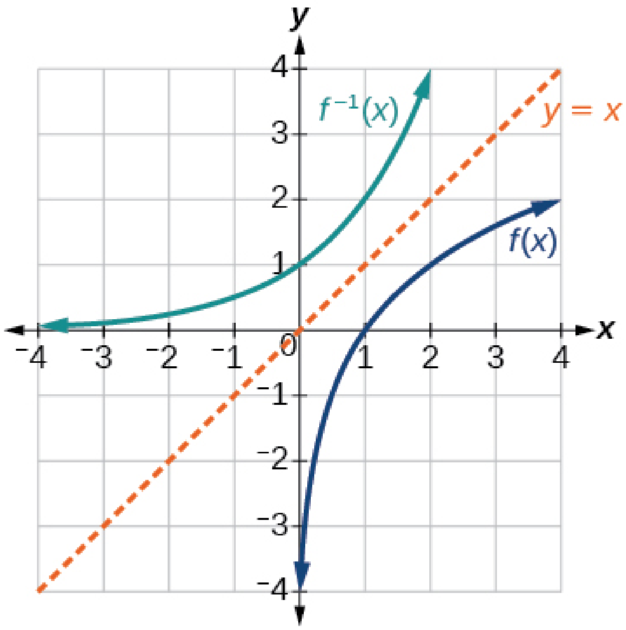
Given the graph of $f(x)$ in [link], sketch a graph of $f^{-1}(x)$.



Solution:

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives [link].

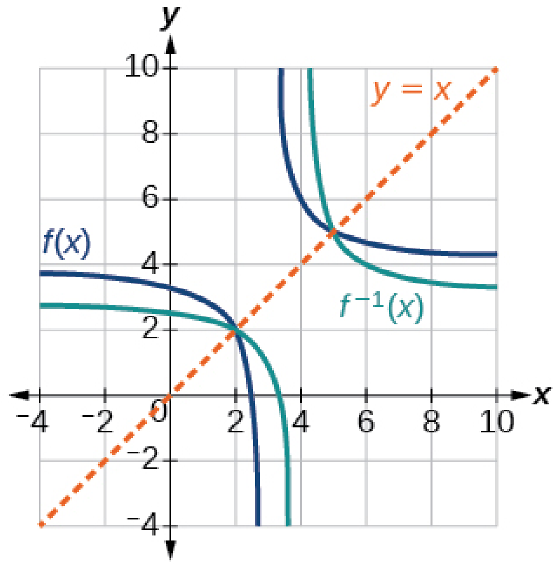


The function and its inverse, showing reflection about the identity line

Note:
Exercise:

Problem: Draw graphs of the functions f and $f^{-1}$from [link].

Solution:



Note:

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$*, then* $f\left(f\left(x\right)\right) = x$*, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because*

Equation:

$$\frac{1}{\frac{1}{x}} = x$$

Any function $f\left(x\right) = c - x$*, where* c *is a constant, is also equal to its own inverse.*

Note:

Access these online resources for additional instruction and practice with inverse functions.

- [Inverse Functions]
- [One-to-one Functions]
- [Inverse Function Values Using Graph]
- [Restricting the Domain and Finding the Inverse]

Visit this website for additional practice questions from Learningpod.

Key Concepts

- If $g(x)$ is the inverse of $f(x),$ then $g(f(x))=f(g(x))=x.$ See [link], [link], and [link].
- Only some of the toolkit functions have an inverse. See [link].
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse. See [link].
- The inverse of a function can be determined at specific points on its graph. See [link].
- To find the inverse of a formula, solve the equation $y=f(x)$ for x as a function of $y.$ Then exchange the labels x and $y.$ See [link], [link], and [link].
- The graph of an inverse function is the reflection of the graph of the original function across the line $y=x.$ See [link].

Section Exercises

Verbal

Exercise:

Problem:

Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

Solution:

Each output of a function must have exactly one output for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that y-values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no y-values repeat and the function is one-to-one.

Exercise:

Problem:

Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?

Exercise:

Problem: Can a function be its own inverse? Explain.

Solution:

Yes. For example, $f(x) = \frac{1}{x}$ is its own inverse.

Exercise:

Problem:

Are one-to-one functions either always increasing or always decreasing? Why or why not?

Exercise:

Problem: How do you find the inverse of a function algebraically?

Solution:

Given a function $y = f(x)$, solve for x in terms of y. Interchange the x and y. Solve the new equation for y. The expression for y is the inverse, $y = f^{-1}(x)$.

Algebraic

Exercise:

Problem:

Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

Exercise:

Problem: $f(x) = x + 3$

Solution:

$f^{-1}(x) = x - 3$

Exercise:

Problem: $f(x) = x + 5$

Exercise:

Problem: $f(x) = 2 - x$

Solution:

$f^{-1}(x) = 2 - x$

Exercise:

Problem: $f(x) = 3 - x$

Exercise:

Problem: $f(x) = \frac{x}{x+2}$

Solution:

$f^{-1}(x) = \frac{-2x}{x-1}$

Exercise:

Problem: $f(x) = \frac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

Exercise:

Problem: $f(x) = (x + 7)^2$

Solution:

domain of $f(x) : [-7, \infty)$; $f^{-1}(x) = \sqrt{x} - 7$

Exercise:

Problem: $f(x) = (x-6)^2$

Exercise:

Problem: $f(x) = x^2 - 5$

Solution:

domain of $f(x):\ [0,\infty);\ f^{-1}(x) = \sqrt{x+5}$

Exercise:

Problem: Given $f\left(x\right) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$:

a. Find $f(g(x))$ and $g(f(x))$.
b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

Solution:

a. $f(g(x)) = x$ and $g(f(x)) = x.$ b. This tells us that f and g are inverse functions

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

Exercise:

Problem: $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$

Solution:

$f(g(x)) = x,\ g(f(x)) = x$

Exercise:

Problem: $f(x) = -3x + 5$ and $g(x) = \frac{x-5}{-3}$

Graphical

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

Exercise:

Problem: $f(x) = \sqrt{x}$

Solution:

one-to-one

Exercise:

Problem: $f(x) = \sqrt[3]{3x+1}$

Exercise:

Problem: $f(x) = -5x+1$

Solution:

one-to-one

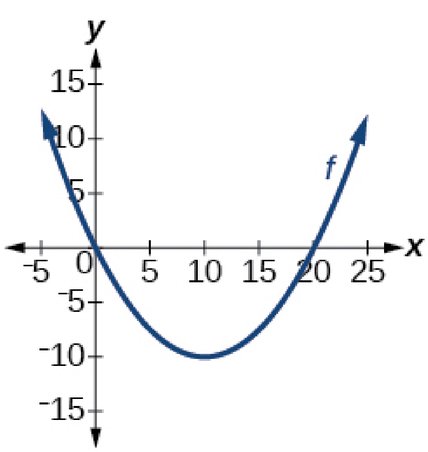
Exercise:

Problem: $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

Exercise:

Problem:

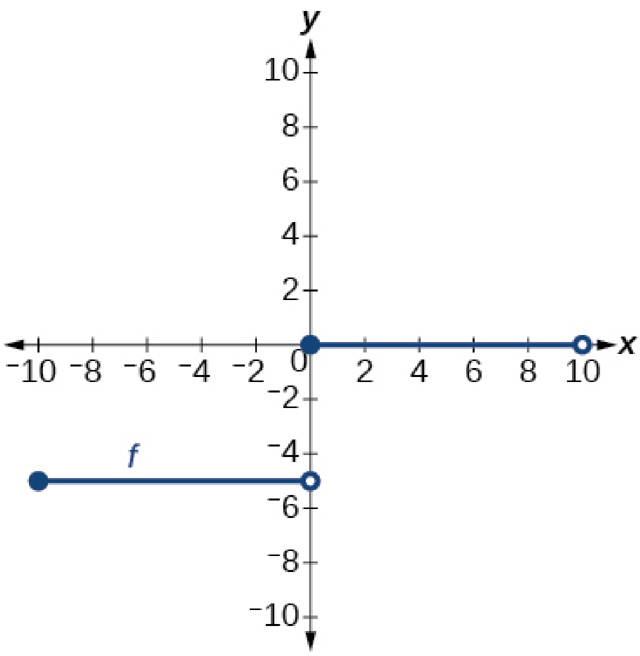


Solution:

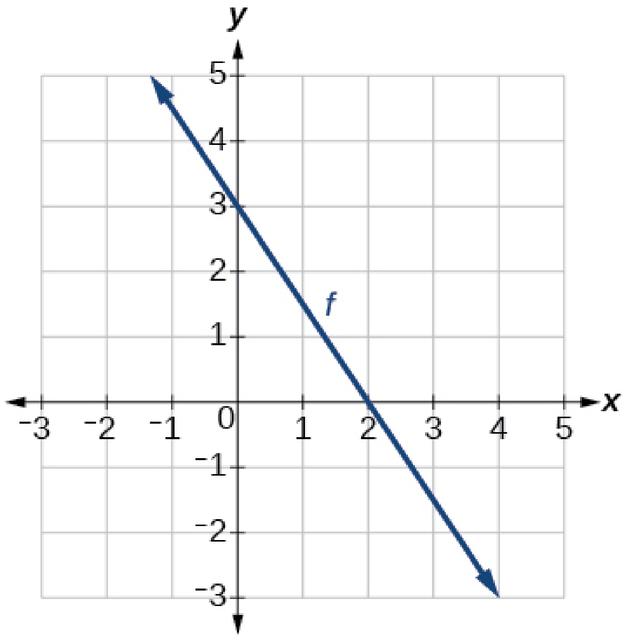
not one-to-one

Exercise:

Problem:



For the following exercises, use the graph of f shown in [link].



Exercise:

Problem: Find $f\left(0\right)$.

Solution:

3

Exercise:

Problem: Solve $f(x)=0$.

Exercise:

Problem: Find $f^{-1}\left(0\right)$.

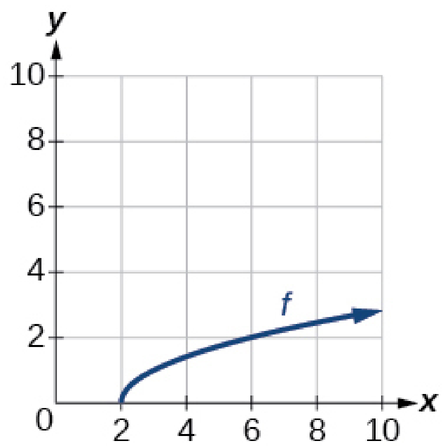
Solution:

2

Exercise:

Problem: Solve $f^{-1}\left(x\right)=0$.

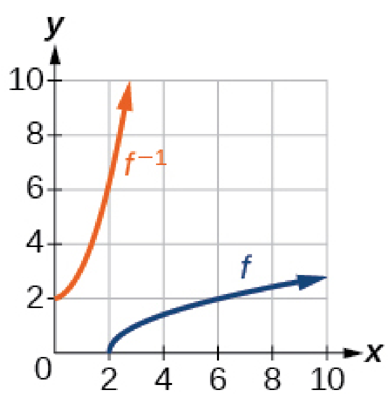
For the following exercises, use the graph of the one-to-one function shown in [link].



Exercise:

Problem: Sketch the graph of f^{-1}.

Solution:



Exercise:

Problem: Find $f(6)$ and $f^{-1}(2)$.

Exercise:

Problem: If the complete graph of f is shown, find the domain of f.

Solution:

$[2, 10]$

Exercise:

Problem: If the complete graph of f is shown, find the range of f.

Numeric

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

Exercise:

Problem: If $f(6)=7,$ find $f^{-1}(7).$

Solution:

6

Exercise:

Problem: If $f(3)=2,$ find $f^{-1}(2).$

Exercise:

Problem: If $f^{-1}(-4)=-8,$ find $f(-8).$

Solution:

-4

Exercise:

Problem: If $f^{-1}(-2)=-1,$ find $f(-1).$

For the following exercises, use the values listed in [link] to evaluate or solve.

x	$f(x)$

x	$f(x)$
0	8
1	0
2	7
3	4
4	2
5	6
6	5
7	3
8	9
9	1

Exercise:

Problem: Find $f(1)$.

Solution:

0

Exercise:

Problem: Solve $f(x) = 3$.

Exercise:

Problem: Find $f^{-1}(0)$.

Solution:

1

Exercise:

Problem: Solve $f^{-1}(x) = 7$.

Exercise:

Problem: Use the tabular representation of f in [link] to create a table for $f^{-1}(x)$.

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Solution:

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

Technology

For the following exercises, find the inverse function. Then, graph the function and its inverse.

Exercise:

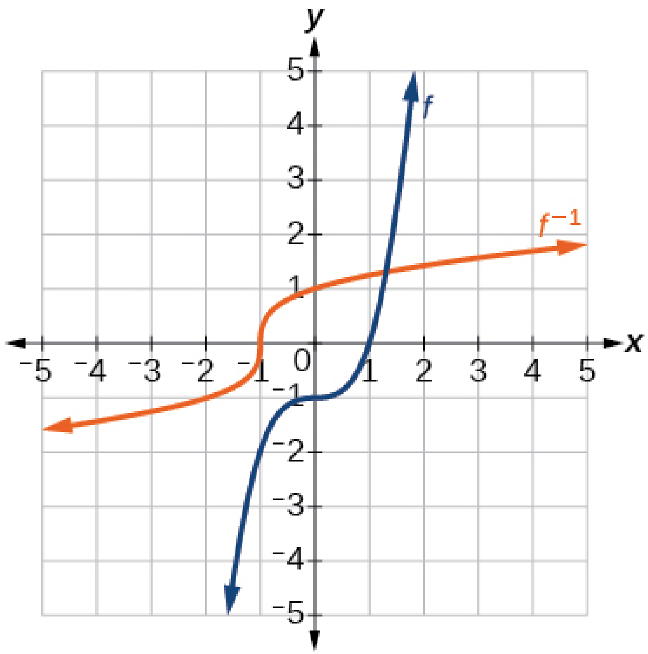
Problem: $f(x) = \frac{3}{x-2}$

Exercise:

Problem: $f(x) = x^3 - 1$

Solution:

$f^{-1}(x) = (1 + x)^{1/3}$



Exercise:

Problem:

Find the inverse function of $f(x) = \frac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

Real-World Applications

Exercise:

Problem:

To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \frac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.

Solution:

$f^{-1}(x) = \frac{5}{9}(x - 32).$ Given the Fahrenheit temperature, $x,$ this formula allows you to calculate the Celsius temperature.

Exercise:

Problem:

The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r.$ Express the radius of a circle as a function of its circumference. Call this function $r(C).$ Find $r(36\pi)$ and interpret its meaning.

Exercise:

Problem:

A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, $t,$ in hours given by $d(t) = 50t.$ Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d).$ Find $t(180)$ and interpret its meaning.

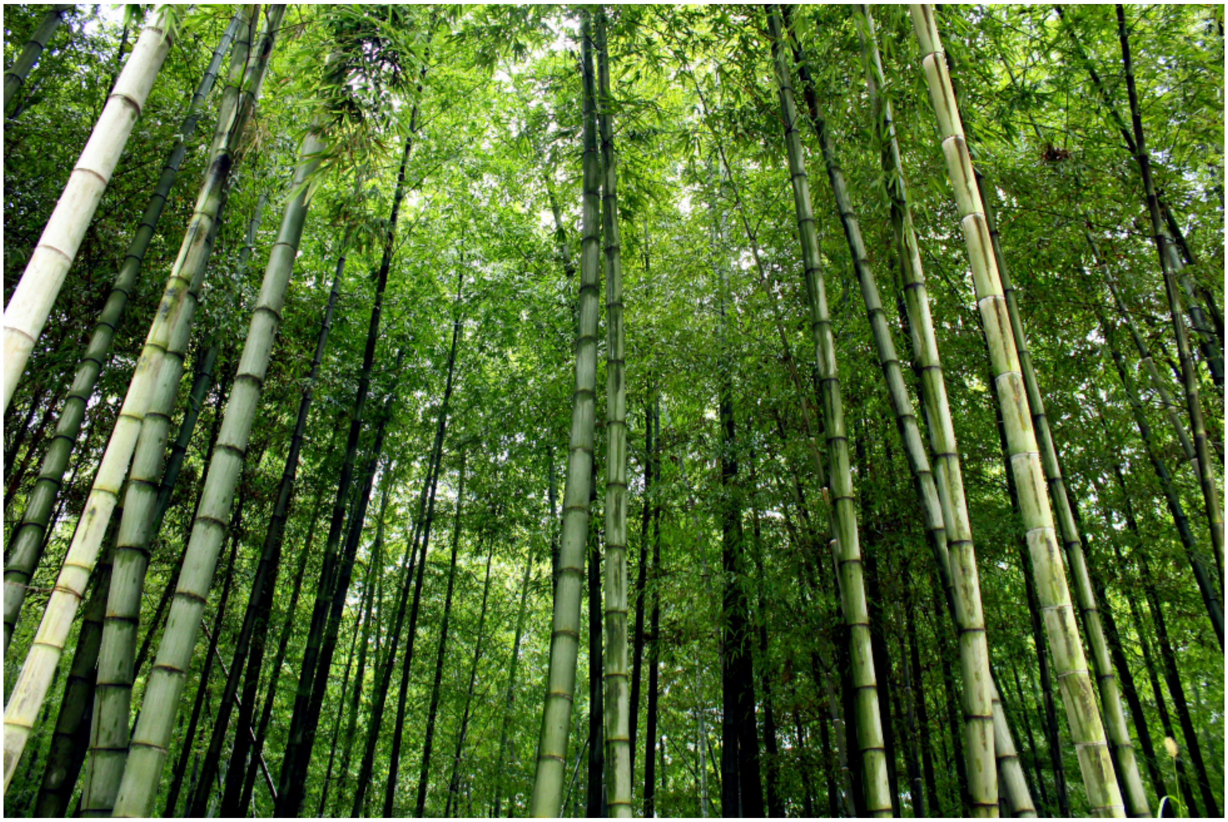
Solution:

$t(d) = \frac{d}{50}$, $t(180) = \frac{180}{50}.$ The time for the car to travel 180 miles is 3.6 hours.

Glossary

inverse function
: for any one-to-one function $f(x),$ the inverse is a function $f^{-1}(x)$ such that $f^{-1}\left(f\left(x\right)\right) = x$ for all x in the domain of $f;$ this also implies that $f\left(f^{-1}\left(x\right)\right) = x$ for all x in the domain of f^{-1}

Introduction to Linear Functions
class="introduction"



A bamboo forest in China (credit: "JFXie"/Flickr)

Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain types of bamboo species. These members of the grass family are the fastest-growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour. [footnote] In a twenty-four hour period, this bamboo plant grows about 36 inches, or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function.
http://www.guinnessworldrecords.com/records-3000/fastest-growing-plant/

Recall from Functions and Function Notation that a function is a relation that assigns to every element in the domain exactly one element in the

range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

Linear Functions
In this section you will:

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Interpret slope as a rate of change.
- Write and interpret an equation for a linear function.
- Graph linear functions.
- Determine whether lines are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Shanghai MagLev Train (credit: "kanegen"/Flickr)

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train ([link]). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes[footnote].
http://www.chinahighlights.com/shanghai/transportation/maglev-train.htm

Suppose a maglev train travels a long distance, and maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train's distance from the station at a given point in time.

Representing Linear Functions

The function describing the train’s motion is a linear function, which is defined as a function with a constant rate of change. This is a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train’s motion as a function using each method.

Representing a Linear Function in Word Form

Let’s begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train’s distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the slope-intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

Equation:

$$\begin{array}{ll} \text{Equation form} & y = mx + b \\ \text{Function notation} & f(x) = mx + b \end{array}$$

In the example of the train, we might use the notation $D(t)$ where the total distance D is a function of the time $t.$ The rate, $m,$ is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

Equation:

$$D(t) = 83t + 250$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in [link]. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

1 second 1 second 1 second

t	0	1	2	3
$D(t)$	250	333	416	499

83 meters 83 meters 83 meters

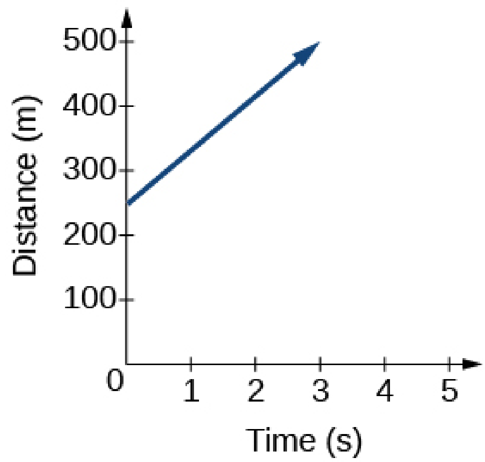
Tabular representation of the function D showing selected input and output values

Note:
Can the input in the previous example be any real number?
No. The input represents time so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250,$ to draw a graph as represented in [link]. Notice the graph is a line**.** When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or slope of the line. The point at which the input value is zero is the vertical intercept, or *y*-intercept, of the line. We can see from the graph that the *y*-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.



The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1.$ Ask yourself what numbers can be input to the function. In other words, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Note:
Linear Function
A **linear function** is a function whose graph is a line. Linear functions can be written in the **slope-intercept form** of a line
Equation:

$$f(x) = mx + b$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or slope of the function. The *y*-intercept is at $(0, b)$.

Example:
Exercise:

Problem:
Using a Linear Function to Find the Pressure on a Diver

The pressure, P,in pounds per square inch (PSI) on the diver in [link] depends upon her depth below the water surface, d,in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.

(credit: Ilse Reijs and Jan-Noud Hutten)

Solution:

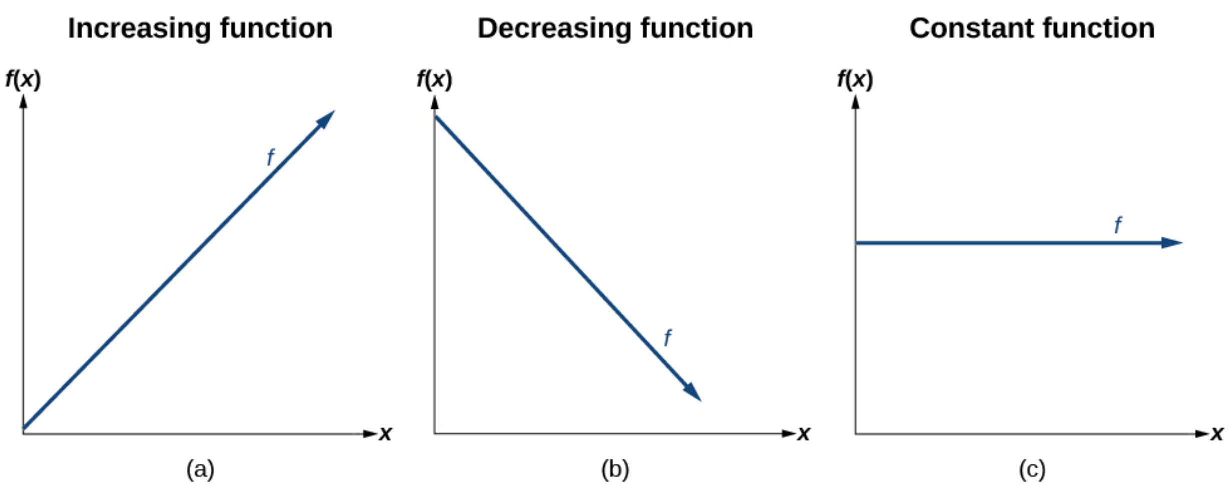
To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining Whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an increasing function has a positive slope. A line with a positive slope slants upward from left to right as in [link]**(a)**. For a decreasing function, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in [link]**(b)**. If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in [link]**(c)**.



Note:
Increasing and Decreasing Functions
The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is a decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

Example:
Exercise:

Problem:
Deciding Whether a Function Is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day[footnote]. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.
http://www.cbsnews.com/8301-501465_162-57400228-501465/teens-are-sending-60-texts-a-day-study-says/

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Solution:

Analyze each function.

a. The function can be represented as $f(x)=60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
b. The function can be represented as $f(x)=500-60x$ where $x$is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
c. The cost function can be represented as $f(x)=50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Interpreting Slope as a Rate of Change

In the examples we have seen so far, the slope was provided to us. However, we often need to calculate the slope given input and output values. Recall that given two values for the input, $x_1$and x_2,and two corresponding values for the output, y_1 and y_2 —which can be represented by a set of points, (x_1, y_1) and (x_2, y_2)—we can calculate the slope m.

Equation:

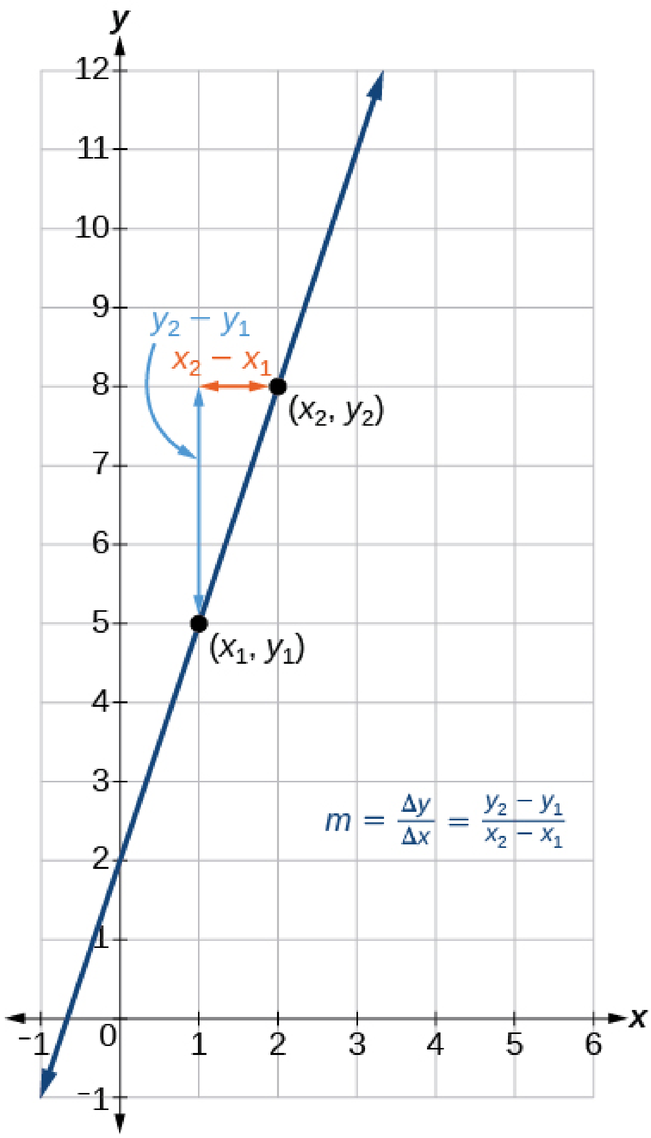
$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that in function notation we can obtain two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$,so we could equivalently write

Equation:

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

[link] indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness, or slant. The greater the absolute value of the slope, the steeper the slant is.



The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the $(x_{2,}y_2)$ and which is the (x_1, y_1), as long as each calculation is started with the elements from the same coordinate pair.

Note:

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Note:
Calculate Slope
The slope, or rate of change, of a function m can be calculated according to the following:
Equation:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

Note:
Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

Example:
Exercise:

Problem:
Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3, -2)$ and $(8, 1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution:

The coordinate pairs are $(3, -2)$ and $(8, 1)$. To find the rate of change, we divide the change in output by the change in input.
Equation:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1-(-2)}{8-3} = \frac{3}{5}$$

We could also write the slope as $m = 0.6$. The function is increasing because $m > 0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used. Note that if we had reversed them, we would have obtained the same slope.

Equation:

$$m = \frac{(-2)-(1)}{3-8} = \frac{-3}{-5} = \frac{3}{5}$$

Note:

Exercise:

Problem:

If $f(x)$ is a linear function, and $(2,3)$ and $(0,4)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution:

$m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; decreasing because $m < 0$.

Example:

Exercise:

Problem:

Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

Solution:

The rate of change relates the change in population to the change in time. The population increased by $27,800 - 23,400 = 4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

Equation:

$$\frac{4\text{,}400 \text{ people}}{4 \text{ years}} = 1\text{,}100 \ \frac{\text{people}}{\text{year}}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

Note:
Exercise:

Problem:

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

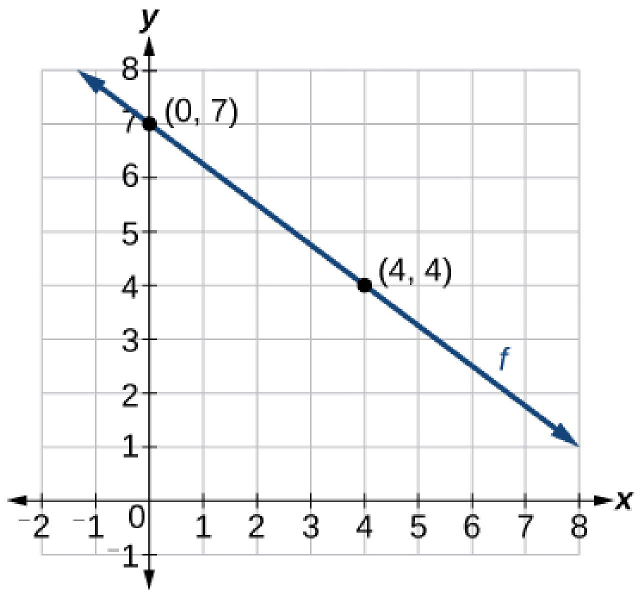
Solution:
Equation:

$$m = \frac{1,868 - 1,442}{2,012 - 2,009} = \frac{426}{3} = 142 \text{ people per year}$$

Writing and Interpreting an Equation for a Linear Function

Recall from Equations and Inequalities that we wrote equations in both the slope-intercept form and the point-slope form. Now we can choose which method to use to write equations for linear functions based on the information we are given. That

information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in [link].



We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0, 7)$ and $(4, 4)$.

Equation:

$$\begin{array}{lll} m & = & \frac{y_2 - y_1}{x_2 - x_1} \\ & = & \frac{4-7}{4-0} \\ & = & -\frac{3}{4} \end{array}$$

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

Equation:

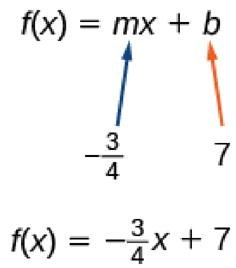
$$\begin{array}{rll} y - y_1 & = & m(x - x_1) \\ y - 4 & = & -\frac{3}{4}(x - 4) \end{array}$$

If we want to rewrite the equation in the slope-intercept form, we would find

Equation:

$$\begin{aligned} y-4 &= -\frac{3}{4}(x-4) \\ y-4 &= -\frac{3}{4}x+3 \\ y &= -\frac{3}{4}x+7 \end{aligned}$$

If we want to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the *y*-axis when the output value is 7. Therefore, $b=7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.



So the function is $f(x)=-\frac{3}{4}x+7$, and the linear equation would be $y=-\frac{3}{4}x+7$.

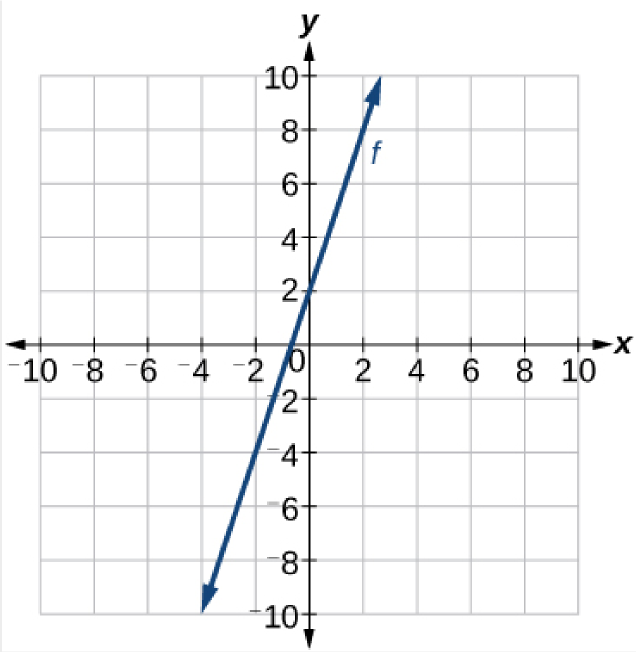
Note:
Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the *y*-axis to identify the *y*-intercept by visual inspection.
4. Substitute the slope and *y*-intercept into the slope-intercept form of a line equation.

Example:
Exercise:

Problem:
Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in [link].



Solution:

Identify two points on the line, such as $(0, 2)$ and $(-2, -4)$. Use the points to calculate the slope.

Equation:

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{-4-2}{-2-0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

Equation:

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-(-4) &= 3(x-(-2)) \\ y+4 &= 3(x+2) \end{aligned}$$

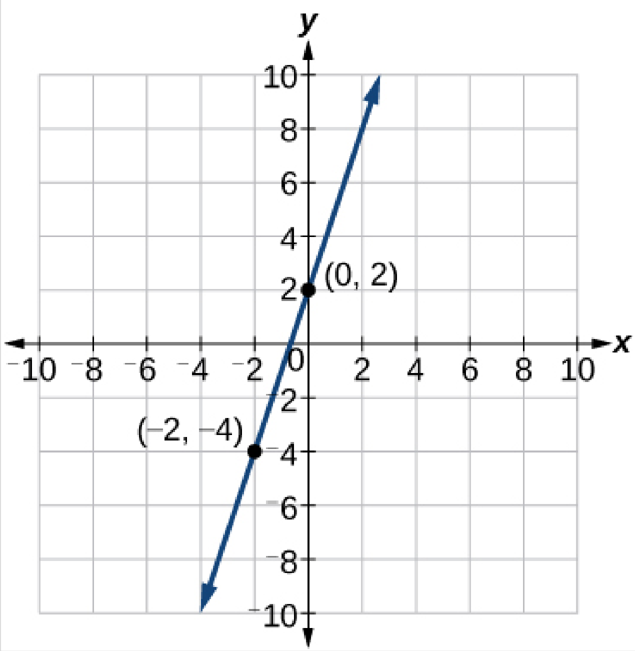
We can use algebra to rewrite the equation in the slope-intercept form.

Equation:

$$\begin{aligned} y+4 &= 3(x+2) \\ y+4 &= 3x+6 \\ y &= 3x+2 \end{aligned}$$

Analysis

This makes sense because we can see from [link] that the line crosses the *y*-axis at the point $(0,\ 2)$, which is the *y*-intercept, so $b=2$.



Example:
Exercise:

Problem:
Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of \$1,250 per month for the overhead, which includes his office rent. His production costs are \$37.50 per item. Write a linear function C where $C\left(x\right)$ is the cost for x items produced in a given month.

Solution:

The fixed cost is present every month, \$1,250. The costs that can vary include the cost to produce each item, which is \$37.50. The variable cost, called the marginal

cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C\left(x\right)=1250+37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is found by substituting 100 for x.

Equation:

$$\begin{array}{ccc}C(100)&=&1250+37.5(100)\\&=&5000\end{array}$$

So his monthly cost would be $5,000.

Example:
Exercise:

Problem:
Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3)=-2$,and $f(8)=1$,find an equation for the function in slope-intercept form.

Solution:

We can write the given points using coordinates.

Equation:

$$\begin{array}{ccc}f(3)&=&-2\to (3,-2)\\f(8)&=&1\to (8,1)\end{array}$$

We can then use the points to calculate the slope.

Equation:

$$\begin{array}{ccc}m&=&\frac{y_2-y_1}{x_2-x_1}\\&=&\frac{1-(-2)}{8-3}\\&=&\frac{3}{5}\end{array}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.
Equation:

$$\begin{aligned} y - y_1 &= m\left(x - x_1\right) \\ y - (-2) &= \tfrac{3}{5}(x - 3) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.
Equation:

$$\begin{aligned} y + 2 &= \tfrac{3}{5}(x - 3) \\ y + 2 &= \tfrac{3}{5}x - \tfrac{9}{5} \\ y &= \tfrac{3}{5}x - \tfrac{19}{5} \end{aligned}$$

Note:
Exercise:

Problem:

If $f(x)$ is a linear function, with $f(2) = -11$,and $f(4) = -25$,write an equation for the function in slope-intercept form.

Solution:

$y = -7x + 3$

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

Note:

Given a linear function f and the initial value and rate of change, evaluate $f\left(c\right)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

Example:
Exercise:

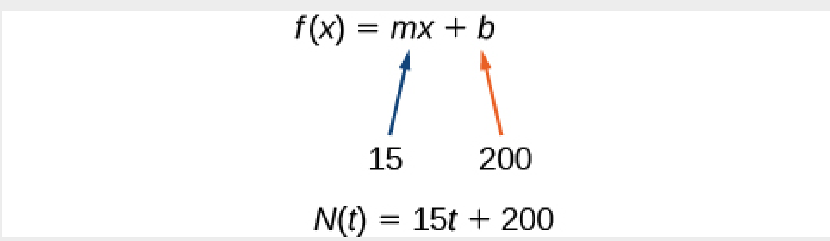
Problem:
Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N,in his collection as a function of time, t,the number of months. How many songs will he own at the end of one year?

Solution:

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$,which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.



We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have at the end of one year (12 months). In other words, we can evaluate the function at $t = 12$.
Equation:

$$\begin{aligned} N(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that *N* is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

Example:
Exercise:

Problem:
Using a Linear Function to Calculate Salary Based on Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya’s weekly income I,depends on the number of new policies, n,he sells during the week. Last week he sold 3 new policies, and earned $760 for the week. The week before, he sold 5 new policies and earned $920. Find an equation for $I(n)$,and interpret the meaning of the components of the equation.

Solution:

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

Equation:

$$\begin{aligned} m &= \frac{920-760}{5-3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by $160 when the number of policies increased by 2, so the rate of change is $80 per policy. Therefore, Ilya earns a commission of $80 for each policy sold during the week.

We can then solve for the initial value.

Equation:

$$\begin{aligned} I(n) &= 80n + b \\ 760 &= 80(3) + b \qquad \text{when } n = 3, I(3) = 760 \\ 760 &- 80(3) = b \\ 520 &= b \end{aligned}$$

The value of b is the starting value for the function and represents Ilya's income when $n = 0,$ or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

Equation:

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya's base salary is $520 per week and he earns an additional $80 commission for each policy sold.

Example:

Exercise:

Problem:

Using Tabular Form to Write an Equation for a Linear Function

[link] relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

number of weeks, *w*	0	2	4	6
number of rats, *P(w)*	1000	1080	1160	1240

Solution:

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m,we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

Equation:

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

Equation:

$$\begin{aligned} m &= \tfrac{1240-1080}{6-2} \\ &= \tfrac{160}{4} \\ &= 40 \end{aligned}$$

Note:

Is the initial value always provided in a table of values like [link]?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for b.

Note:

Exercise:

Problem:

A new plant food was introduced to a young tree to test its effect on the height of the tree. [link] shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$,where x is the number of months since the start of the experiment.

x	0	2	4	8	12
***H*(*x*)**	12.5	13.5	14.5	16.5	18.5

Solution:

$H\left(x\right)=0.5x+12.5$

Graphing Linear Functions

Now that we've seen and interpreted graphs of linear functions, let's take a look at how to create the graphs. There are three basic methods of graphing linear functions. The first is by plotting points and then drawing a line through the points. The second is by using the *y*-intercept and slope. And the third method is by using transformations of the identity function $f(x)=x$.

Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x)=2x$,we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1,2).$ Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2,4).$ Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

Note:
Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.

5. Draw a line through the points.

Example:
Exercise:

Problem:
Graphing by Plotting Points

Graph $f(x) = -\frac{2}{3}x + 5$ by plotting points.

Solution:

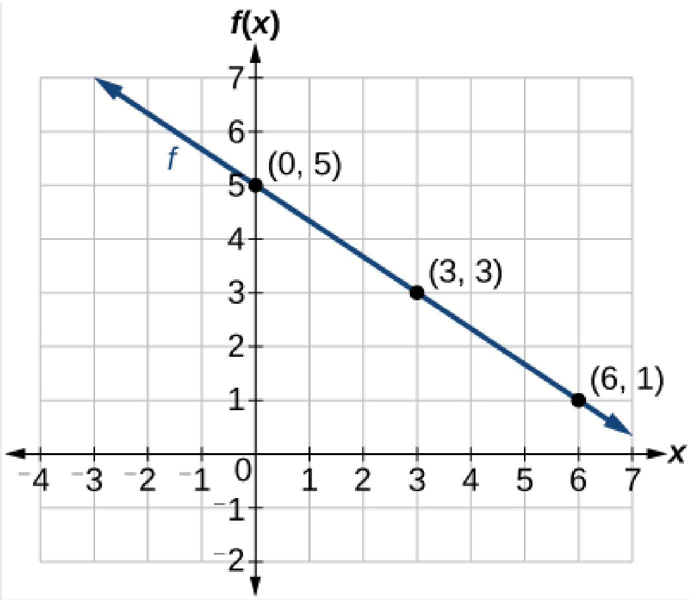
Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.

Equation:

$$\begin{array}{ll} x = 0 & f(0) = -\frac{2}{3}(0) + 5 = 5 \Rightarrow (0, 5) \\ x = 3 & f(3) = -\frac{2}{3}(3) + 5 = 3 \Rightarrow (3, 3) \\ x = 6 & f(6) = -\frac{2}{3}(6) + 5 = 1 \Rightarrow (6, 1) \end{array}$$

Plot the coordinate pairs and draw a line through the points. [link] represents the graph of the function $f(x) = -\frac{2}{3}x + 5$.



The graph of the linear function
$f(x) = -\frac{2}{3}x + 5.$

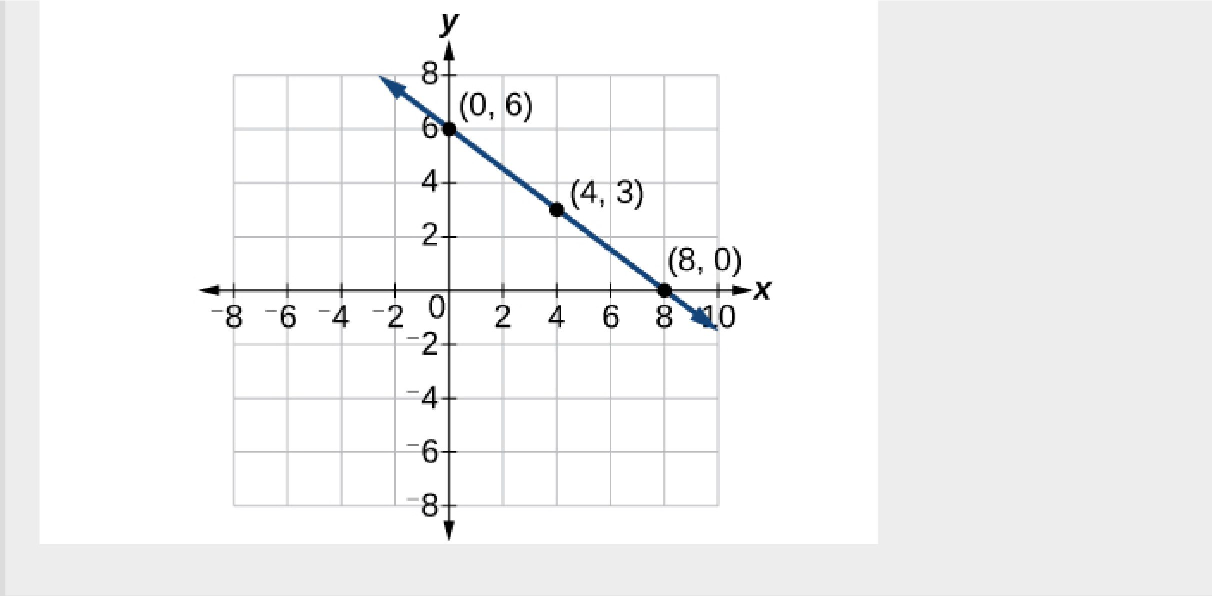
Analysis

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative, constant rate of change in the equation for the function.

Note:
Exercise:

Problem: Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.

Solution:



Graphing a Function Using *y*-intercept and Slope

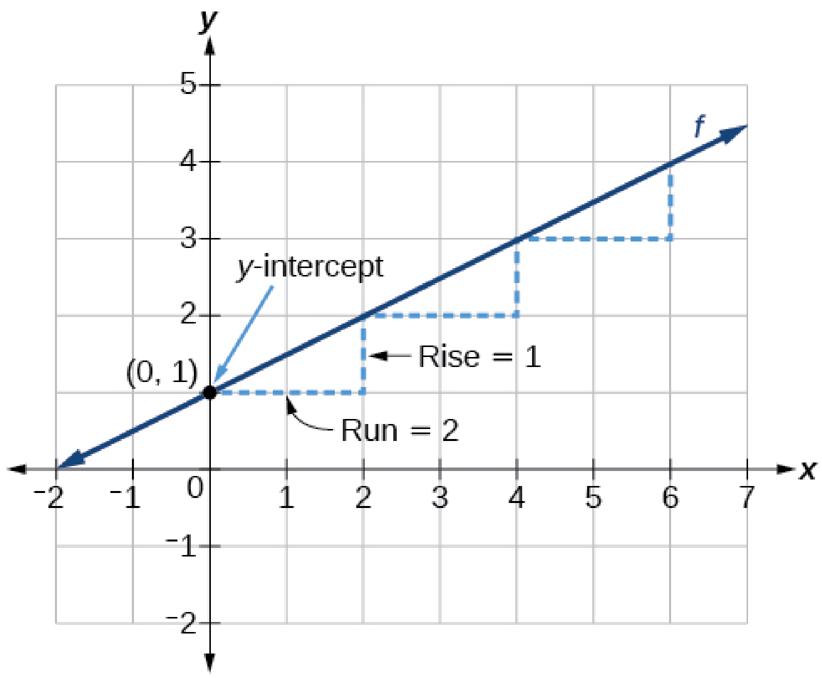
Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its *y*-intercept, which is the point at which the input value is zero. To find the *y*-intercept, we can set $x = 0$ in the equation.

The other characteristic of the linear function is its slope**.**

Let’s consider the following function.
Equation:

$$f(x) = \frac{1}{2}x + 1$$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The *y*-intercept is the point on the graph when $x = 0$. The graph crosses the *y*-axis at $(0, 1)$. Now we know the slope and the *y*-intercept. We can begin graphing by plotting the point $(0, 1)$. We know that the slope is the change in the *y*-coordinate over the change in the *x*-coordinate. This is commonly referred to as rise over run, $m = \frac{\text{rise}}{\text{run}}$. From our example, we have $m = \frac{1}{2}$,which means that the rise is 1 and the run is 2. So starting from our *y*-intercept $(0, 1)$,we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in [link].



Note:
Graphical Interpretation of a Linear Function
In the equation $f(x) = mx + b$

- b is the *y*-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the *y*-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

Equation:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note:
Do all linear functions have *y*-intercepts?
Yes. All linear functions cross the y-axis and therefore have y-intercepts. (Note: *A vertical line is parallel to the y-axis does not have a y-intercept, but it is not a function.*)

Note:
Given the equation for a linear function, graph the function using the *y*-intercept and slope.

1. Evaluate the function at an input value of zero to find the *y*-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the *y*-intercept.
4. Use $\frac{\text{rise}}{\text{run}}$ to determine at least two more points on the line.
5. Sketch the line that passes through the points.

Example:
Exercise:

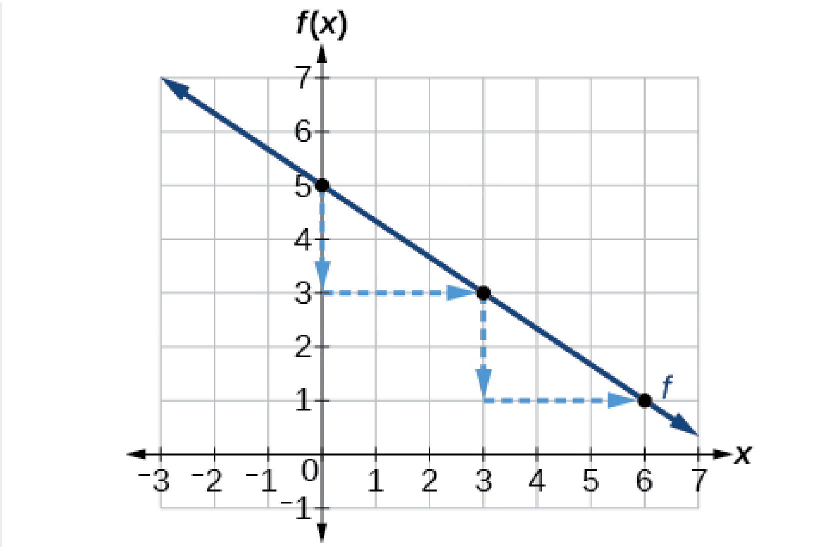
Problem:
Graphing by Using the *y*-intercept and Slope

Graph $f(x) = -\frac{2}{3}x + 5$ using the *y*-intercept and slope.

Solution:

Evaluate the function at $x = 0$ to find the *y*-intercept. The output value when $x = 0$ is 5, so the graph will cross the *y*-axis at $(0, 5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$. This tells us that for each vertical decrease in the "rise" of -2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the *y*-intercept on the graph in [link]. From the initial value $(0, 5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then drawing a line through the points.



Graph of $f(x) = -2/3x + 5$ and shows how to calculate the rise over run for the slope.

Analysis

The graph slants downward from left to right, which means it has a negative slope as expected.

Note:
Exercise:

Problem: Find a point on the graph we drew in [link] that has a negative *x*-value.

Solution:

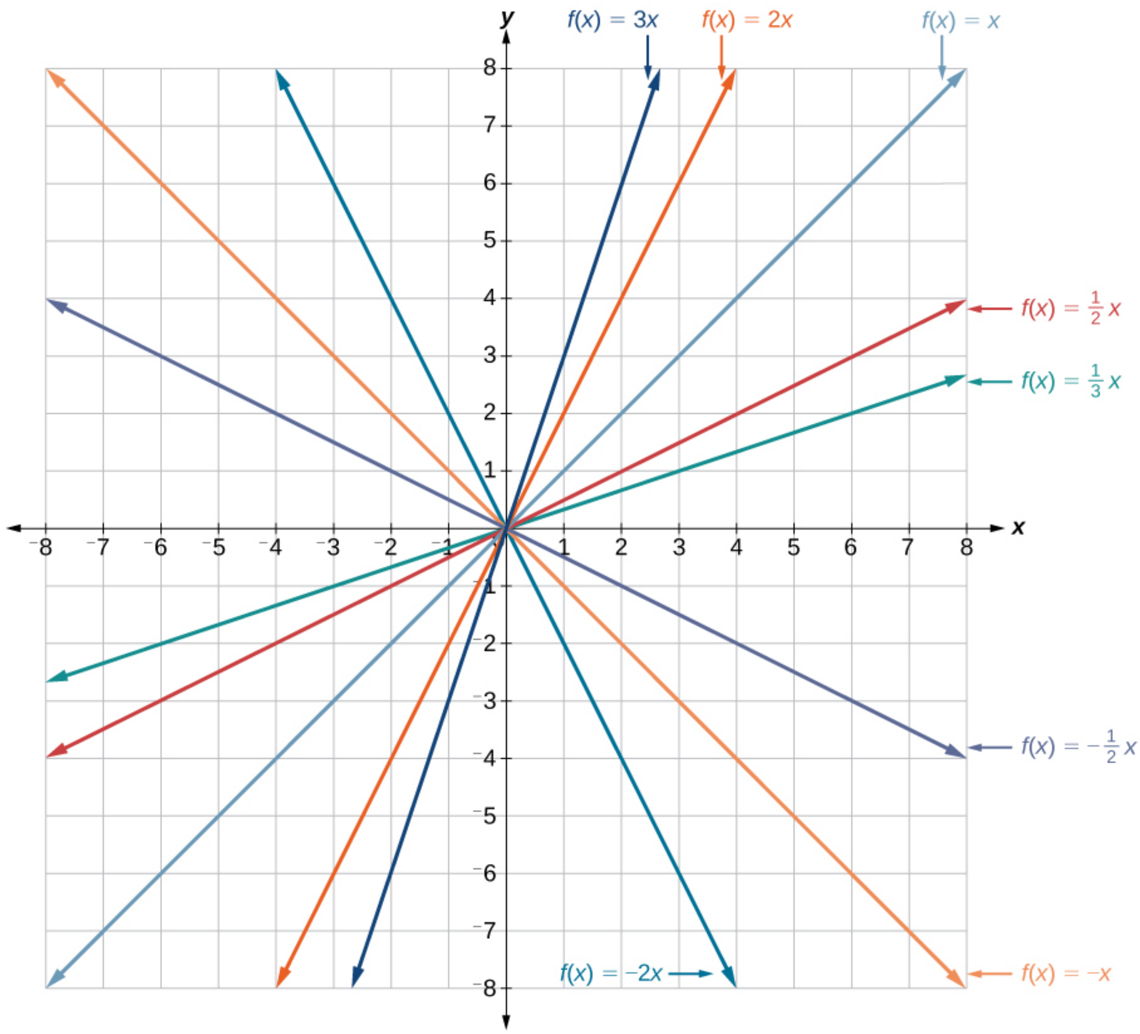
Possible answers include $(-3, 7)$, $(-6, 9)$, or $(-9, 11)$.

Graphing a Function Using Transformations

Another option for graphing is to use a transformation of the identity function $f(x) = x.$ A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

Vertical Stretch or Compression

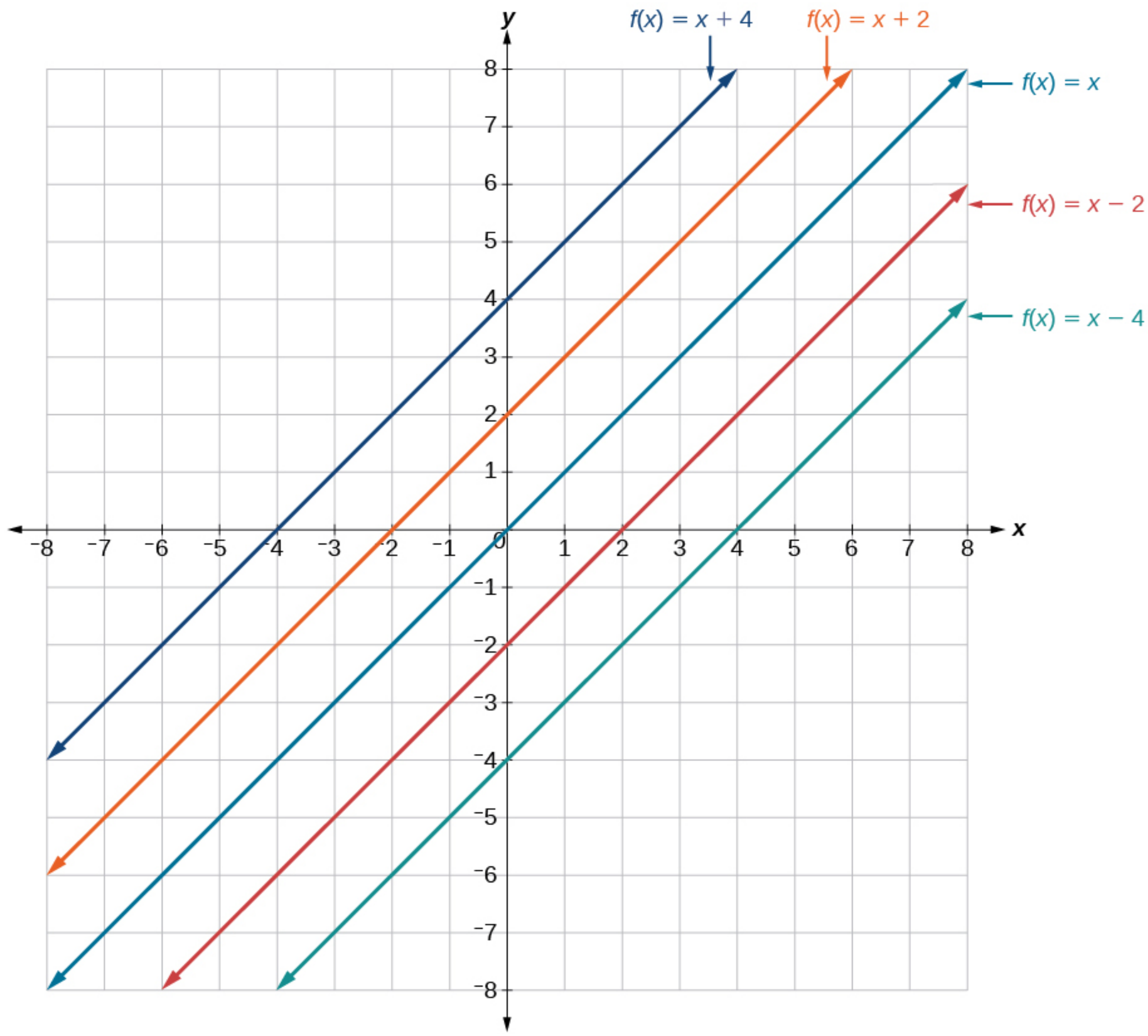
In the equation $f(x) = mx$,the m is acting as the vertical stretch or compression of the identity function. When m is negative, there is also a vertical reflection of the graph. Notice in [link] that multiplying the equation of $f(x) = x$ by m stretches the graph of f by a factor of m units if $m > 1$ and compresses the graph of f by a factor of m units if $0 < m < 1$. This means the larger the absolute value of m, the steeper the slope.



Vertical stretches and compressions and reflections on the function $f(x) = x$

Vertical Shift

In $f(x) = mx + b$,the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Notice in [link] that adding a value of b to the equation of $f\left(x\right) = x$ shifts the graph of f a total of b units up if b is positive and $|b|$ units down if b is negative.



This graph illustrates vertical shifts of the function $f(x) = x$.

Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

Note:
Given the equation of a linear function, use transformations to graph the linear function in the form $f(x) = mx + b$.

1. Graph $f(x) = x$.
2. Vertically stretch or compress the graph by a factor m.
3. Shift the graph up or down b units.

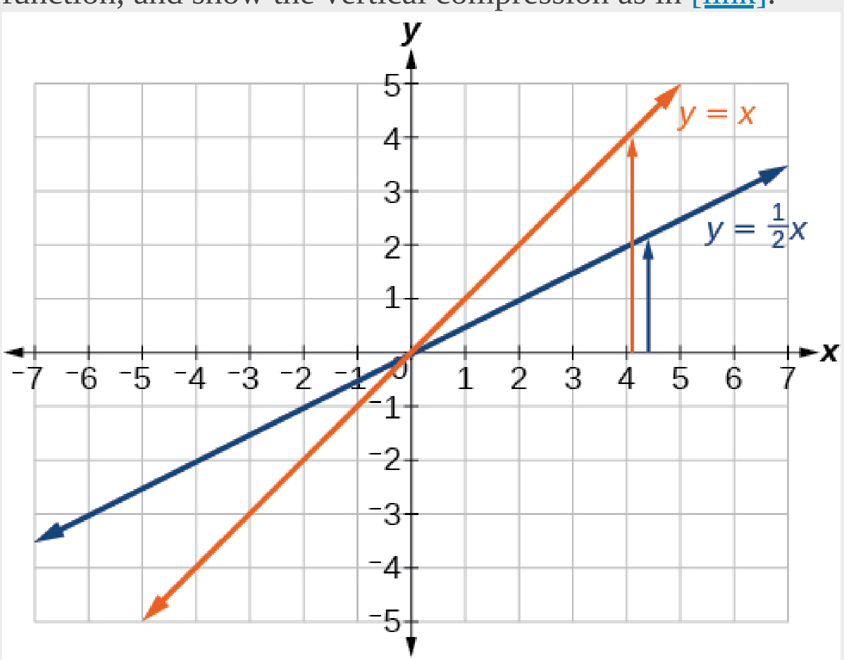
Example:
Exercise:

Problem:
Graphing by Using Transformations

Graph $f(x) = \frac{1}{2}x - 3$ using transformations.

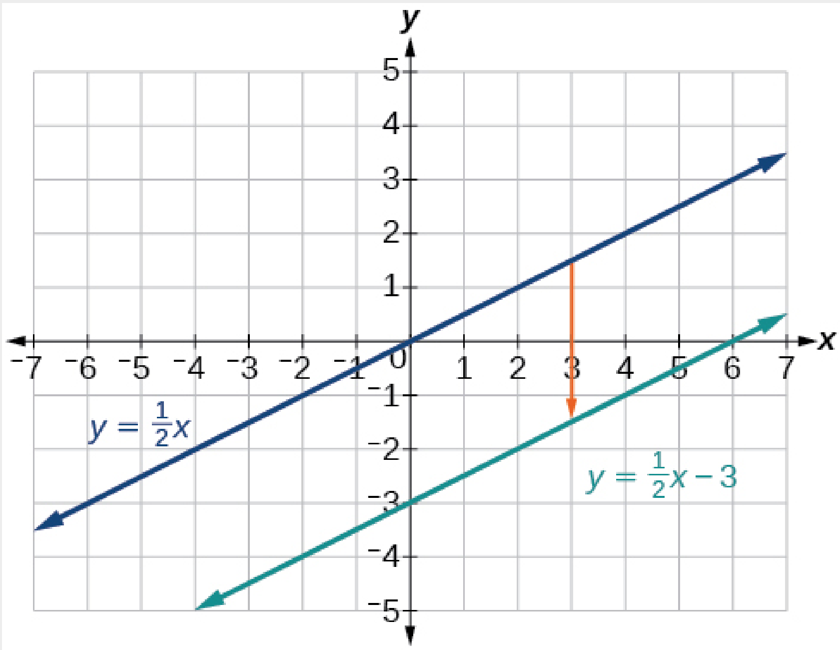
Solution:

The equation for the function shows that $m = \frac{1}{2}$ so the identity function is vertically compressed by $\frac{1}{2}$. The equation for the function also shows that $b = -3$ so the identity function is vertically shifted down 3 units. First, graph the identity function, and show the vertical compression as in [link].



The function, $y = x$,compressed by a factor of $\frac{1}{2}$

Then show the vertical shift as in [link].

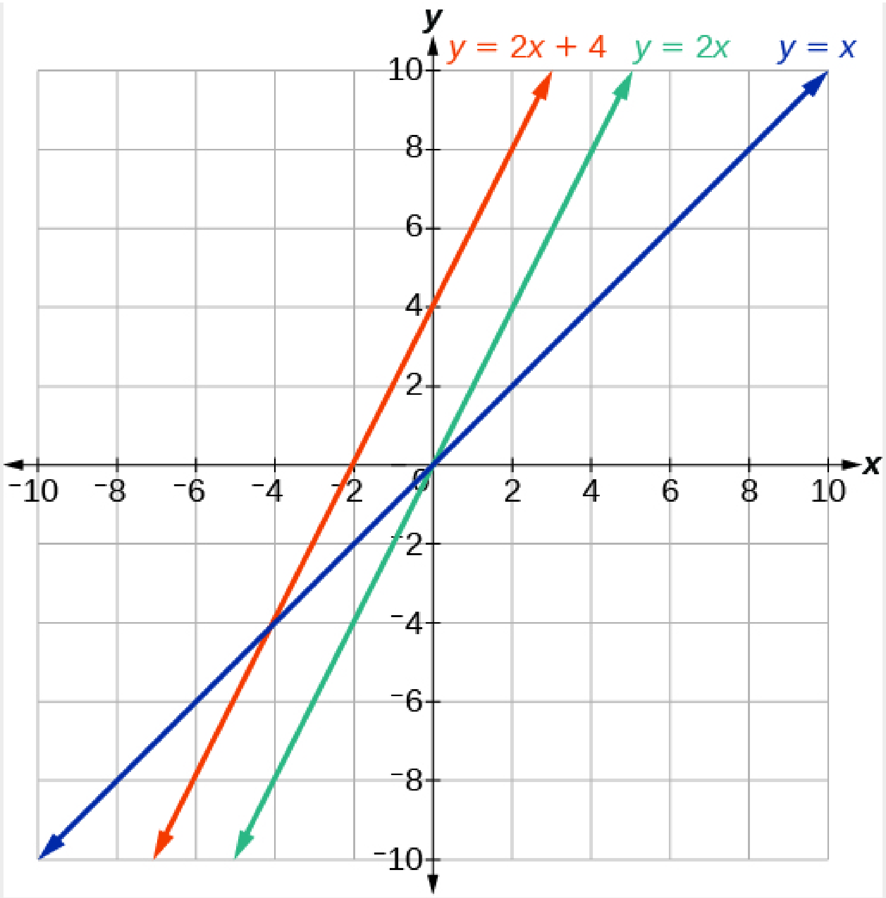


The function $y=\frac{1}{2}x$,shifted down 3 units

Note:
Exercise:

Problem: Graph $f(x)=4+2x$ using transformations.

Solution:

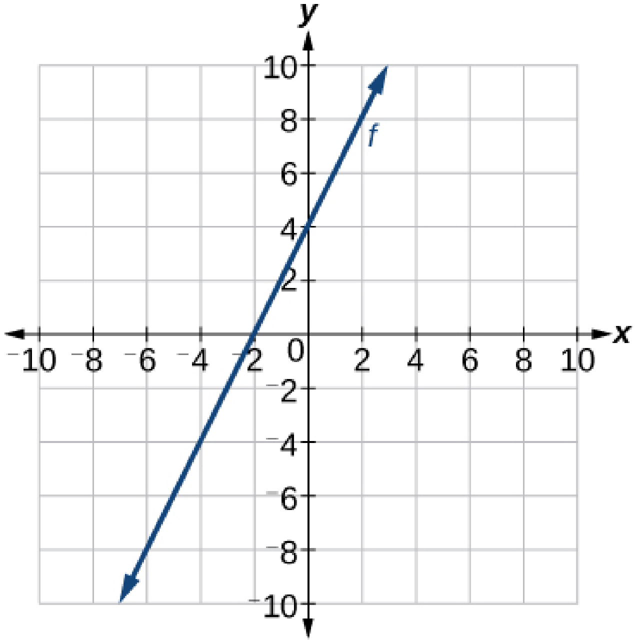


Note:
In [link], could we have sketched the graph by reversing the order of the transformations?
No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.
Equation:

$$\begin{aligned} f(2) &= \tfrac{1}{2}(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Writing the Equation for a Function from the Graph of a Line

Earlier, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at [link]. We can see right away that the graph crosses the *y*-axis at the point $(0, 4)$ so this is the *y*-intercept.



Then we can calculate the slope by finding the rise and run. We can choose any two points, but let’s look at the point $(–2, 0)$. To get from this point to the *y*-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

Equation:

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2$$

Substituting the slope and *y*-intercept into the slope-intercept form of a line gives

Equation:

$$y = 2x + 4$$

Note:

Given a graph of linear function, find the equation to describe the function.

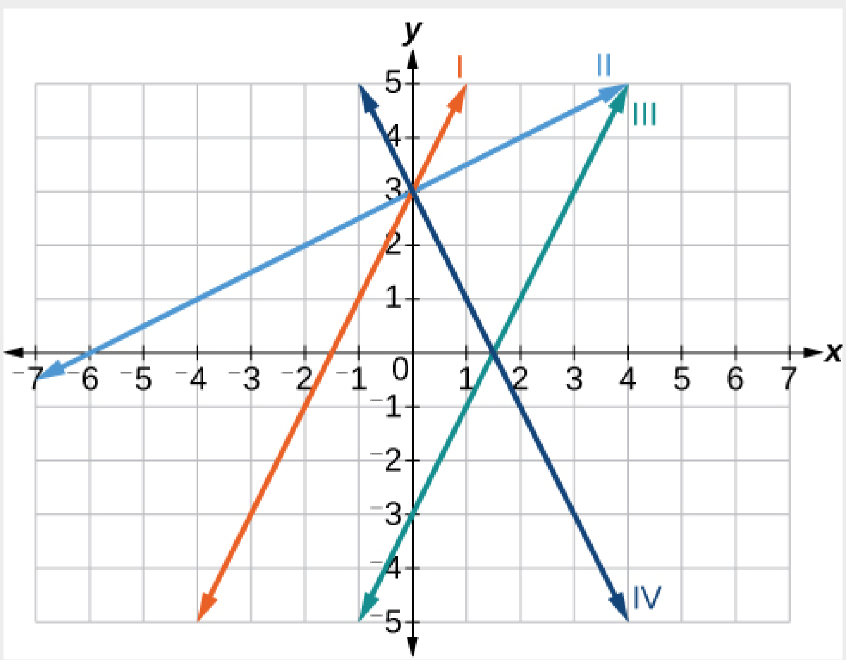
1. Identify the *y*-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the *y*-intercept and slope into the slope-intercept form of a line.

Example:
Exercise:

Problem:
Matching Linear Functions to Their Graphs

Match each equation of the linear functions with one of the lines in [link].
Equation:

$$\begin{array}{llll} a. & f(x) & = & 2x+3 \\ b. & g(x) & = & 2x-3 \\ c. & h(x) & = & -2x+3 \\ d. & j(x) & = & \frac{1}{2}x+3 \end{array}$$

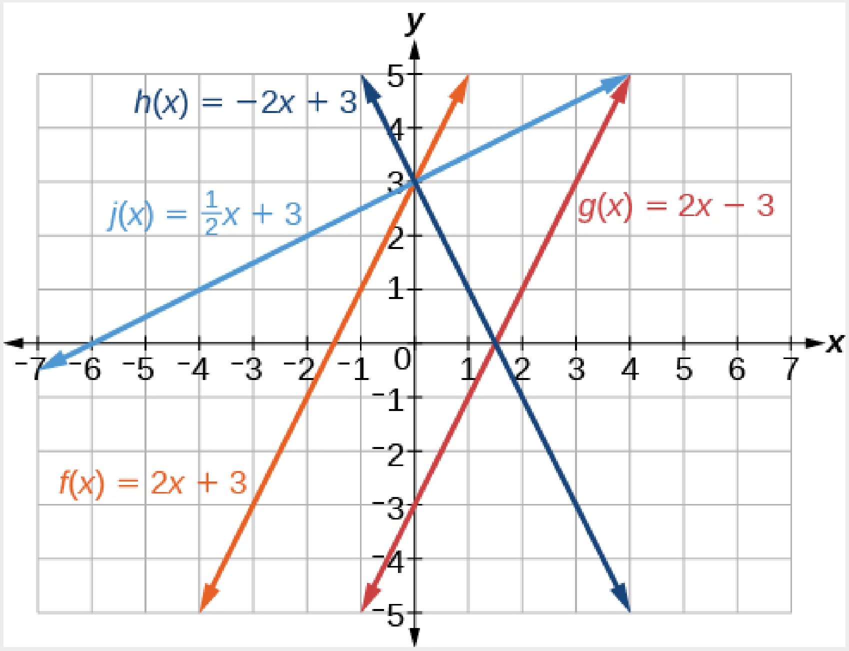


Solution:

Analyze the information for each function.

a. This function has a slope of 2 and a y-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function g has the same slope, but a different y-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through $(0, 3)$ so f must be represented by line I.
b. This function also has a slope of 2, but a y-intercept of -3. It must pass through the point $(0, -3)$ and slant upward from left to right. It must be represented by line III.
c. This function has a slope of –2 and a y-intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.
d. This function has a slope of $\frac{1}{2}$ and a y-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through $(0, 3)$,but the slope of j is less than the slope of f so the line for j must be flatter. This function is represented by Line II.

Now we can re-label the lines as in [link].



Finding the *x*-intercept of a Line

So far we have been finding the y-intercepts of a function: the point at which the graph of the function crosses the y-axis. Recall that a function may also have an x-intercept,

which is the *x*-coordinate of the point where the graph of the function crosses the *x*-axis. In other words, it is the input value when the output value is zero.

To find the *x*-intercept, set a function $f(x)$ equal to zero and solve for the value of x. For example, consider the function shown.

Equation:

$$f(x) = 3x - 6$$

Set the function equal to 0 and solve for x.

Equation:

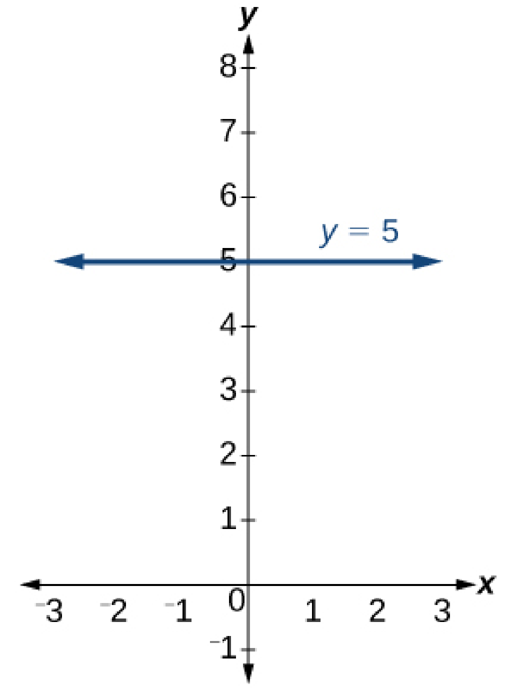
$$\begin{aligned} 0 &= 3x - 6 \\ 6 &= 3x \\ 2 &= x \\ x &= 2 \end{aligned}$$

The graph of the function crosses the *x*-axis at the point $(2, 0)$.

Note:

Do all linear functions have *x*-intercepts?

No. However, linear functions of the form $y = c$*,where* c *is a nonzero real number are the only examples of linear functions with no x-intercept. For example,* $y = 5$ *is a horizontal line 5 units above the x-axis. This function has no x-intercepts, as shown in* ***[link]***.



Note:
x-intercept
The *x*-intercept of the function is value of x when $f(x) = 0$. It can be solved by the equation $0 = mx + b$.

Example:
Exercise:

Problem:
Finding an *x*-intercept

Find the *x*-intercept of $f(x) = \frac{1}{2}x - 3$.

Solution:

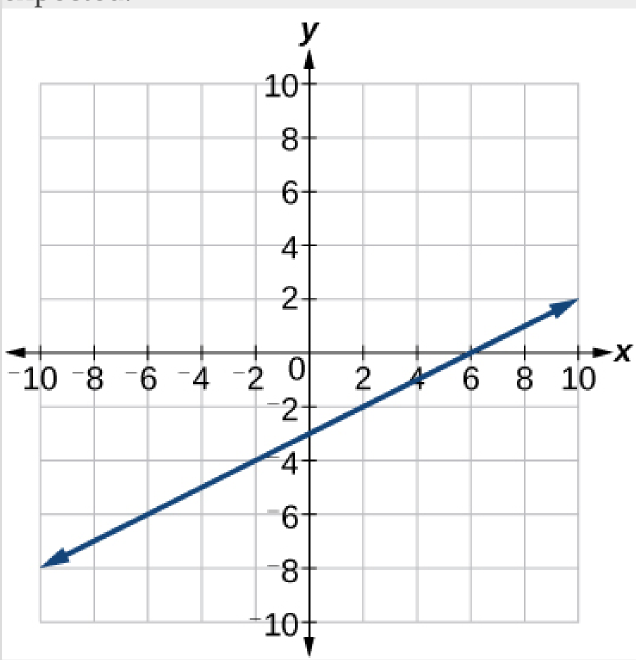
Set the function equal to zero to solve for x.
Equation:

$$\begin{array}{rcl} 0 & = & \frac{1}{2}x - 3 \\ 3 & = & \frac{1}{2}x \\ 6 & = & x \\ x & = & 6 \end{array}$$

The graph crosses the *x*-axis at the point $(6, 0)$.

Analysis

A graph of the function is shown in [link]. We can see that the *x*-intercept is $(6, 0)$ as we expected.



Note:
Exercise:

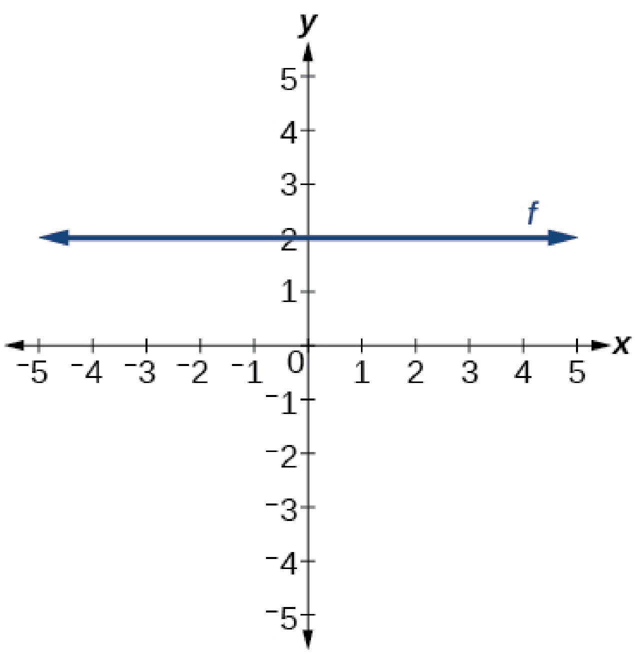
Problem: Find the *x*-intercept of $f(x) = \frac{1}{4}x - 4$.

Solution:

$(16,\ 0)$

Describing Horizontal and Vertical Lines

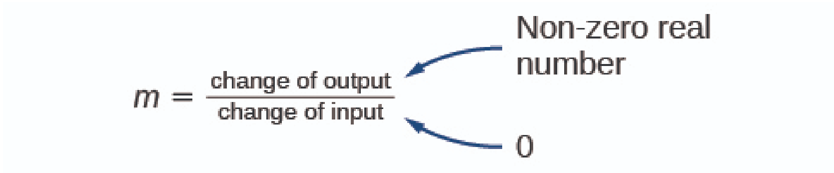
There are two special cases of lines on a graph—horizontal and vertical lines. A horizontal line indicates a constant output, or y-value. In [link], we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m = 0$ in the equation $f(x) = mx + b$,the equation simplifies to $f(x) = b$. In other words, the value of the function is a constant. This graph represents the function $f(x) = 2$.



x	−4	−2	0	2	4
y	2	2	2	2	2

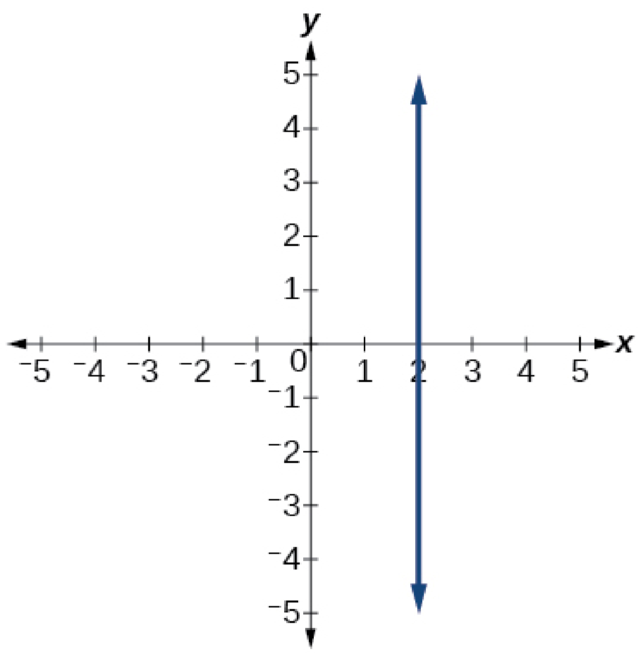
A horizontal line representing the function $f(x) = 2$

A vertical line indicates a constant input, or *x*-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.



Example of how a line has a vertical slope. 0 in the denominator of the slope.

A vertical line, such as the one in [link], has an *x*-intercept, but no *y*-intercept unless it's the line $x = 0$. This graph represents the line $x = 2$.



x	2	2	2	2	2
y	−4	−2	0	2	4

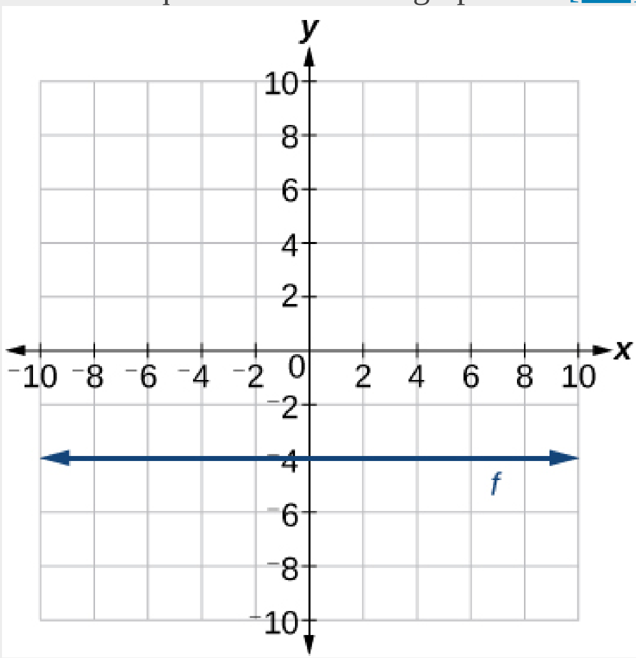
The vertical line, $x = 2,$ which does not represent a function

Note:
Horizontal and Vertical Lines
Lines can be horizontal or vertical.
A **horizontal line** is a line defined by an equation in the form $f(x) = b$.
A **vertical line** is a line defined by an equation in the form $x = a$.

Example:
Exercise:

Problem:
Writing the Equation of a Horizontal Line

Write the equation of the line graphed in [link].



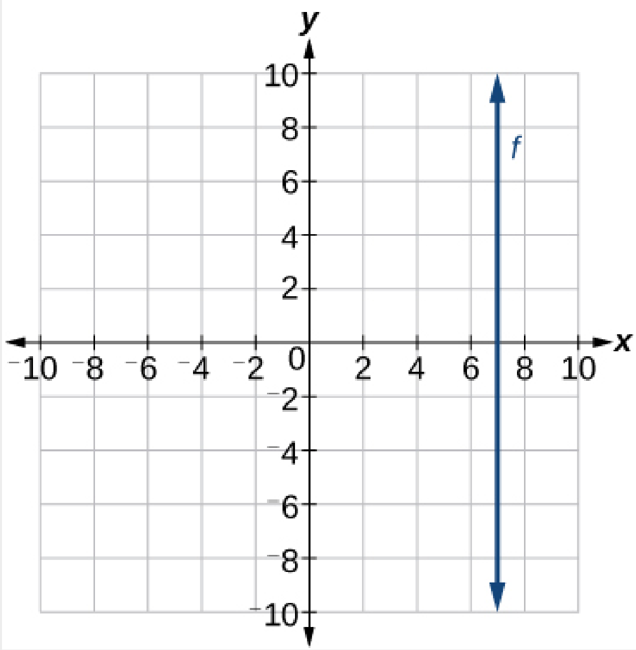
Solution:

For any *x*-value, the *y*-value is $-4,$ so the equation is $y = -4.$

Example:
Exercise:

Problem:
Writing the Equation of a Vertical Line

Write the equation of the line graphed in [link].

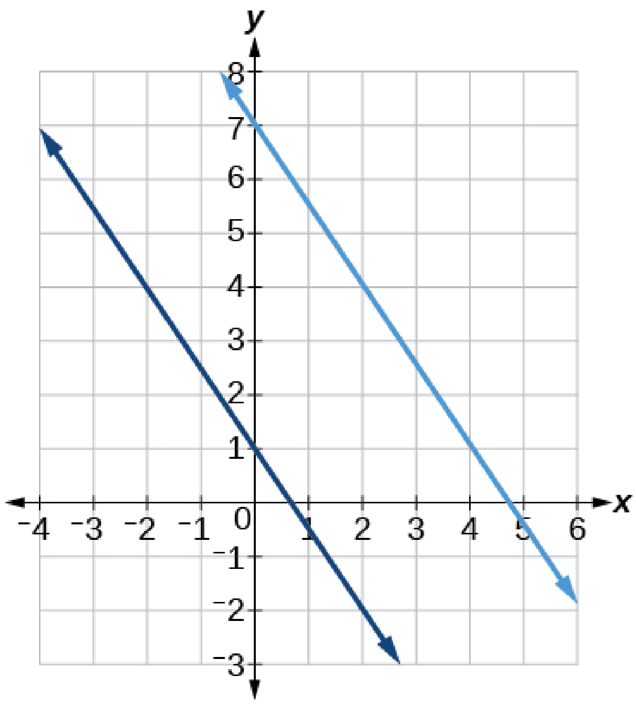


Solution:

The constant x-value is 7,so the equation is $x = 7$.

Determining Whether Lines are Parallel or Perpendicular

The two lines in [link] are parallel lines: they will never intersect. They have exactly the same steepness, which means their slopes are identical. The only difference between the two lines is the y-intercept. If we shifted one line vertically toward the other, they would become coincident.



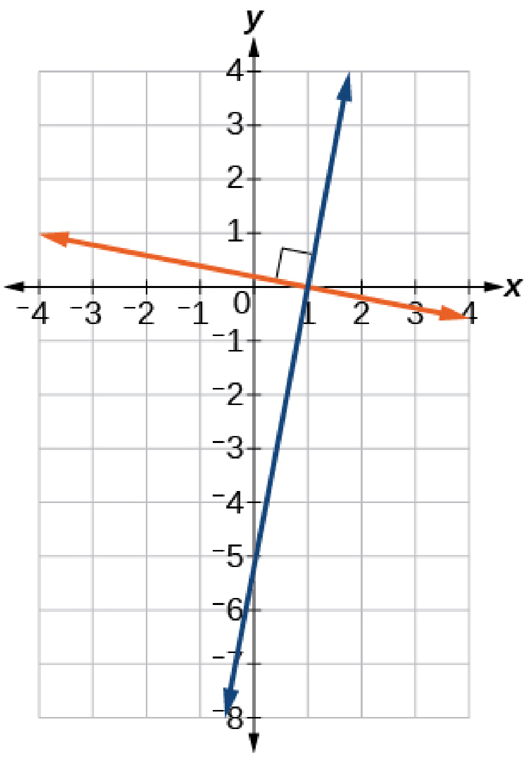
Parallel lines

We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the *y*-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

Equation:

$$\left.\begin{array}{l} f(x) = -2x + 6 \\ f(x) = -2x - 4 \end{array}\right\} \text{parallel} \qquad \left.\begin{array}{l} f(x) = 3x + 2 \\ f(x) = 2x + 2 \end{array}\right\} \text{not parallel}$$

Unlike parallel lines, perpendicular lines do intersect. Their intersection forms a right, or 90-degree, angle. The two lines in [link] are perpendicular.



Perpendicular lines

Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1. So, if m_1 and m_2 are negative reciprocals of one another, they can be multiplied together to yield -1.

Equation:

$$m_1 m_2 = -1$$

To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is $\frac{1}{8}$, and the reciprocal of $\frac{1}{8}$ is 8. To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes, assuming that the lines are neither horizontal nor vertical. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.

Equation:

$$\begin{aligned} f(x) &= \tfrac{1}{4}x + 2 && \text{negative reciprocal of } \tfrac{1}{4} \text{ is } -4 \\ f(x) &= -4x + 3 && \text{negative reciprocal of } -4 \text{ is } \tfrac{1}{4} \end{aligned}$$

The product of the slopes is –1.

Equation:

$$-4\left(\frac{1}{4}\right) = -1$$

Note:

Parallel and Perpendicular Lines

Two lines are **parallel lines** if they do not intersect. The slopes of the lines are the same.

Equation:

$$f(x) = m_1x + b_1 \text{ and } g(x) = m_2x + b_2 \text{ are parallel if and only if } m_1 = m_2$$

If and only if $b_1 = b_2$ and $m_1 = m_2,$ we say the lines coincide. Coincident lines are the same line.

Two lines are **perpendicular lines** if they intersect to form a right angle.

Equation:

$$f(x) = m_1x + b_1 \text{ and } g(x) = m_2x + b_2 \text{ are perpendicular if and only if}$$

Equation:

$$m_1m_2 = -1, \text{so } m_2 = -\frac{1}{m_1}$$

Example:

Exercise:

Problem:

Identifying Parallel and Perpendicular Lines

Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.

Equation:

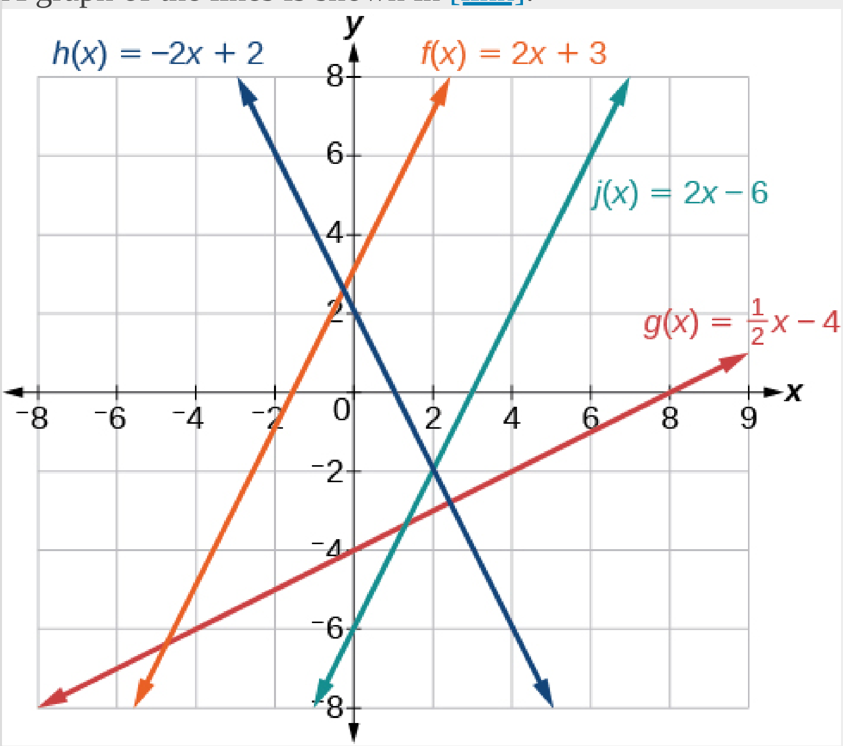
$$\begin{aligned} f(x) &= 2x+3 & \qquad h(x) &= -2x+2 \\ g(x) &= \tfrac{1}{2}x-4 & \qquad j(x) &= 2x-6 \end{aligned}$$

Solution:

Parallel lines have the same slope. Because the functions $f(x) = 2x + 3$ and $j(x) = 2x - 6$ each have a slope of 2, they represent parallel lines. Perpendicular lines have negative reciprocal slopes. Because −2 and $\frac{1}{2}$ are negative reciprocals, the functions $g(x) = \frac{1}{2}x - 4$ and $h(x) = -2x + 2$ represent perpendicular lines.

Analysis

A graph of the lines is shown in [link].



The graph shows that the lines $f(x) = 2x + 3$ and $j(x) = 2x – 6$ are parallel, and the lines $g(x) = \frac{1}{2}x – 4$ and $h(x) = -2x + 2$ are perpendicular.

Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Writing Equations of Parallel Lines

Suppose for example, we are given the equation shown.
Equation:

$$f(x) = 3x + 1$$

We know that the slope of the line formed by the function is 3. We also know that the *y*-intercept is $(0, 1)$. Any other line with a slope of 3 will be parallel to $f(x)$. So the lines formed by all of the following functions will be parallel to $f(x)$.
Equation:

$$\begin{array}{rcl} g(x) & = & 3x + 6 \\ h(x) & = & 3x + 1 \\ p(x) & = & 3x + \frac{2}{3} \end{array}$$

Suppose then we want to write the equation of a line that is parallel to f and passes through the point $(1, 7)$. This type of problem is often described as a point-slope problem because we have a point and a slope. In our example, we know that the slope is 3. We need to determine which value of b will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.
Equation:

$$\begin{array}{rcl} y - y_1 & = & m(x - x_1) \\ y - 7 & = & 3(x - 1) \\ y - 7 & = & 3x - 3 \\ y & = & 3x + 4 \end{array}$$

So $g(x) = 3x + 4$ is parallel to $f\ (x) = 3x + 1$ and passes through the point $(1, 7)$.

Note:
Given the equation of a function and a point through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.

3. Simplify.

Example:
Exercise:

Problem:
Finding a Line Parallel to a Given Line

Find a line parallel to the graph of $f(x) = 3x + 6$ that passes through the point $(3, 0)$.

Solution:

The slope of the given line is 3. If we choose the slope-intercept form, we can substitute $m = 3$, $x = 3$,and $f(x) = 0$ into the slope-intercept form to find the *y*-intercept.

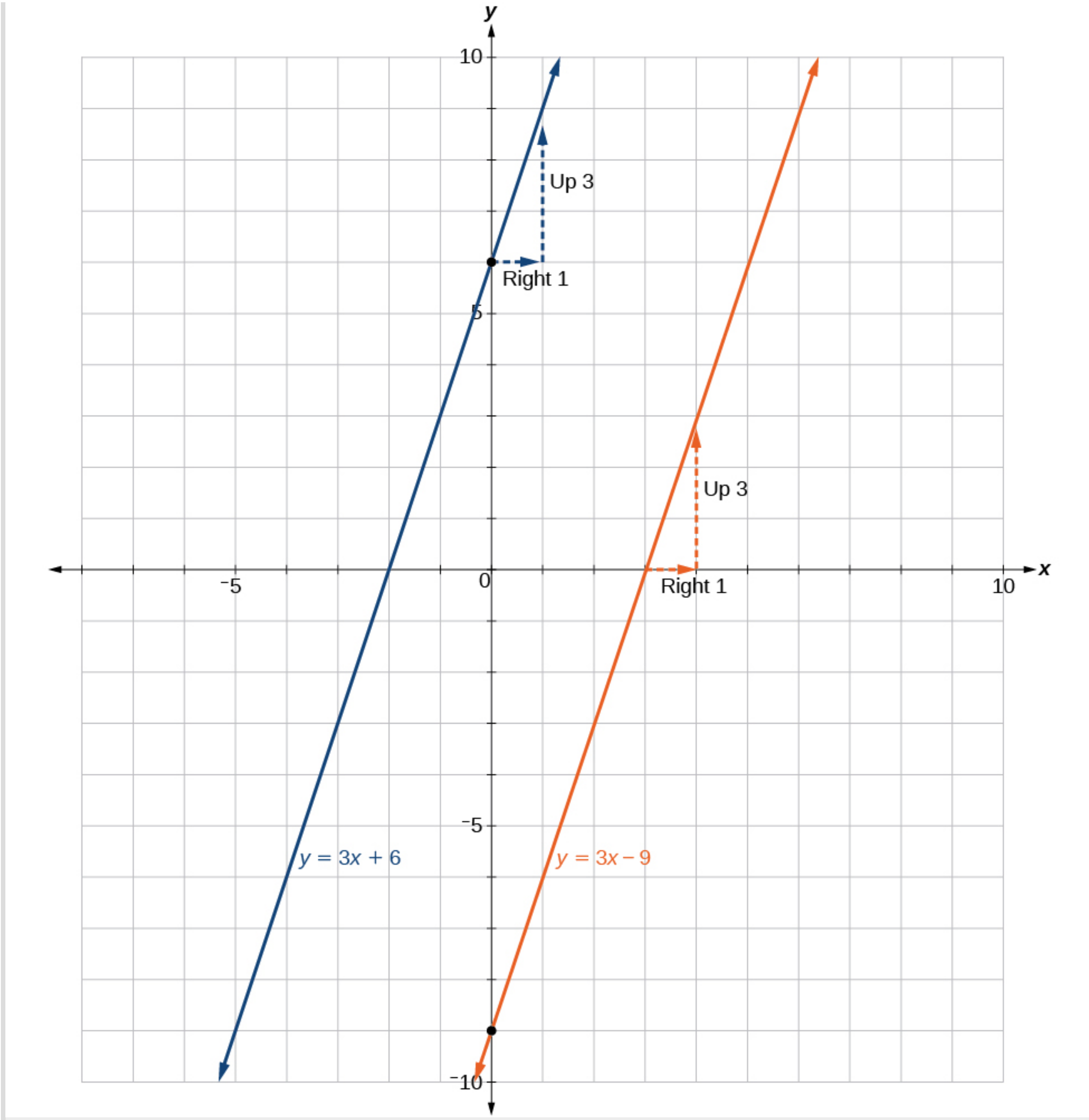
Equation:

$$\begin{array}{rcl} g(x) & = & 3x + b \\ 0 & = & 3(3) + b \\ b & = & -9 \end{array}$$

The line parallel to $f(x)$ that passes through $(3, 0)$ is $g(x) = 3x - 9$.

Analysis

We can confirm that the two lines are parallel by graphing them. [link] shows that the two lines will never intersect.



Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the function shown.

Equation:

$$f(x) = 2x + 4$$

The slope of the line is 2, and its negative reciprocal is $-\frac{1}{2}$. Any function with a slope of $-\frac{1}{2}$ will be perpendicular to $f(x)$. So the lines formed by all of the following functions will be perpendicular to $f(x)$.

Equation:

$$\begin{array}{rcl} g(x) & = & -\frac{1}{2}x + 4 \\ h(x) & = & -\frac{1}{2}x + 2 \\ p(x) & = & -\frac{1}{2}x - \frac{1}{2} \end{array}$$

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to $f(x)$ and passes through the point $(4, 0)$. We already know that the slope is $-\frac{1}{2}$. Now we can use the point to find the *y*-intercept by substituting the given values into the slope-intercept form of a line and solving for *b*.

Equation:

$$\begin{array}{rcl} g(x) & = & mx + b \\ 0 & = & -\frac{1}{2}(4) + b \\ 0 & = & -2 + b \\ 2 & = & b \\ b & = & 2 \end{array}$$

The equation for the function with a slope of $-\frac{1}{2}$ and a *y*-intercept of 2 is

Equation:

$$g(x) = -\frac{1}{2}x + 2$$

So $g(x) = -\frac{1}{2}x + 2$ is perpendicular to $f(x) = 2x + 4$ and passes through the point $(4, 0)$. Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

Note:
A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not –1. Doesn't this fact contradict the definition of perpendicular lines?

No. For two perpendicular linear functions, the product of their slopes is –1. However, a vertical line is not a function so the definition is not contradicted.

Note:
Given the equation of a function and a point through which its graph passes, write the equation of a line perpendicular to the given line.

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for x and y from the coordinate pair provided into $g\left(x\right)=mx+b$.
4. Solve for b.
5. Write the equation of the line.

Example:
Exercise:

Problem:
Finding the Equation of a Perpendicular Line

Find the equation of a line perpendicular to $f(x)=3x+3$ that passes through the point $(3,0)$.

Solution:

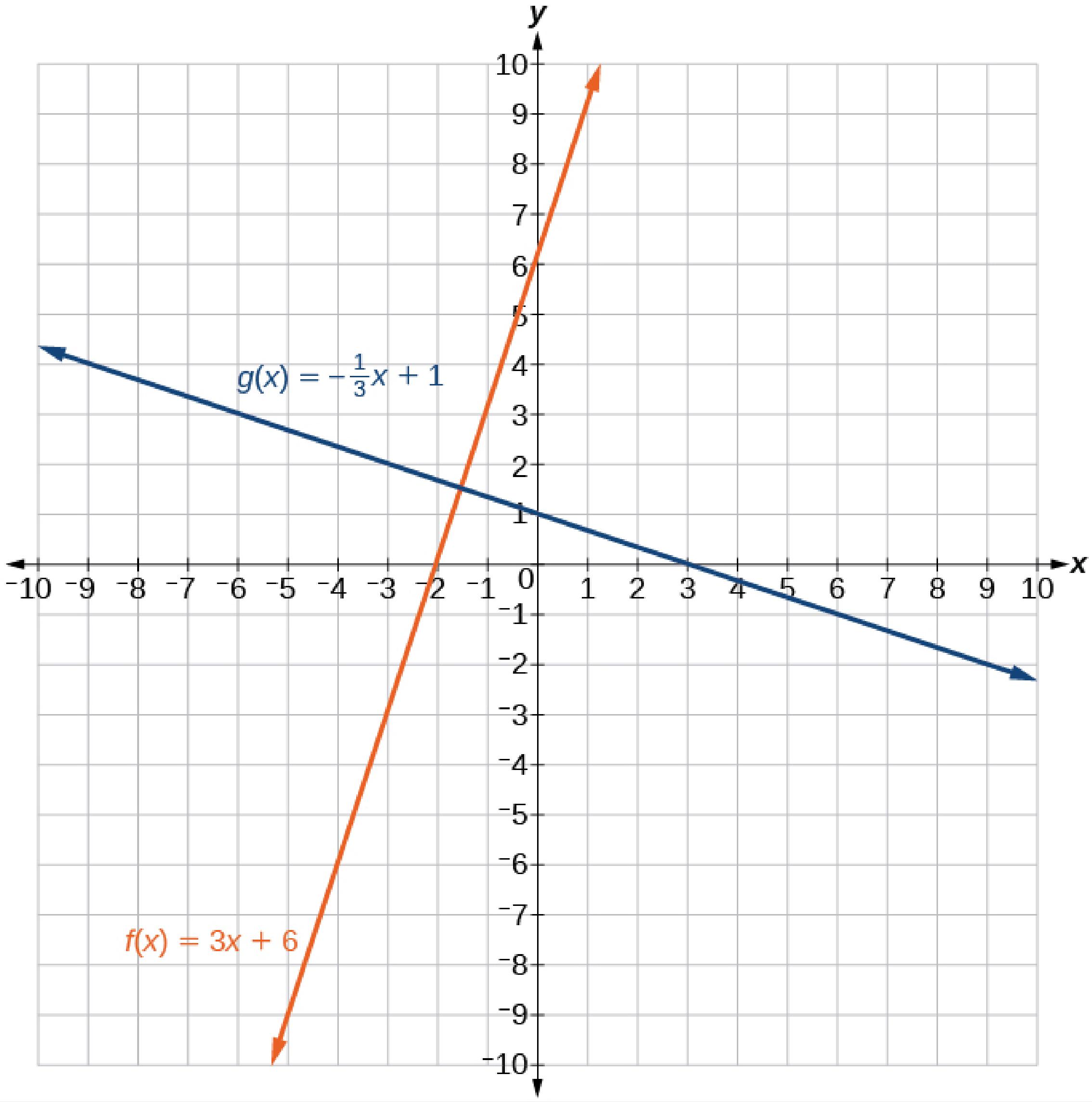
The original line has slope $m=3,$ so the slope of the perpendicular line will be its negative reciprocal, or $-\frac{1}{3}$. Using this slope and the given point, we can find the equation of the line.
Equation:

$$\begin{array}{rcl}g(x)&=&-\frac{1}{3}x+b\\ 0&=&-\frac{1}{3}(3)+b\\ 1&=&b\\ b&=&1\end{array}$$

The line perpendicular to $f(x)$ that passes through $(3,0)$ is $g(x)=-\frac{1}{3}x+1$.

Analysis

A graph of the two lines is shown in [link].



Note that that if we graph perpendicular lines on a graphing calculator using standard zoom, the lines may not appear to be perpendicular. Adjusting the window will make it possible to zoom in further to see the intersection more closely.

Note:
Exercise:

Problem:

Given the function $h(x) = 2x - 4$,write an equation for the line passing through $(0, 0)$ that is

a. parallel to $h(x)$
b. perpendicular to $h(x)$

Solution:

a. $f(x) = 2x$;b. $g(x) = -\frac{1}{2}x$

Note:
Given two points on a line and a third point, write the equation of the perpendicular line that passes through the point.

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Use the slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

Example:
Exercise:

Problem:
Finding the Equation of a Line Perpendicular to a Given Line Passing through a Point

A line passes through the points $(-2, 6)$ and $(4, 5)$. Find the equation of a perpendicular line that passes through the point $(4, 5)$.

Solution:

From the two points of the given line, we can calculate the slope of that line.
Equation:

$$\begin{aligned} m_1 &= \frac{5-6}{4-(-2)} \\ &= \frac{-1}{6} \\ &= -\frac{1}{6} \end{aligned}$$

Find the negative reciprocal of the slope.
Equation:

$$\begin{aligned} m_2 &= \frac{-1}{-\frac{1}{6}} \\ &= -1\left(-\frac{6}{1}\right) \\ &= 6 \end{aligned}$$

We can then solve for the *y*-intercept of the line passing through the point $(4, 5)$.
Equation:

$$\begin{aligned} g(x) &= 6x + b \\ 5 &= 6(4) + b \\ 5 &= 24 + b \\ -19 &= b \\ b &= -19 \end{aligned}$$

The equation for the line that is perpendicular to the line passing through the two given points and also passes through point $(4, 5)$ is
Equation:

$$y = 6x - 19$$

Note:
Exercise:

Problem:

A line passes through the points, $(-2, -15)$ and $(2, -3)$. Find the equation of a perpendicular line that passes through the point, $(6, 4)$.

Solution:

$y = -\frac{1}{3}x + 6$

Note:

Access this online resource for additional instruction and practice with linear functions.

- Linear Functions
- Finding Input of Function from the Output and Graph
- Graphing Functions using Tables

Key Concepts

- Linear functions can be represented in words, function notation, tabular form, and graphical form. See [link].
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope. A decreasing linear function results in a graph that slants downward from left to right and has a negative slope. A constant linear function results in a graph that is a horizontal line. See [link].
- Slope is a rate of change. The slope of a linear function can be calculated by dividing the difference between *y*-values by the difference in corresponding *x*-values of any two points on the line. See [link] and [link].
- An equation for a linear function can be written from a graph. See [link].
- The equation for a linear function can be written if the slope m and initial value b are known. See [link] and [link].
- A linear function can be used to solve real-world problems given information in different forms. See [link], [link], and [link].
- Linear functions can be graphed by plotting points or by using the *y*-intercept and slope. See [link] and [link].
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections. See [link].
- The equation for a linear function can be written by interpreting the graph. See [link].
- The *x*-intercept is the point at which the graph of a linear function crosses the *x*-axis. See [link].
- Horizontal lines are written in the form, $f(x) = b.$ See [link].
- Vertical lines are written in the form, $x = b.$ See [link].
- Parallel lines have the same slope. Perpendicular lines have negative reciprocal slopes, assuming neither is vertical. See [link].
- A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the *x*- and *y*-values of the given point into the equation, $f(x) = mx + b$, and using the b that results. Similarly, the point-slope form of an equation can also be used. See [link].
- A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope. See

[link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Terry is skiing down a steep hill. Terry's elevation, $E(t)$,in feet after t seconds is given by $E(t) = 3000 - 70t.$ Write a complete sentence describing Terry’s starting elevation and how it is changing over time.

Solution:

Terry starts at an elevation of 3000 feet and descends 70 feet per second.

Exercise:

Problem:

Jessica is walking home from a friend’s house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate in miles per hour?

Exercise:

Problem:

A boat is 100 miles away from the marina, sailing directly toward it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.

Solution:

$d\left(t\right) = 100 - 10t$

Exercise:

Problem:

If the graphs of two linear functions are perpendicular, describe the relationship between the slopes and the y-intercepts.

Exercise:

Problem:

If a horizontal line has the equation $f\left(x\right)=a$ and a vertical line has the equation $x=a,$ what is the point of intersection? Explain why what you found is the point of intersection.

Solution:

The point of intersection is $\left(a,\text{ }a\right).$ This is because for the horizontal line, all of the y coordinates are a and for the vertical line, all of the x coordinates are $a.$ The point of intersection is on both lines and therefore will have these two characteristics.

Algebraic

For the following exercises, determine whether the equation of the curve can be written as a linear function.

Exercise:

Problem:$y=\frac{1}{4}x+6$

Exercise:

Problem:$y=3x-5$

Solution:

Yes

Exercise:

Problem:$y=3{x}^{2}-2$

Exercise:

Problem:$3x+5y=15$

Solution:

Yes

Exercise:

Problem: $3x^2 + 5y = 15$

Exercise:

Problem: $3x + 5y^2 = 15$

Solution:

No

Exercise:

Problem: $-2x^2 + 3y^2 = 6$

Exercise:

Problem: $-\frac{x-3}{5} = 2y$

Solution:

Yes

For the following exercises, determine whether each function is increasing or decreasing.

Exercise:

Problem: $f\left(x\right) = 4x + 3$

Exercise:

Problem: $g\left(x\right) = 5x + 6$

Solution:

Increasing

Exercise:

Problem: $a\left(x\right) = 5 - 2x$

Exercise:

Problem: $b\left(x\right) = 8 - 3x$

Solution:

Decreasing

Exercise:

Problem: $h\left(x\right)=-2x+4$

Exercise:

Problem: $k\left(x\right)=-4x+1$

Solution:

Decreasing

Exercise:

Problem: $j\left(x\right)=\frac{1}{2}x-3$

Exercise:

Problem: $p\left(x\right)=\frac{1}{4}x-5$

Solution:

Increasing

Exercise:

Problem: $n\left(x\right)=-\frac{1}{3}x-2$

Exercise:

Problem: $m\left(x\right)=-\frac{3}{8}x+3$

Solution:

Decreasing

For the following exercises, find the slope of the line that passes through the two given points.

Exercise:

Problem: $(2, 4)$ and $(4, 10)$

Exercise:

Problem: $(1, 5)$ and $(4, 11)$

Solution:

2

Exercise:

Problem: $(–1, 4)$ and $(5, 2)$

Exercise:

Problem: $(8, –2)$ and $(4, 6)$

Solution:

–2

Exercise:

Problem: $(6, 11)$ and $(–4, 3)$

For the following exercises, given each set of information, find a linear equation satisfying the conditions, if possible.

Exercise:

Problem: $f(-5) = -4,$ and $f(5) = 2$

Solution:

$y = \frac{3}{5}x - 1$

Exercise:

Problem: $f(-1) = 4,$ and $f(5) = 1$

Exercise:

Problem: Passes through $(2, 4)$ and $(4, 10)$

Solution:

$y = 3x - 2$

Exercise:

Problem: Passes through $(1, 5)$ and $(4, 11)$

Exercise:

Problem: Passes through $(-1, 4)$ and $(5, 2)$

Solution:

$y = -\frac{1}{3}x + \frac{11}{3}$

Exercise:

Problem: Passes through $(-2, 8)$ and $(4, 6)$

Exercise:

Problem: *x* intercept at $(-2, 0)$ and *y* intercept at $(0, -3)$

Solution:

$y = -1.5x - 3$

Exercise:

Problem: *x* intercept at $(-5, 0)$ and *y* intercept at $(0, 4)$

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither.

Exercise:

Problem:

$$4x - 7y = 10$$
$$7x + 4y = 1$$

Solution:

perpendicular

Exercise:

Problem: $$\begin{array}{l}3y+x=12\\ -y=8x+1\end{array}$$

Exercise:

Problem: $$\begin{array}{l}3y+4x=12\\ -6y=8x+1\end{array}$$

Solution:

parallel

Exercise:

Problem: $$\begin{array}{l}6x-9y=10\\ 3x+2y=1\end{array}$$

For the following exercises, find the *x*- and *y*-intercepts of each equation.

Exercise:

Problem: $f\left(x\right)=-x+2$

Solution:

$$\begin{array}{l}f(0)=-(0)+2\\ f(0)=2\\ y-\text{int}:(0,2)\\ 0=-x+2\\ x-\text{int}:(2,0)\end{array}$$

Exercise:

Problem: $g\left(x\right)=2x+4$

Exercise:

Problem: $h\left(x\right)=3x-5$

Solution:

$$h(0)=3(0)-5$$
$$h(0)=-5$$
$$y-\text{int}:(0,-5)$$
$$0=3x-5$$
$$x-\text{int}:\left(\frac{5}{3},0\right)$$

Exercise:

Problem:$k\left(x\right)=-5x+1$

Exercise:

Problem:$-2x+5y=20$

Solution:

$$-2x+5y=20$$
$$-2(0)+5y=20$$
$$5y=20$$
$$y=4$$
$$y-\text{int}:(0,4)$$
$$-2x+5(0)=20$$
$$x=-10$$
$$x-\text{int}:(-10,0)$$

Exercise:

Problem:$7x+2y=56$

For the following exercises, use the descriptions of each pair of lines given below to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?
Exercise:

Problem: Line 1: Passes through $(0,6)$ and $(3,-24)$

Line 2: Passes through $(-1,19)$ and $(8,-71)$

Solution:

Line 1: *m* = –10 Line 2: *m* = –10 Parallel

Exercise:

Problem: Line 1: Passes through $(-8, -55)$ and $(10, 89)$

Line 2: Passes through $(9, -44)$ and $(4, -14)$

Exercise:

Problem: Line 1: Passes through $(2, 3)$ and $(4, -1)$

Line 2: Passes through $(6, 3)$ and $(8, 5)$

Solution:

Line 1: m = –2 Line 2: m = 1 Neither

Exercise:

Problem: Line 1: Passes through $(1, 7)$ and $(5, 5)$

Line 2: Passes through $(-1, -3)$ and $(1, 1)$

Exercise:

Problem: Line 1: Passes through $(2, 5)$ and $(5, -1)$

Line 2: Passes through $(-3, 7)$ and $(3, -5)$

Solution:

$\text{Line 1}: \ m =-2 \quad \text{Line 2}: \ m =-2 \quad \text{Parallel}$

For the following exercises, write an equation for the line described.

Exercise:

Problem:

Write an equation for a line parallel to $f(x) = -5x - 3$ and passing through the point $(2, –12)$.

Exercise:

Problem:

Write an equation for a line parallel to $g(x) = 3x - 1$ and passing through the point $(4, 9)$.

Solution:

$y = 3x - 3$

Exercise:

Problem:

Write an equation for a line perpendicular to $h(t) = -2t + 4$ and passing through the point $(-4, -1)$.

Exercise:

Problem:

Write an equation for a line perpendicular to $p(t) = 3t + 4$ and passing through the point $(3, 1)$.

Solution:

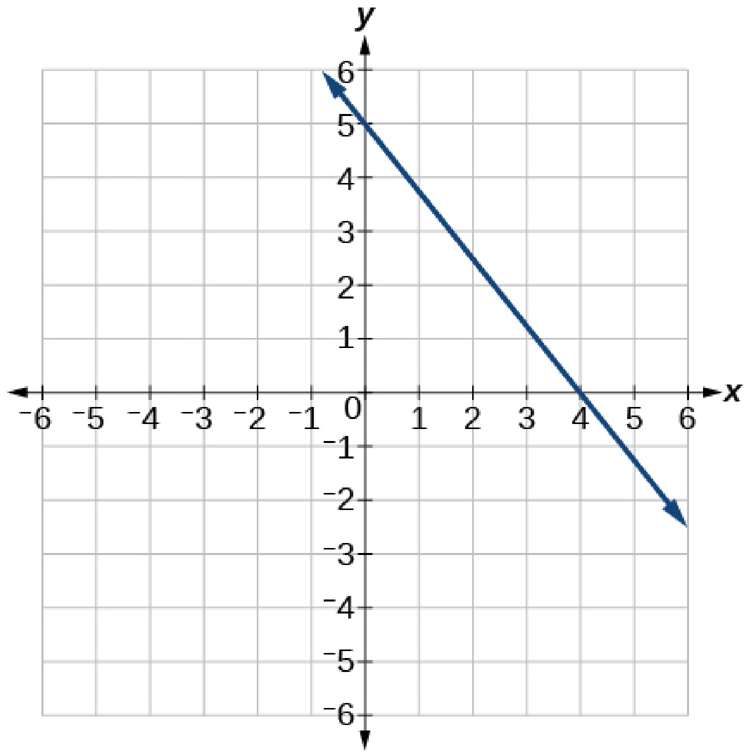
$y = -\frac{1}{3}t + 2$

Graphical

For the following exercises, find the slope of the line graphed.

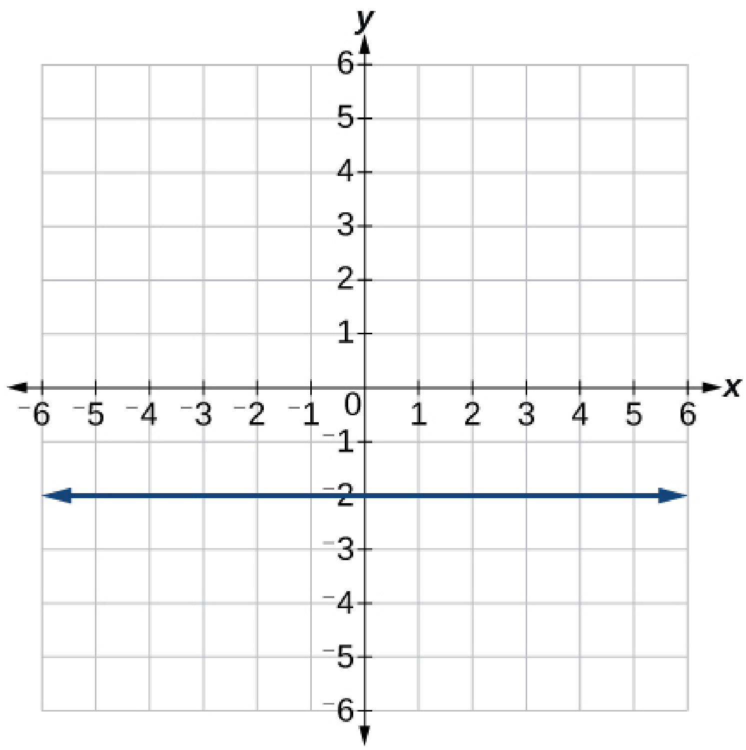
Exercise:

Problem:



Exercise:

Problem:



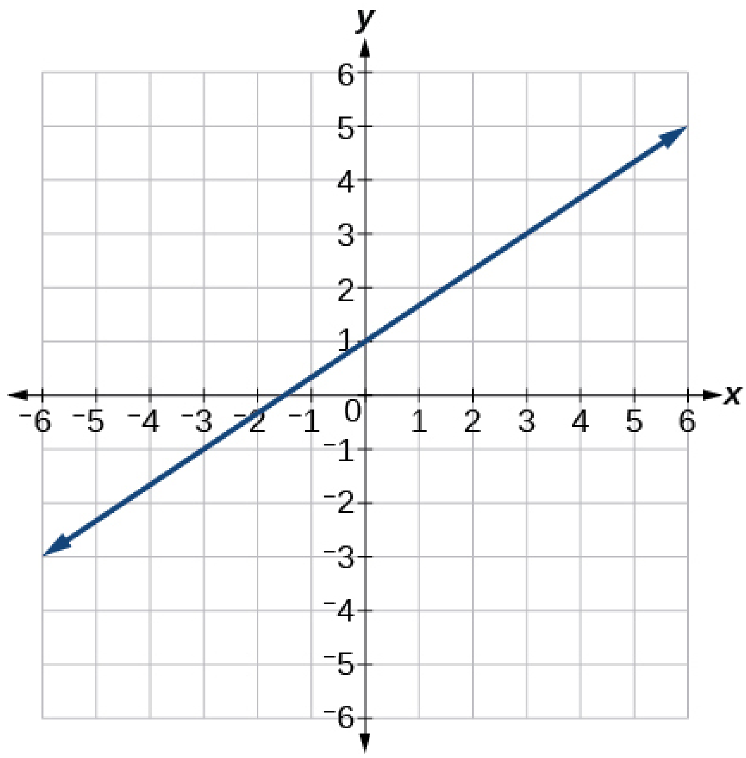
Solution:

0

For the following exercises, write an equation for the line graphed.

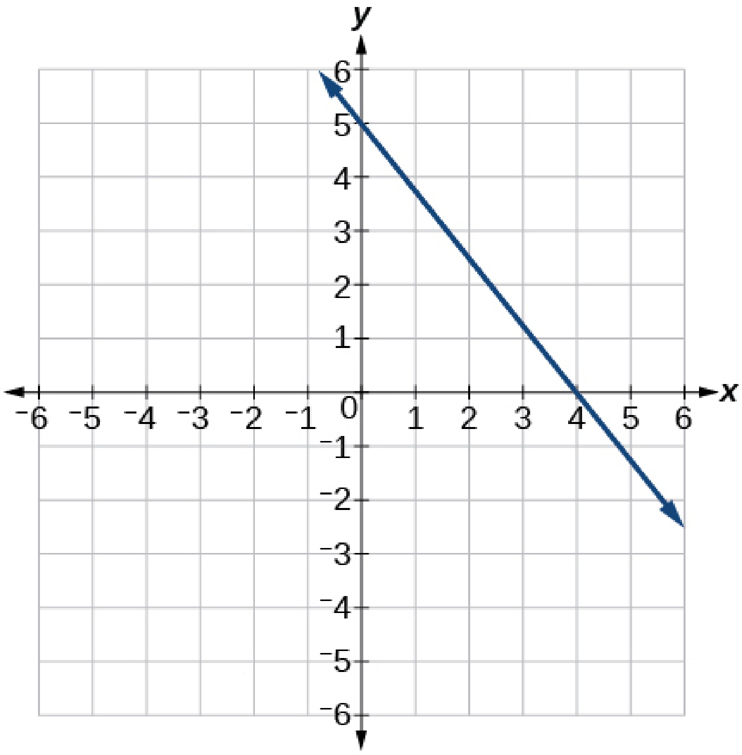
Exercise:

Problem:



Exercise:

Problem:

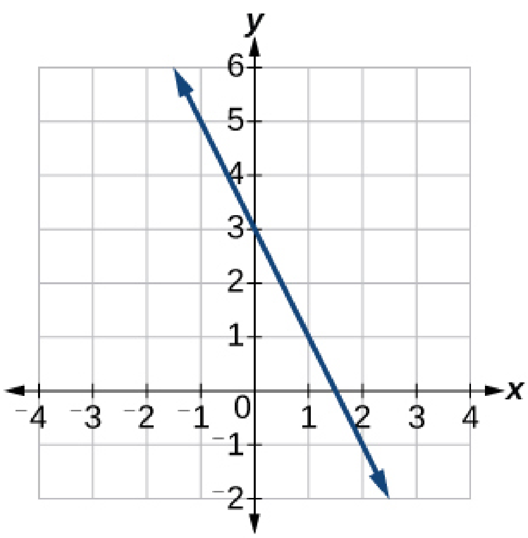


Solution:

$y = -\frac{5}{4}x + 5$

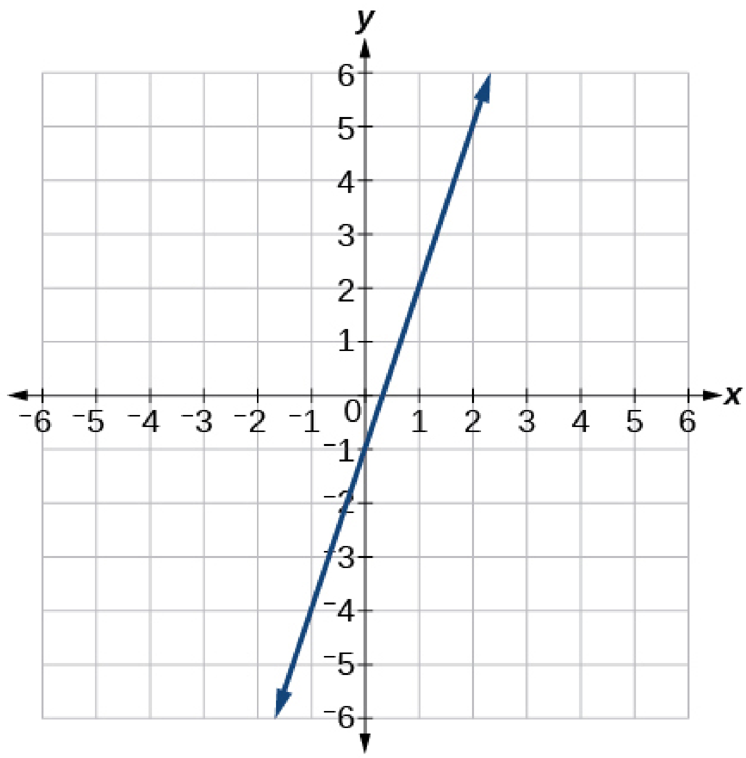
Exercise:

Problem:



Exercise:

Problem:

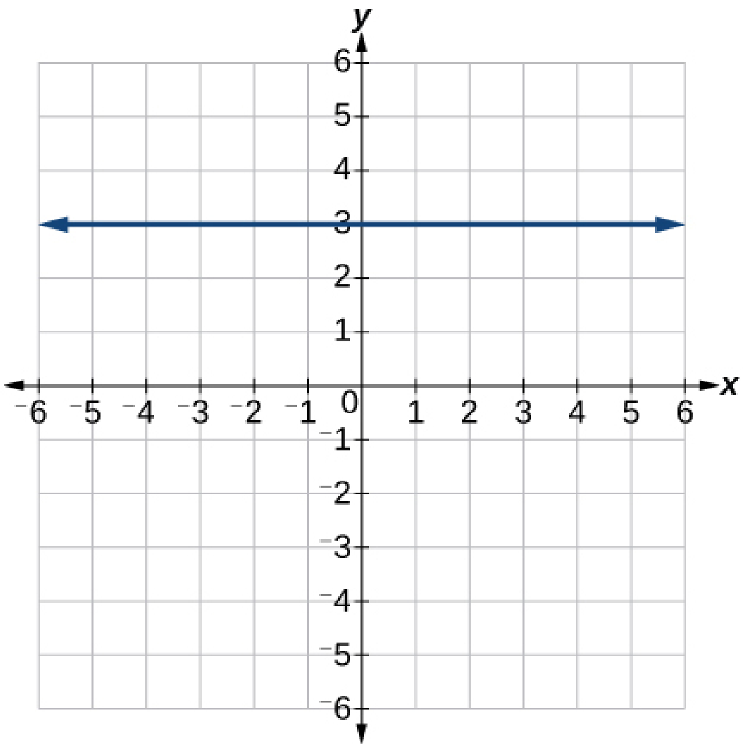


Solution:

$y = 3x - 1$

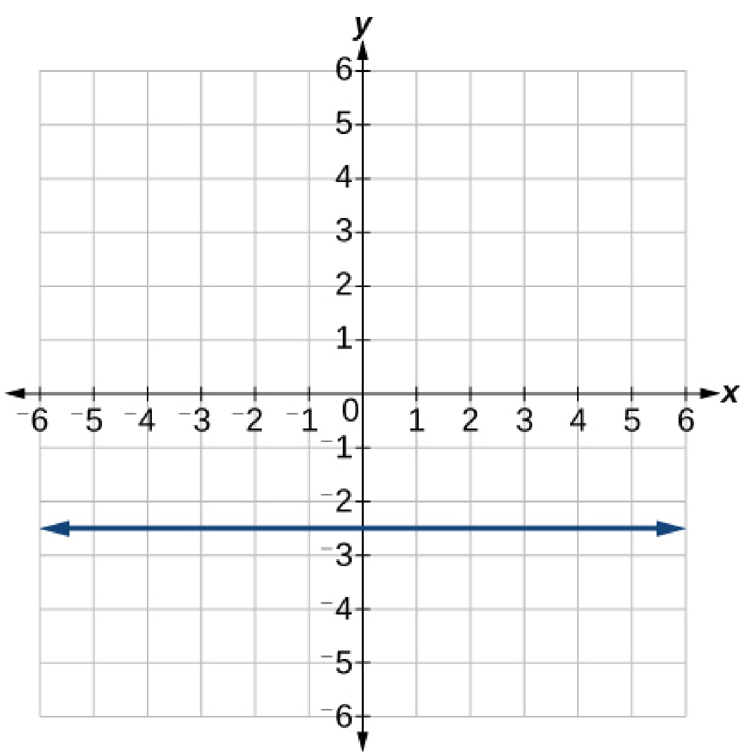
Exercise:

Problem:



Exercise:

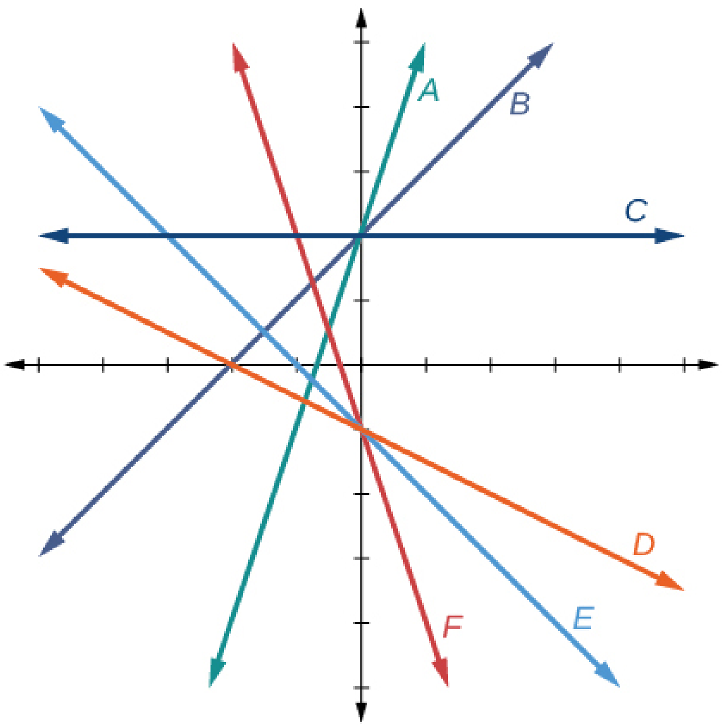
Problem:



Solution:

$y = -2.5$

For the following exercises, match the given linear equation with its graph in [link].



Exercise:

Problem: $f(x) = -x - 1$

Exercise:

Problem: $f(x) = -2x - 1$

Solution:

F

Exercise:

Problem: $f(x) = -\frac{1}{2}x - 1$

Exercise:

Problem: $f\left(x\right)=2$

Solution:

C

Exercise:

Problem: $f\left(x\right)=2+x$

Exercise:

Problem: $f\left(x\right)=3x+2$

Solution:

A

For the following exercises, sketch a line with the given features.

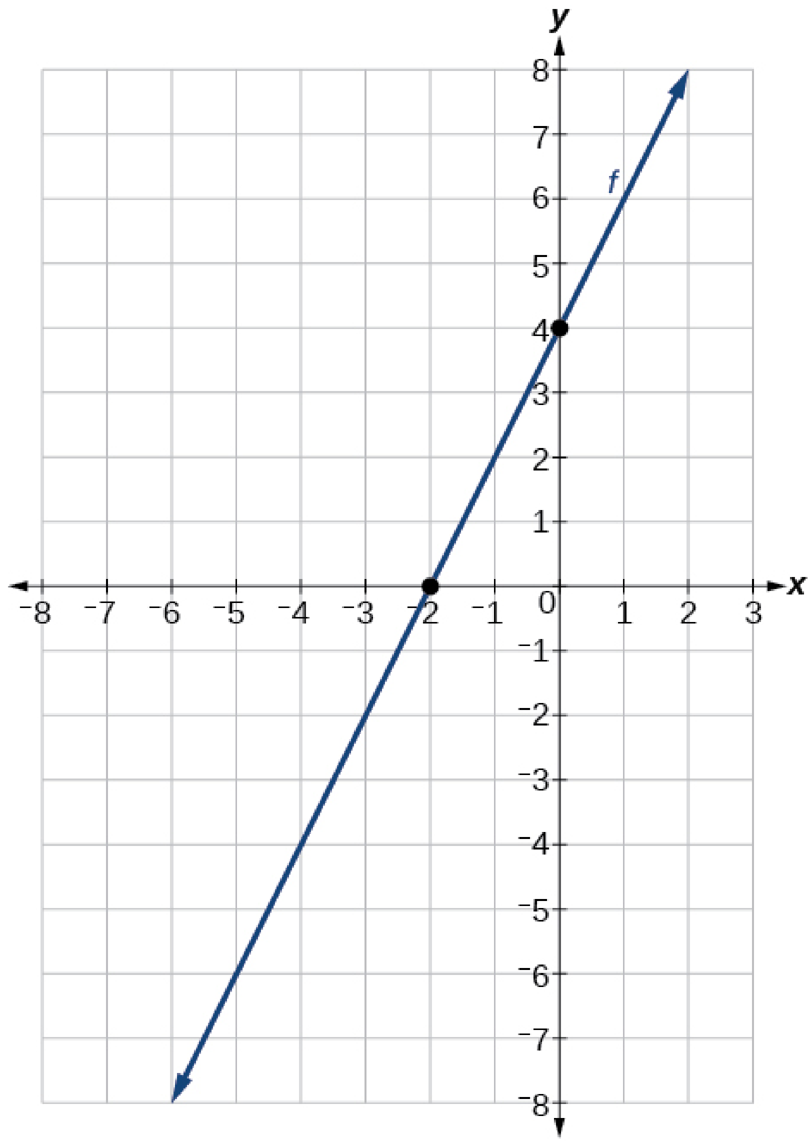
Exercise:

Problem: An *x*-intercept of $\left(–4,\text{ }0\right)$ and *y*-intercept of $\left(0,\text{ }–2\right)$

Exercise:

Problem: An *x*-intercept $\left(–2,\text{ }0\right)$ and *y*-intercept of $\left(0,\text{ }4\right)$

Solution:



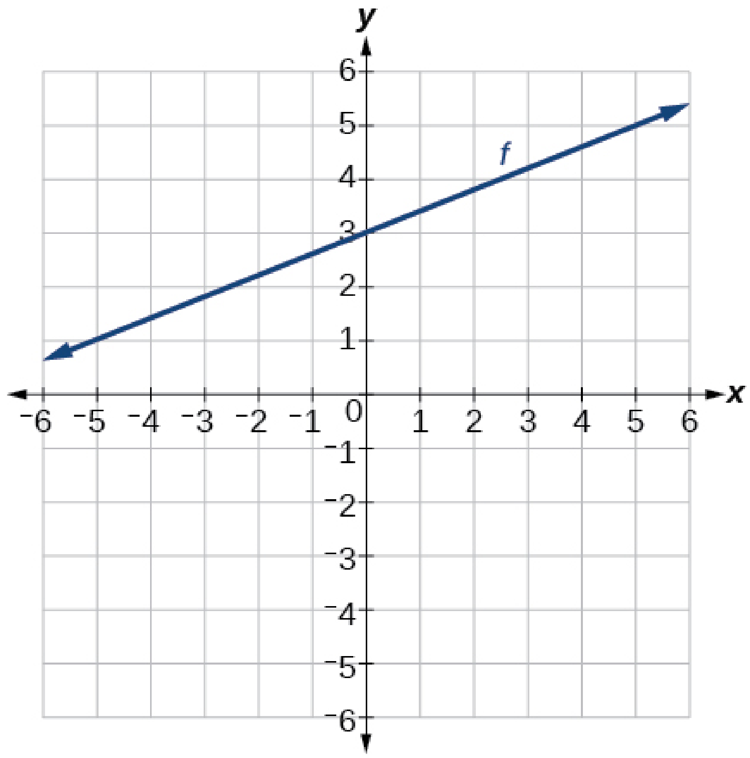
Exercise:

Problem:A y-intercept of $(0,7)$ and slope $-\frac{3}{2}$

Exercise:

Problem:A y-intercept of $(0,3)$ and slope $\frac{2}{5}$

Solution:



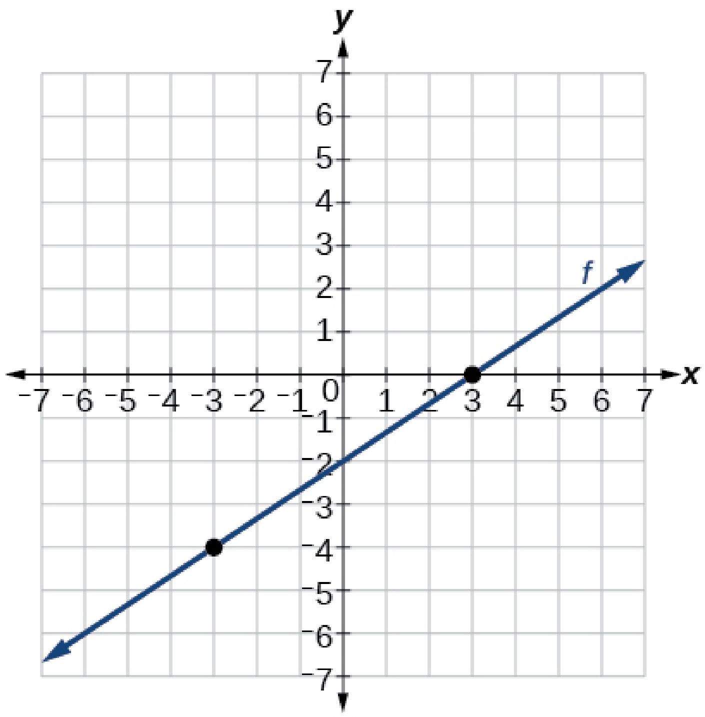
Exercise:

Problem: Passing through the points $(-6, -2)$ and $(6, -6)$

Exercise:

Problem: Passing through the points $(-3, -4)$ and $(3, 0)$

Solution:



For the following exercises, sketch the graph of each equation.

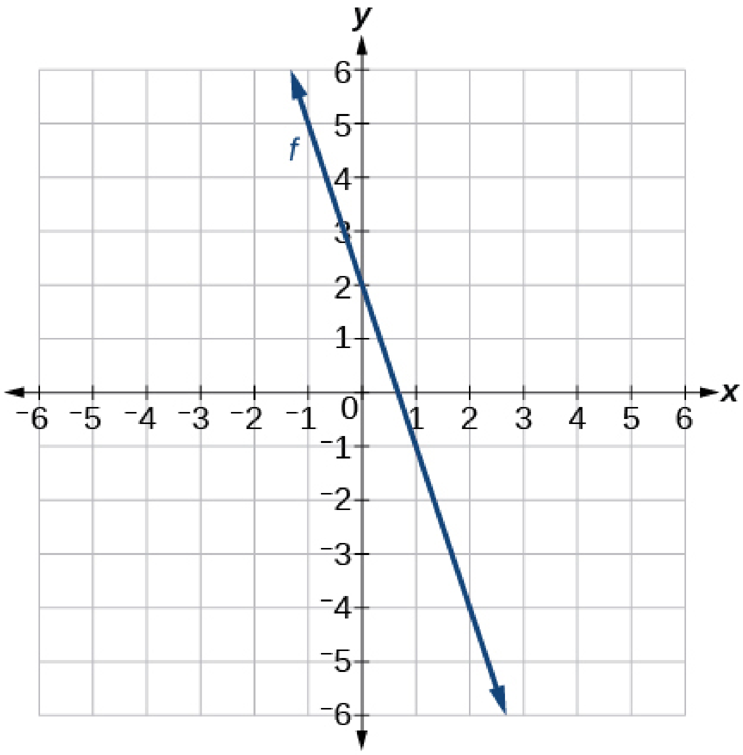
Exercise:

Problem: $f(x) = -2x - 1$

Exercise:

Problem: $f(x) = -3x + 2$

Solution:



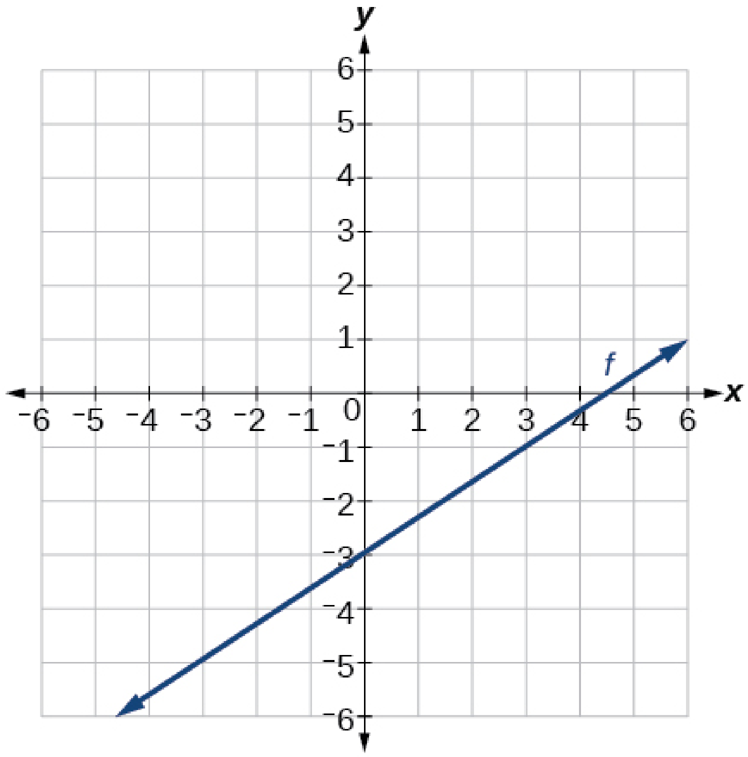
Exercise:

Problem: $f(x) = \frac{1}{3}x + 2$

Exercise:

Problem: $f(x) = \frac{2}{3}x - 3$

Solution:



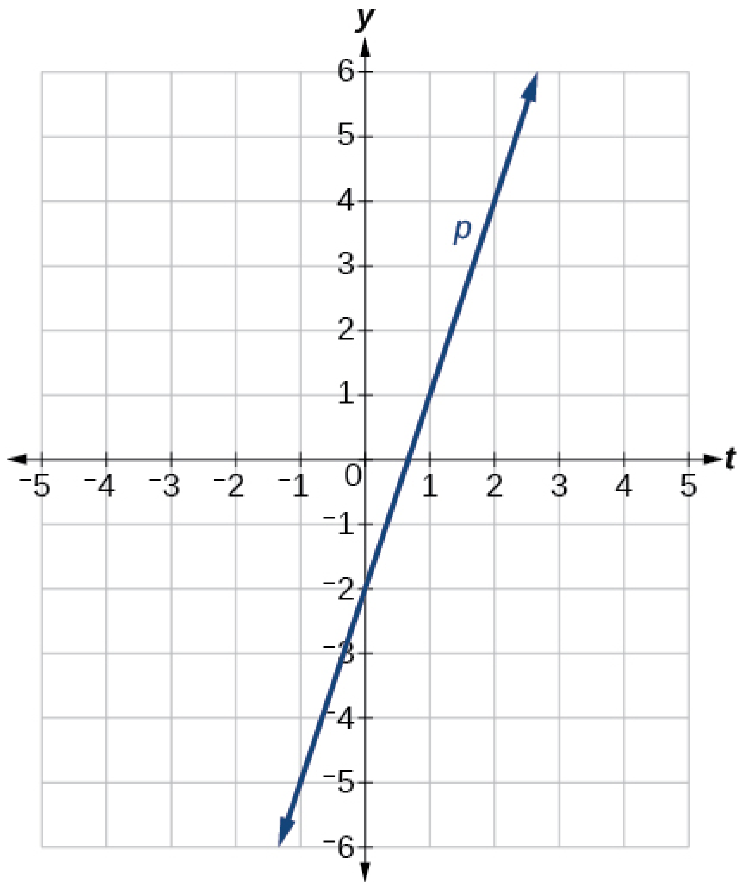
Exercise:

Problem: $f\left(t\right)=3+2t$

Exercise:

Problem: $p\left(t\right)=-2+3t$

Solution:



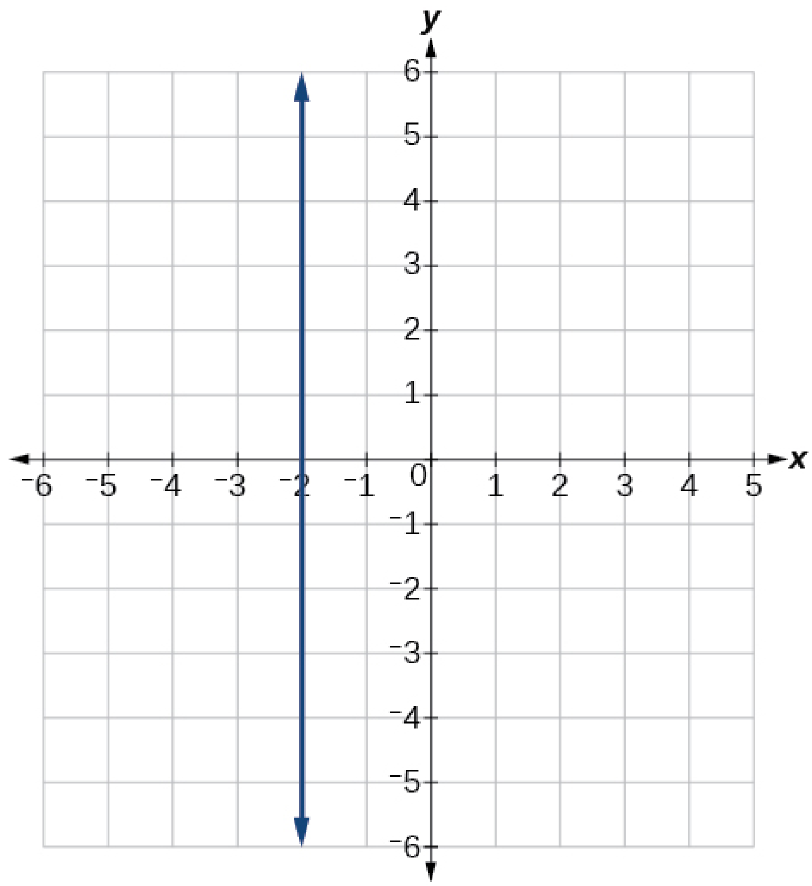
Exercise:

Problem: $x = 3$

Exercise:

Problem: $x = -2$

Solution:



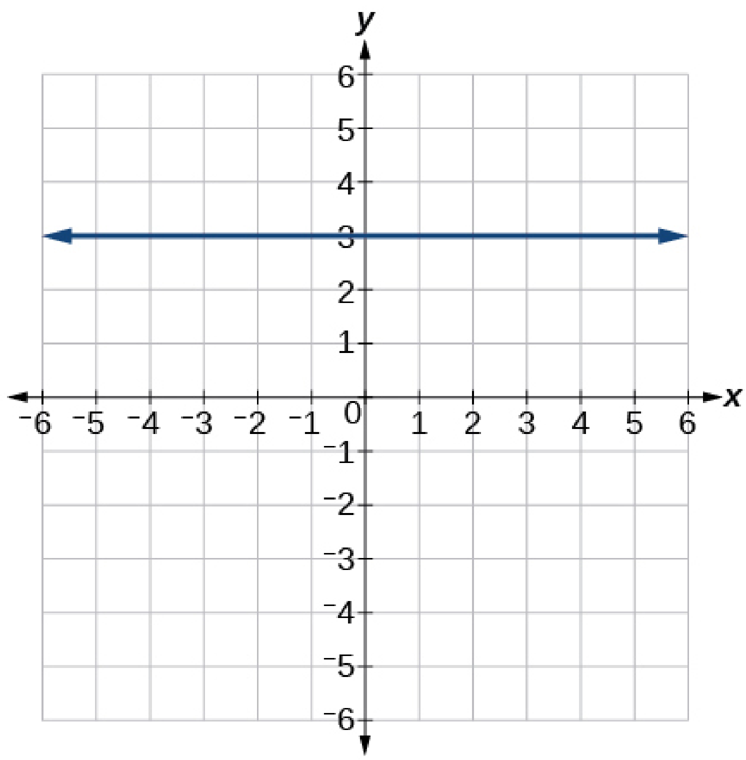
Exercise:

Problem: $r\left(x\right)=4$

For the following exercises, write the equation of the line shown in the graph.

Exercise:

Problem:

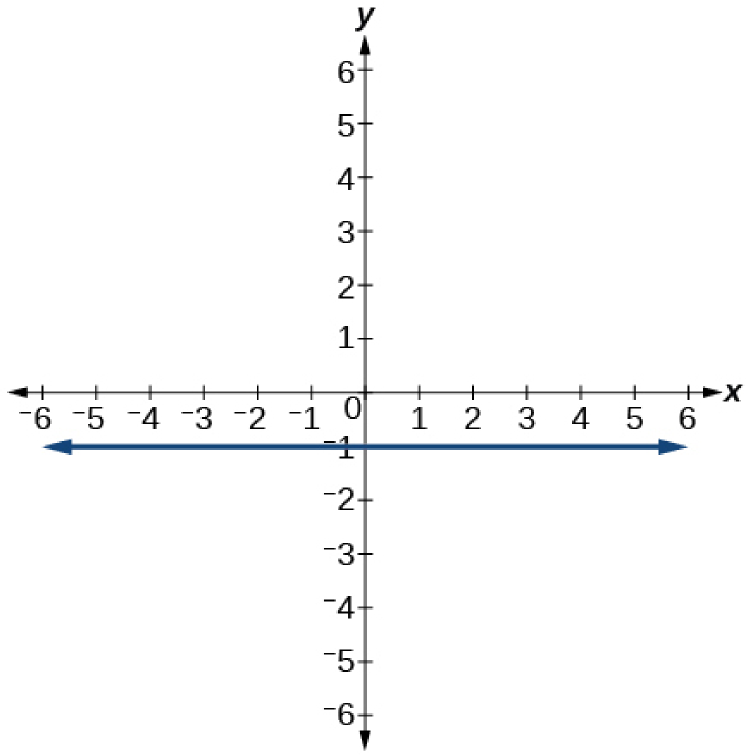


Solution:

$y = 3$

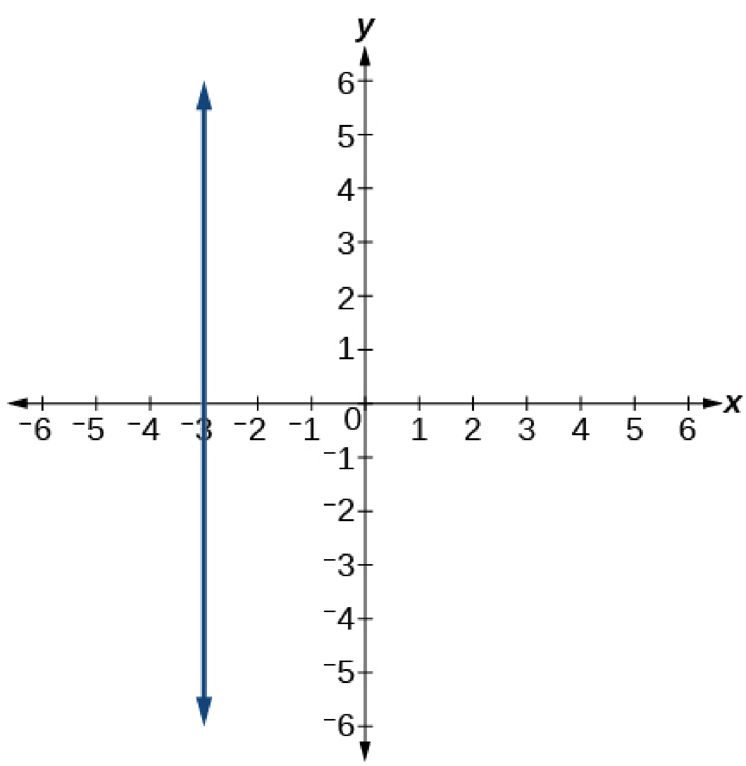
Exercise:

Problem:



Exercise:

Problem:

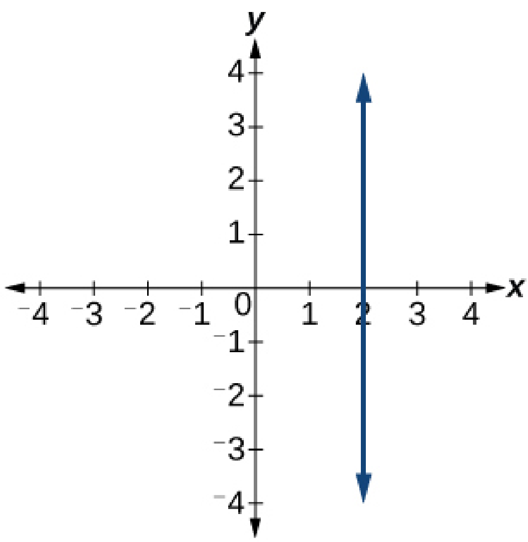


Solution:

$x = -3$

Exercise:

Problem:



Numeric

For the following exercises, which of the tables could represent a linear function? For each that could be linear, find a linear equation that models the data.

Exercise:

Problem:

x	0	5	10	15
$g(x)$	5	–10	–25	–40

Solution:

Linear, $g(x) = -3x + 5$

Exercise:

Problem:

x	0	5	10	15
$h(x)$	5	30	105	230

Exercise:

Problem:

x	0	5	10	15
$f(x)$	–5	20	45	70

Solution:

Linear, $f(x) = 5x - 5$

Exercise:

Problem:

x	5	10	20	25

$k(x)$	13	28	58	73

Exercise:

Problem:

x	0	2	4	6
$g(x)$	6	–19	–44	–69

Solution:

Linear, $g(x) = -\frac{25}{2}x + 6$

Exercise:

Problem:

x	2	4	8	10
$h(x)$	13	23	43	53

Exercise:

Problem:

x	2	4	6	8

$f(x)$	–4	16	36	56

Solution:

Linear, $f(x) = 10x - 24$

Exercise:

Problem:

x	0	2	6	8
$k(x)$	6	31	106	231

Technology

For the following exercises, use a calculator or graphing technology to complete the task.

Exercise:

Problem:

If f is a linear function, $f(0.1) = 11.5, \text{ and } f(0.4) = –5.9,$ find an equation for the function.

Solution:

$f(x) = -58x + 17.3$

Exercise:

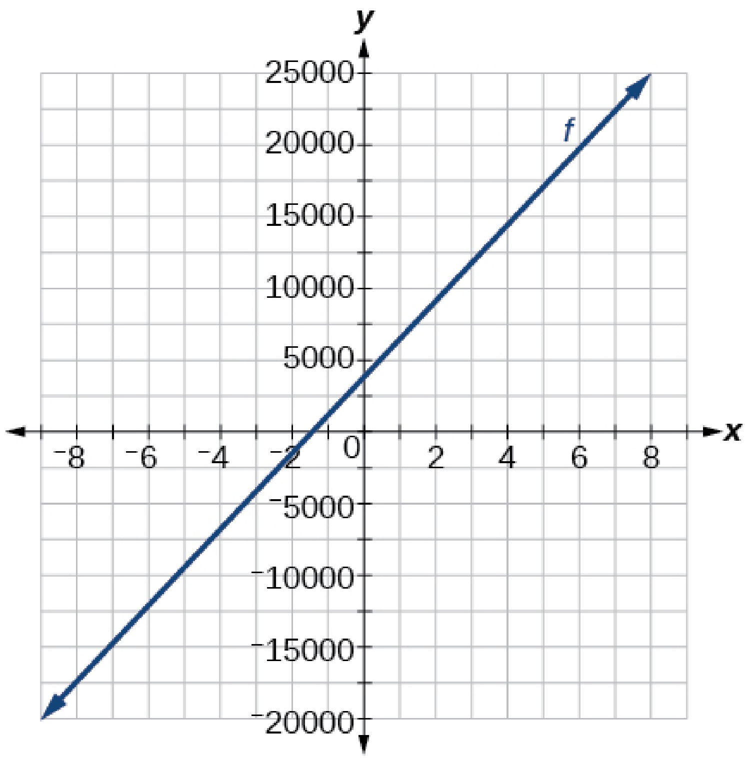
Problem:

Graph the function f on a domain of $[–10, 10] : f(x) = 0.02x - 0.01.$ Enter the function in a graphing utility. For the viewing window, set the minimum value of x to be -10 and the maximum value of x to be $10.$

Exercise:

Problem: Graph the function f on a domain of $[-10, 10] : fx) = 2,500x + 4,000$

Solution:



Exercise:

Problem:

[link] shows the input, w,and output, k,for a linear function k. a. Fill in the missing values of the table. b. Write the linear function k,round to 3 decimal places.

w	–10	5.5	67.5	*b*
k	30	–26	*a*	–44

Solution:

$y = 3.613x - 6.129$

Exercise:

Problem:

[link] shows the input, p,and output, q,for a linear function q. a. Fill in the missing values of the table. b. Write the linear function k.

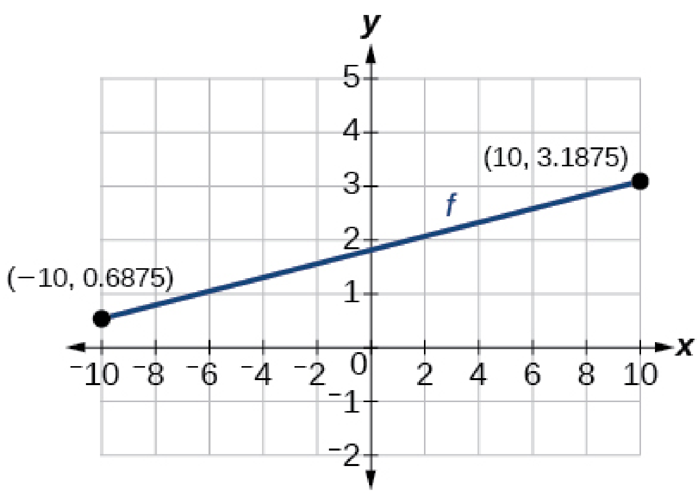
p	0.5	0.8	12	*b*
q	400	700	*a*	1,000,000

Exercise:

Problem:

Graph the linear function f on a domain of $[-10, 10]$ for the function whose slope is $\frac{1}{8}$ and y-intercept is $\frac{31}{16}$. Label the points for the input values of -10 and 10.

Solution:



Exercise:

Problem:

Graph the linear function f on a domain of $[-0.1, 0.1]$ for the function whose slope is 75 and y-intercept is -22.5. Label the points for the input values of -0.1 and 0.1.

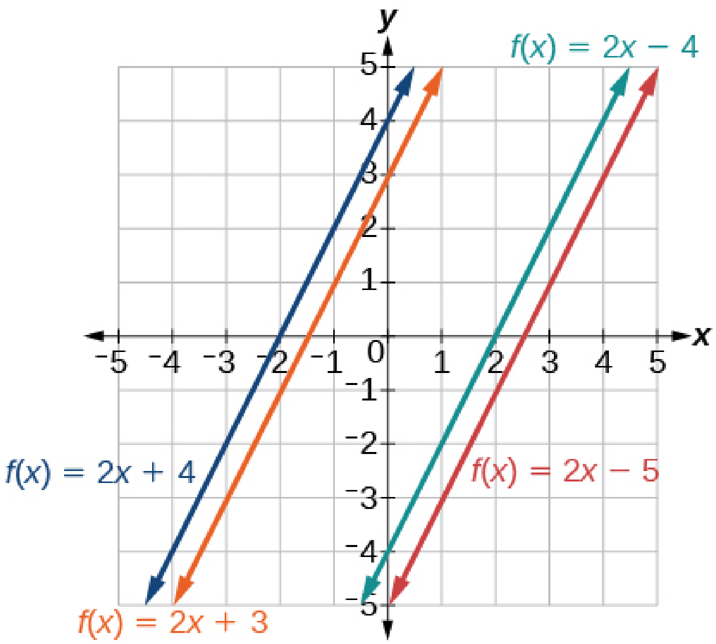
Exercise:

Problem:

Graph the linear function f where $f(x) = ax + b$ on the same set of axes on a domain of $[-4, 4]$ for the following values of a and b.

i. $a = 2; b = 3$
ii. $a = 2; b = 4$
iii. $a = 2; b = –4$
iv. $a = 2; b = –5$

Solution:



Extensions

Exercise:

Problem:

Find the value of x if a linear function goes through the following points and has the following slope: $(x, 2), (-4, 6),\ m = 3$

Exercise:

Problem:

Find the value of y if a linear function goes through the following points and has the following slope: $(10, y), (25, 100),\ m = -5$

Solution:

$y = 175$

Exercise:

Problem: Find the equation of the line that passes through the following points:

$(a,\ b)$ and $(a,\ b+1)$

Exercise:

Problem: Find the equation of the line that passes through the following points:

$(2a, b)$ and $(a, b+1)$

Solution:

$y = -\frac{1}{2}x + b + 2$

Exercise:

Problem: Find the equation of the line that passes through the following points:

$(a, 0)$and (c, d)

Exercise:

Problem:

Find the equation of the line parallel to the line $g(x) = -0.01x+2.01$ through the point $(1, 2)$.

Solution:

$y = –0.01x + 2.01$

Exercise:

Problem:

Find the equation of the line perpendicular to the line $g\left(x\right)=-0.01x+2.01$ through the point $\left(1,2\right)$.

For the following exercises, use the functions $f\left(x\right)=-0.1x+200$ and $g\left(x\right)=20x+0.1$.

Exercise:

Problem: Find the point of intersection of the lines f and g.

Solution:

$\left(\frac{1999}{201},\frac{400,001}{2010}\right)$

Exercise:

Problem: Where is $f\left(x\right)$ greater than $g\left(x\right)$? Where is $g\left(x\right)$ greater than $f\left(x\right)$?

Real-World Applications

Exercise:

Problem:

At noon, a barista notices that she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

Solution:

$20+0.5n$

Exercise:

Problem:

A gym membership with two personal training sessions costs $125, while gym membership with five personal training sessions costs $260. What is cost per session?

Exercise:

Problem:

A clothing business finds there is a linear relationship between the number of shirts, n,it can sell and the price, p,it can charge per shirt. In particular, historical data shows that 1,000 shirts can be sold at a price of $\$30$,while 3,000 shirts can be sold at a price of $22. Find a linear equation in the form $p(n) = mn + b$ that gives the price p they can charge for n shirts.

Solution:

$p(n) = -0.004n + 34$

Exercise:

Problem:

A phone company charges for service according to the formula: $C(n) = 24 + 0.1n$,where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

Exercise:

Problem:

A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y,each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when n stalks are planted.

Solution:

$y = -0.5n + 45$

Exercise:

Problem:

A city’s population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the rate of growth of the population and make a statement about the population rate of change in people per year.

Exercise:

Problem:

A town’s population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1,700 people each year. Write an equation, $P(t)$,for the population t years after 2003.

Solution:

$P(t) = 1700t + 45,000$

Exercise:

Problem:

Suppose that average annual income (in dollars) for the years 1990 through 1999 is given by the linear function: $I(x) = 1054x + 23,286$,where x is the number of years after 1990. Which of the following interprets the slope in the context of the problem?

a. As of 1990, average annual income was $23,286.
b. In the ten-year period from 1990–1999, average annual income increased by a total of $1,054.
c. Each year in the decade of the 1990s, average annual income increased by $1,054.
d. Average annual income rose to a level of $23,286 by the end of 1999.

Exercise:

Problem:

When temperature is 0 degrees Celsius, the Fahrenheit temperature is 32. When the Celsius temperature is 100, the corresponding Fahrenheit temperature is 212. Express the Fahrenheit temperature as a linear function of C,the Celsius temperature, $F(C)$.

a. Find the rate of change of Fahrenheit temperature for each unit change temperature of Celsius.
b. Find and interpret $F(28)$.

c. Find and interpret $F(-40)$.

Solution:

a.
Rate of change $= \frac{\Delta F}{\Delta C} = \frac{212-32}{100-0} = 1.8$ degrees F for one degree change in C
b. $F(28) = 1.8(28) + 32 = 82.4$ degrees F is 28 degrees C
c. $F(-40) = 1.8(-40) + 32 = -40$ degrees F is -40 degrees C

Glossary

decreasing linear function
: a function with a negative slope: If $f(x) = mx + b,\ \text{then}\ m < 0.$

horizontal line
: a line defined by $f(x) = b$,where b is a real number. The slope of a horizontal line is 0.

increasing linear function
: a function with a positive slope: If $f(x) = mx + b,\ \text{then}\ m > 0.$

linear function
: a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

parallel lines
: two or more lines with the same slope

perpendicular lines
: two lines that intersect at right angles and have slopes that are negative reciprocals of each other

point-slope form
: the equation for a line that represents a linear function of the form $y - y_1 = m\left(x - x_1\right)$

slope
: the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form

the equation for a line that represents a linear function in the form $f(x) = mx + b$

vertical line
a line defined by $x = a$,where a is a real number. The slope of a vertical line is undefined.

Modeling with Linear Functions
In this section you will:

- Build linear models from verbal descriptions.
- Model a set of data with a linear function.

(credit: EEK Photography/Flickr)

Emily is a college student who plans to spend a summer in Seattle. She has saved $3,500 for her trip and anticipates spending $400 each week on rent, food, and activities. How can we write a linear model to represent her situation? What would be the *x*-intercept, and what can she learn from it? To answer these and related questions, we can create a model using a linear function. Models such as this one can be extremely useful for analyzing relationships and making predictions based on those relationships. In this section, we will explore examples of linear function models.

Building Linear Models from Verbal Descriptions

When building linear models to solve problems involving quantities with a constant rate of change, we typically follow the same problem strategies that we would use for any type of function. Let’s briefly review them:

1. Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
3. Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
5. When needed, write a formula for the function.
6. Solve or evaluate the function using the formula.
7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Now let’s take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.
Equation:

$$\begin{array}{l} \text{Output: } M\text{, money remaining, in dollars} \\ \text{Input: } t\text{, time, in weeks} \end{array}$$

So, the amount of money remaining depends on the number of weeks: $M\left(t\right)$.
Equation:

We can also identify the initial value and the rate of change.

$$\begin{array}{l} \text{Initial Value: She saved \$3,500, so \$3,500 is the initial value for } M. \\ \text{Rate of Change: She anticipates spending \$400 each week, so } -\text{\$400 per week is the rate of chang} \end{array}$$

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The rate of change is constant, so we can start with the linear model $M\left(t\right) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = \underbrace{m}_{-400}t + \underbrace{b}_{3500}$$

$$M(t) = -400t + 3500$$

To find the *x*-intercept, we set the output to zero, and solve for the input.
Equation:

$$\begin{array}{rcl} 0 & = & -400t + 3500 \\ t & = & \frac{3500}{400} \\ & = & 8.75 \end{array}$$

The *x*-intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved $3,500, but the scenario discussed poses the question once she saved $3,500 because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the *x*-intercept, unless Emily uses a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \leq t \leq 8.75$.

In this example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Using a Given Intercept to Build a Model

Some real-world problems provide the *y*-intercept, which is the constant or initial value. Once the *y*-intercept is known, the *x*-intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $1,000. She plans to pay $250 per month until her balance is $0. The *y*-

intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is -$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.
Equation:

$$\begin{aligned} f(x) &= mx + b \\ &= -250x + 1000 \end{aligned}$$

Now we can set the function equal to 0, and solve for x to find the *x*-intercept.
Equation:

$$\begin{aligned} 0 &= -250x + 1000 \\ 1000 &= 250x \\ 4 &= x \\ x &= 4 \end{aligned}$$

The *x*-intercept is the number of months it takes her to reach a balance of $0. The *x*-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

Note:
Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given *x*-value.
6. Use the model to identify an *x*-value that results in a given *y*-value.
7. Answer the question posed.

Example:
Exercise:

Problem:
Using a Linear Model to Investigate a Town's Population

A town's population has been growing linearly. In 2004, the population was 6,200. By 2009, the population had grown to 8,100. Assume this trend continues.

a. Predict the population in 2013.
b. Identify the year in which the population will reach 15,000.

Solution:

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the *y*-intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004.
Equation:

$$\begin{array}{l}\text{Input: } t\text{, years since 2004}\\ \text{Output: } P(t)\text{, the town's population}\end{array}$$

To predict the population in 2013 ($t = 9$), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.
Equation:

$$m = \frac{\text{change in output}}{\text{change in input}}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$,giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the *y*-intercept of the function. The year 2009 would correspond to $t = 5$,giving the point $(5, 8100)$.

The two coordinate pairs are $(0, 6200)$ and $(5, 8100)$. Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.
Equation:

$$\begin{array}{lll} m & = & \frac{8100-6200}{5-0} \\ & = & \frac{1900}{5} \\ & = & 380 \text{ people per year} \end{array}$$

We already know the *y*-intercept of the line, so we can immediately write the equation:
Equation:

$$P(t) = 380t + 6200$$

To predict the population in 2013, we evaluate our function at $t = 9$.
Equation:

$$\begin{array}{lll} P(9) & = & 380(9) + 6{,}200 \\ & = & 9{,}620 \end{array}$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.
Equation:

$$\begin{aligned} 15000 &= 380t + 6200 \\ 8800 &= 380t \\ t &\approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Note:
Exercise:

Problem:

A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

a. Write a linear model to represent the cost C of the company as a function of x,the number of doughnuts produced.
b. Find and interpret the *y*-intercept.

Solution:

a. $C\left(x\right)=0.25x+25,000$ b. The *y*-intercept is $\left(0,25,000\right).$ If the company does not produce a single doughnut, they still incur a cost of $25,000.

Note:
Exercise:

Problem:

A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

a. Predict the population in 2014.
b. Identify the year in which the population will reach 54,000.

Solution:

a. 41,100
b. 2020

Using a Diagram to Build a Model

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

Example:
Exercise:

Problem:
Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

Solution:

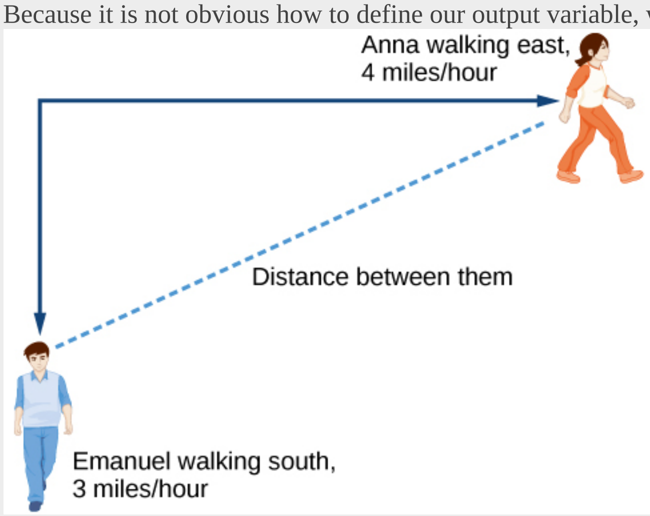
In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question:

"How long will it take them to be 2 miles apart"?

In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.
Equation:

$$\begin{array}{l}\text{Input:}\,t,\,\text{time in hours.}\\ \text{Output:}\,A(t),\,\text{distance in miles, and}\,E(t),\,\text{distance in miles}\end{array}$$

Because it is not obvious how to define our output variable, we'll start by drawing a picture such as [link].



Initial Value: They both start at the same intersection so when $t=0$,the distance traveled by each person should also be 0. Thus the initial value for each is 0.

Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for A is 4 and the slope for E is 3.

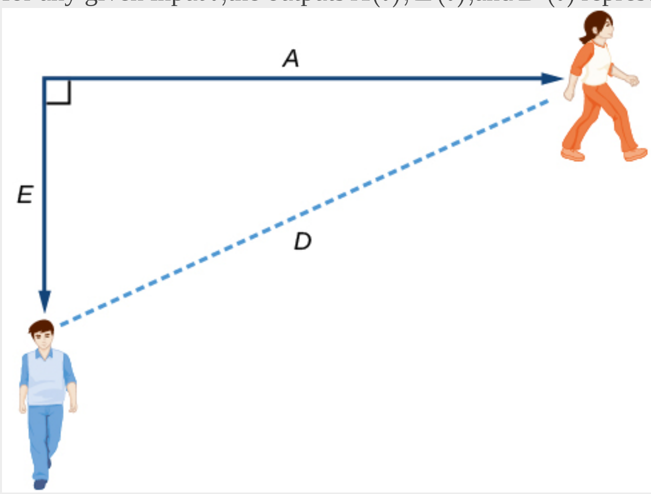
Using those values, we can write formulas for the distance each person has walked.
Equation:

$$\begin{array}{rcl}A(t) & = & 4t\\ E(t) & = & 3t\end{array}$$

For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the “starting point” at the intersection where they both started. Then we can use the variable, A,which we introduced above, to represent Anna’s position, and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, E,to represent Emanuel’s position, measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.

We can then define a third variable, D,to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful, as we can see from [link].

Recall that we need to know how long it takes for D,the distance between them, to equal 2 miles. Notice that for any given input t,the outputs $A(t)$, $E(t)$,and $D(t)$ represent distances.



[link] shows us that we can use the Pythagorean Theorem because we have drawn a right angle.

Using the Pythagorean Theorem, we get:
Equation:

$$\begin{aligned} D(t)^2 &= A(t)^2 + E(t)^2 \\ &= (4t)^2 + (3t)^2 \\ &= 16t^2 + 9t^2 \\ &= 25t^2 \\ D(t) &= \pm\sqrt{25t^2} && \text{Solve for } D(t) \text{ using the square root.} \\ &= \pm 5|t| \end{aligned}$$

In this scenario we are considering only positive values of t, so our distance $D(t)$ will always be positive. We can simplify this answer to $D(t) = 5t$. This means that the distance between Anna and Emanuel is also a linear function. Because D is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output $D(t) = 2$ and solve for t.
Equation:

$$\begin{aligned} D(t) &= 2 \\ 5t &= 2 \\ t &= \frac{2}{5} = 0.4 \end{aligned}$$

They will fall out of radio contact in 0.4 hour, or 24 minutes.

Note:
Should I draw diagrams when given information based on a geometric shape?
Yes. Sketch the figure and label the quantities and unknowns on the sketch.

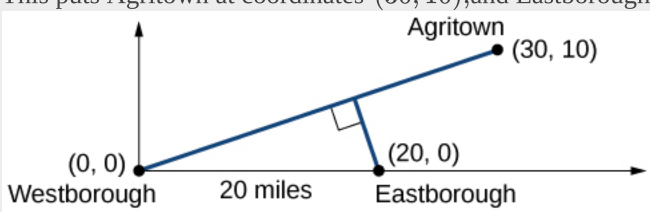
Example:
Exercise:

Problem:
Using a Diagram to Model Distance Between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Solution:

It might help here to draw a picture of the situation. See [link]. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates $(30, 10)$,and Eastborough at $(20, 0)$.



Using this point along with the origin, we can find the slope of the line from Westborough to Agritown.
Equation:

$$m = \frac{10-0}{30-0} = \frac{1}{3}$$

Now we can write an equation to describe the road from Westborough to Agritown.
Equation:

$$W(x) = \frac{1}{3}x$$

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Because the town of Eastborough is at the point (20, 0), we can find the equation.
Equation:

$$\begin{array}{rcll} E(x) & = & -3x+b & \\ 0 & = & -3(20)+b & \text{Substitute } (20,0) \text{into the equation.} \\ b & = & 60 & \\ E(x) & = & -3x+60 & \end{array}$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,
Equation:

$$\begin{aligned}
\tfrac{1}{3}x &= -3x+60 \\
\tfrac{10}{3}x &= 60 \\
10x &= 180 \\
x &= 18 && \text{Substitute this back into } W(x). \\
y &= W(18) \\
&= \tfrac{1}{3}(18) \\
&= 6
\end{aligned}$$

The roads intersect at the point (18, 6). Using the distance formula, we can now find the distance from Westborough to the junction.

Equation:

$$\begin{aligned}
\text{distance} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
&= \sqrt{(18-0)^2+(6-0)^2} \\
&\approx 18.974 \text{ miles}
\end{aligned}$$

Analysis

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

Note:
Exercise:

Problem:

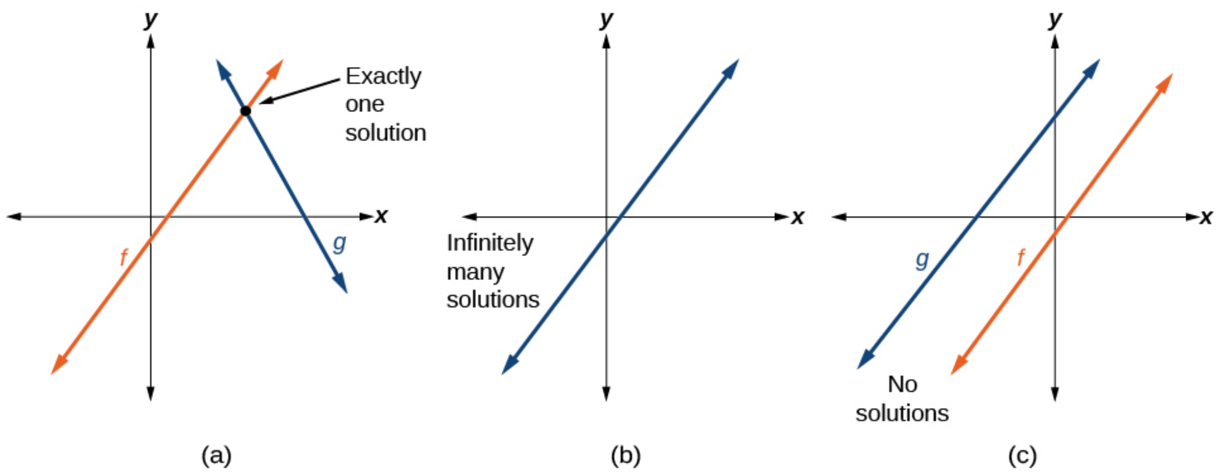
There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

Solution:

21.15 miles

Modeling a Set of Data with Linear Functions

Real-world situations including two or more linear functions may be modeled with a system of linear equations. Remember, when solving a system of linear equations, we are looking for points the two lines have in common. Typically, there are three types of answers possible, as shown in [link].



Note:
Given a situation that represents a system of linear equations, write the system of equations and identify the solution.

1. Identify the input and output of each linear model.
2. Identify the slope and y-intercept of each linear model.
3. Find the solution by setting the two linear functions equal to another and solving for x,or find the point of intersection on a graph.

Example:
Exercise:

Problem:
Building a System of Linear Models to Choose a Truck Rental Company

Jamal is choosing between two truck-rental companies. The first, Keep on Trucking, Inc., charges an up-front fee of $20, then 59 cents a mile. The second, Move It Your Way, charges an up-front fee of $16, then 63 cents a mile[footnote]. When will Keep on Trucking, Inc. be the better choice for Jamal?
Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/

Solution:

The two important quantities in this problem are the cost and the number of miles driven. Because we have two companies to consider, we will define two functions in [link].

Input	d,distance driven in miles
Outputs	$K(d)$: cost, in dollars, for renting from Keep on Trucking $M\,(d)$ cost, in dollars, for renting from Move It Your Way

Initial Value	Up-front fee: $K(0) = 20$ and $M(0) = 16$
Rate of Change	$K(d) = \$0.59$ /mile and $P(d) = \$0.63$ /mile

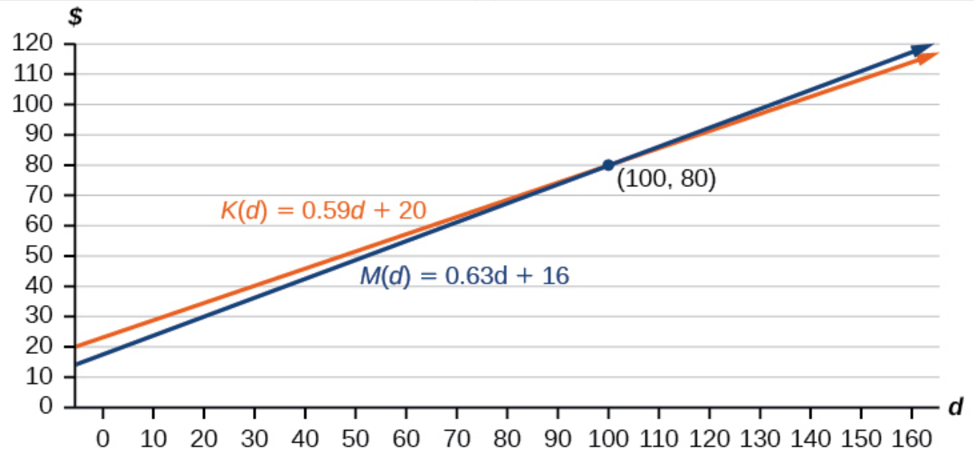
A linear function is of the form $f(x) = mx + b$. Using the rates of change and initial charges, we can write the equations

Equation:

$$\begin{array}{rcl} K(d) & = & 0.59d + 20 \\ M(d) & = & 0.63d + 16 \end{array}$$

Using these equations, we can determine when Keep on Trucking, Inc., will be the better choice. Because all we have to make that decision from is the costs, we are looking for when Move It Your Way, will cost less, or when $K(d) < M(d)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, and then see where the $K(d)$ function is smaller.

These graphs are sketched in [link], with $K(d)$ in blue.



To find the intersection, we set the equations equal and solve:

Equation:

$$\begin{array}{rcl} K(d) & = & M(d) \\ 0.59d + 20 & = & 0.63d + 16 \\ 4 & = & 0.04d \\ 100 & = & d \\ d & = & 100 \end{array}$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $K(d)$ is growing at a slower rate, we can conclude that Keep on Trucking, Inc. will be the cheaper price when more than 100 miles are driven, that is $d > 100$.

Note:
Access this online resource for additional instruction and practice with linear function models.

- Interpreting a Linear Function

Key Concepts

- We can use the same problem strategies that we would use for any type of function.
- When modeling and solving a problem, identify the variables and look for key values, including the slope and *y*-intercept. See [link].
- Draw a diagram, where appropriate. See [link] and [link].
- Check for reasonableness of the answer.
- Linear models may be built by identifying or calculating the slope and using the *y*-intercept.
    - The *x*-intercept may be found by setting $y = 0$,which is setting the expression $mx + b$ equal to 0.
    - The point of intersection of a system of linear equations is the point where the *x*- and *y*-values are the same. See [link].
    - A graph of the system may be used to identify the points where one line falls below (or above) the other line.

Section Exercises

Verbal

Exercise:

Problem: Explain how to find the input variable in a word problem that uses a linear function.

Solution:

Determine the independent variable. This is the variable upon which the output depends.

Exercise:

Problem: Explain how to find the output variable in a word problem that uses a linear function.

Exercise:

Problem: Explain how to interpret the initial value in a word problem that uses a linear function.

Solution:

To determine the initial value, find the output when the input is equal to zero.

Exercise:

Problem: Explain how to determine the slope in a word problem that uses a linear function.

Algebraic

Exercise:

Problem:

Find the area of a parallelogram bounded by the *y*-axis, the line $x = 3$,the line $f(x) = 1 + 2x$,and the line parallel to $f(x)$ passing through $(2, 7)$.

Solution:

6 square units

Exercise:

Problem:

Find the area of a triangle bounded by the *x*-axis, the line $f(x) = 12– \frac{1}{3}x$,and the line perpendicular to $f(x)$ that passes through the origin.

Exercise:

Problem:

Find the area of a triangle bounded by the *y*-axis, the line $f(x) = 9– \frac{6}{7}x$,and the line perpendicular to $f(x)$ that passes through the origin.

Solution:

20.01 square units

Exercise:

Problem:

Find the area of a parallelogram bounded by the *x*-axis, the line $g(x) = 2$,the line $f(x) = 3x$,and the line parallel to $f(x)$ passing through $(6, 1)$.

For the following exercises, consider this scenario: A town's population has been decreasing at a constant rate. In 2010 the population was 5,900. By 2012 the population had dropped 4,700. Assume this trend continues.

Exercise:

Problem: Predict the population in 2016.

Solution:

2,300

Exercise:

Problem: Identify the year in which the population will reach 0.

For the following exercises, consider this scenario: A town's population has been increased at a constant rate. In 2010 the population was 46,020. By 2012 the population had increased to 52,070. Assume this trend continues.

Exercise:

Problem: Predict the population in 2016.

Solution:

64,170

Exercise:

Problem: Identify the year in which the population will reach 75,000.

For the following exercises, consider this scenario: A town has an initial population of 75,000. It grows at a constant rate of 2,500 per year for 5 years.

Exercise:

Problem:

Find the linear function that models the town's population P as a function of the year, t,where t is the number of years since the model began.

Solution:

$P\left(t\right)=75,000+2500t$

Exercise:

Problem: Find a reasonable domain and range for the function P.

Exercise:

Problem: If the function P is graphed, find and interpret the *x*- and *y*-intercepts.

Solution:

(–30, 0) Thirty years before the start of this model, the town had no citizens. (0, 75,000) Initially, the town had a population of 75,000.

Exercise:

Problem: If the function P is graphed, find and interpret the slope of the function.

Exercise:

Problem: When will the population reach 100,000?

Solution:

Ten years after the model began

Exercise:

Problem: What is the population in the year 12 years from the onset of the model?

For the following exercises, consider this scenario: The weight of a newborn is 7.5 pounds. The baby gained one-half pound a month for its first year.

Exercise:

Problem:

Find the linear function that models the baby's weight W as a function of the age of the baby, in months, t.

Solution:

$W(t)=0.5t+7.5$

Exercise:

Problem: Find a reasonable domain and range for the function W.

Exercise:

Problem: If the function W is graphed, find and interpret the *x*- and *y*-intercepts.

Solution:

$(-15, 0)$: The *x*-intercept is not a plausible set of data for this model because it means the baby weighed 0 pounds 15 months prior to birth. $(0, 7.5)$: The baby weighed 7.5 pounds at birth.

Exercise:

Problem: If the function *W* is graphed, find and interpret the slope of the function.

Exercise:

Problem: When did the baby weight 10.4 pounds?

Solution:

At age 5.8 months

Exercise:

Problem: What is the output when the input is 6.2?

For the following exercises, consider this scenario: The number of people afflicted with the common cold in the winter months steadily decreased by 205 each year from 2005 until 2010. In 2005, 12,025 people were inflicted.

Exercise:

Problem:

Find the linear function that models the number of people inflicted with the common cold C as a function of the year, t.

Solution:

$C(t) = 12,025 - 205t$

Exercise:

Problem: Find a reasonable domain and range for the function C.

Exercise:

Problem: If the function C is graphed, find and interpret the *x*- and *y*-intercepts.

Solution:

$(58.7, 0)$: In roughly 59 years, the number of people inflicted with the common cold would be 0. $(0, 12,025)$ Initially there were 12,025 people afflicted by the common cold.

Exercise:

Problem: If the function C is graphed, find and interpret the slope of the function.

Exercise:

Problem: When will the output reach 0?

Solution:

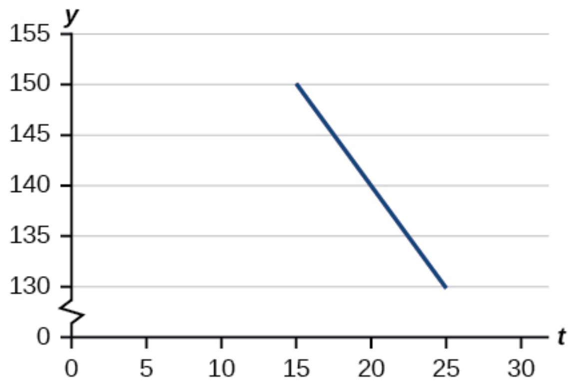
2063

Exercise:

Problem: In what year will the number of people be 9,700?

Graphical

For the following exercises, use the graph in [link], which shows the profit, y,in thousands of dollars, of a company in a given year, t,where t represents the number of years since 1980.



Exercise:

Problem: Find the linear function y,where y depends on t,the number of years since 1980.

Solution:

$y=-2t+180$

Exercise:

Problem: Find and interpret the *y*-intercept.

Exercise:

Problem: Find and interpret the *x*-intercept.

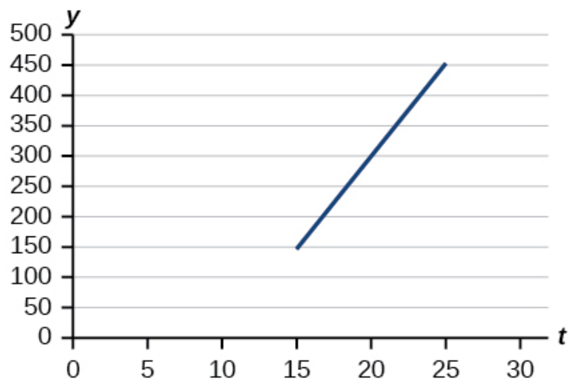
Solution:

In 2070, the company's profit will be zero.

Exercise:

Problem: Find and interpret the slope.

For the following exercises, use the graph in [link], which shows the profit, y,in thousands of dollars, of a company in a given year, t,where t represents the number of years since 1980.



Exercise:

Problem: Find the linear function y,where y depends on t,the number of years since 1980.

Solution:

$y=30t-300$

Exercise:

Problem: Find and interpret the *y*-intercept.

Solution:

$(0,-300)$;In 1980, the company lost $300,000.

Exercise:

Problem: Find and interpret the *x*-intercept.

Exercise:

Problem: Find and interpret the slope.

Solution:

$y=30t-300$ of form $y=mx+b$, $\mathrm{m}=30$.

For each year after 1980, the company's profits increased $30,000 per year

Numeric

For the following exercises, use the median home values in Mississippi and Hawaii (adjusted for inflation) shown in [link]. Assume that the house values are changing linearly.

Year	Mississippi	Hawaii

Year	Mississippi	Hawaii
1950	$25,200	$74,400
2000	$71,400	$272,700

Exercise:

Problem: In which state have home values increased at a higher rate?

Exercise:

Problem: If these trends were to continue, what would be the median home value in Mississippi in 2010?

Solution:

$80,640

Exercise:

Problem:

If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

For the following exercises, use the median home values in Indiana and Alabama (adjusted for inflation) shown in [link]. Assume that the house values are changing linearly.

Year	Indiana	Alabama
1950	$37,700	$27,100
2000	$94,300	$85,100

Exercise:

Problem: In which state have home values increased at a higher rate?

Solution:

Alabama

Exercise:

Problem: If these trends were to continue, what would be the median home value in Indiana in 2010?

Exercise:

Problem:

If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

Solution:

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Real-World Applications

Exercise:

Problem:

In 2004, a school population was 1001. By 2008 the population had grown to 1697. Assume the population is changing linearly.

a. How much did the population grow between the year 2004 and 2008?
b. How long did it take the population to grow from 1001 students to 1697 students?
c. What is the average population growth per year?
d. What was the population in the year 2000?
e. Find an equation for the population, P,of the school t years after 2000.
f. Using your equation, predict the population of the school in 2011.

Exercise:

Problem:

In 2003, a town's population was 1431. By 2007 the population had grown to 2134. Assume the population is changing linearly.

a. How much did the population grow between the year 2003 and 2007?
b. How long did it take the population to grow from 1431 people to 2134 people?
c. What is the average population growth per year?
d. What was the population in the year 2000?
e. Find an equation for the population, P,of the town t years after 2000.
f. Using your equation, predict the population of the town in 2014.

Solution:

a. $2134-1431=703$ people
b. $2007-2003=4$ years
c. Average rate of growth $=\frac{703}{4}=175.75$ people per year

   So, using $y=mx+b$,we have $y=175.75x+1431$.
d. The year 2000 corresponds to $t=-3$.

   So, $y=175.75(-3)+1431=903.75$ or roughly 904 people in year 2000
e. If the year 2000 corresponds to t=0,then we have ordered pair $(0,903.75)$

   $y=175.75x+903.75$ corresponds to $P(t)=175.75t+903.75$
f. The year 2014 corresponds to $t=14$. Therefore, $P(14)=175.75(14)+903.75=3364.25$.

   So, a population of 3364.

Exercise:

Problem:

A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used on the phone. If a customer uses 410 minutes, the monthly cost will be \$71.50. If the customer uses 720 minutes, the monthly cost will be \$118.

a. Find a linear equation for the monthly cost of the cell plan as a function of *x*, the number of monthly minutes used.
b. Interpret the slope and *y*-intercept of the equation.
c. Use your equation to find the total monthly cost if 687 minutes are used.

Exercise:

Problem:

A phone company has a monthly cellular data plan where a customer pays a flat monthly fee of \$10 and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be \$11.20. If the customer uses 130 MB, the monthly cost will be \$17.80.

a. Find a linear equation for the monthly cost of the data plan as a function of x,the number of MB used.
b. Interpret the slope and *y*-intercept of the equation.
c. Use your equation to find the total monthly cost if 250 MB are used.

Solution:

$$\text{Ordered pairs are } (20, 11.20) \text{ and } (130, 17.80)$$

a. $$\begin{aligned} m &= \tfrac{17.80-11.20}{130-20} = 0.06 \text{ and } (0, 10) \\ y &= mx + b \\ y &= 0.06x + 10 \text{ or } C(x) = 0.06x + 10 \end{aligned}$$
b. 0.06 For every MB, the client is charged 6 cents. $(0, 10)$ If no usage occurs, the client is charged \$10
c. $$\begin{aligned} C(250) &= 0.06(250) + 10 \\ &= \$25 \end{aligned}$$

Exercise:

Problem:

In 1991, the moose population in a park was measured to be 4,360. By 1999, the population was measured again to be 5,880. Assume the population continues to change linearly.

a. Find a formula for the moose population, *P* since 1990.
b. What does your model predict the moose population to be in 2003?

Exercise:

Problem:

In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. The population changes linearly. Let the input be years since 1990.

a. Find a formula for the owl population, $P.$ Let the input be years since 2003.
b. What does your model predict the owl population to be in 2012?

Solution:

a. Ordered pairs are $(0, 340)$ and $(4, 285)$

$$m = \frac{285-340}{4-0} = -13.75 \text{ and } (0, 340)$$
$$y = mx + b$$
$$y = -13.75x + 340 \text{ or } P(t) = -13.75t + 340$$

b. The year 2012 corresponds to $\text{t} = 9$

$$P(9) = -13.75(9) + 340$$
$$= 216.25 \text{ or } 216 \text{ owls}$$

Exercise:

Problem:

The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010 and is being depleted by about 2.1 billion cubic feet each year.

a. Give a linear equation for the remaining federal helium reserves, R,in terms of t,the number of years since 2010.
b. In 2015, what will the helium reserves be?
c. If the rate of depletion doesn't change, in what year will the Federal Helium Reserve be depleted?

Exercise:

Problem:

Suppose the world's oil reserves in 2014 are 1,820 billion barrels. If, on average, the total reserves are decreasing by 25 billion barrels of oil each year:

a. Give a linear equation for the remaining oil reserves, R,in terms of t,the number of years since now.
b. Seven years from now, what will the oil reserves be?
c. If the rate at which the reserves are decreasing is constant, when will the world's oil reserves be depleted?

Solution:

a. The year 2014 corresponds to $\text{t} = 0$.

We have $m = -25$ and $(0, 1820)$

$$y = mx + b$$
$$y = -25x + 1820 \text{ or } R(t) = -25t + 1820$$

b.
$$R(7) = -25(7) + 1820$$
$$= 645 \text{ billion cubic feet}$$

c.
$$0 = -25t + 1820$$
$$-1820 = -25t$$
$$72.8 = t \Rightarrow 2014 + 72.8 = 2086.8. \text{ So, in the year } 2086$$

Exercise:

Problem:

You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of $19.95 *plus* 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

Exercise:

Problem:

You are choosing between two different window washing companies. The first charges $5 per window. The second charges a base fee of $40 plus $3 per window. How many windows would you need to have for the second company to be preferable?

Solution:
Equation:

$$\begin{aligned}
&\text{Plan 1: } y = 5x \text{ where x is number of windows}\\
&\text{Plan 2: } y = 3x + 40 \text{ where x is number of windows}\\
3x + 40 &\leq 5x\\
40 &\leq 2x\\
20 &\leq x\\
&\text{So, more than 20 windows}
\end{aligned}$$

Exercise:

Problem: When hired at a new job selling jewelry, you are given two pay options:

Option A: Base salary of $17,000 a year with a commission of 12% of your sales

Option B: Base salary of $20,000 a year with a commission of 5% of your sales

How much jewelry would you need to sell for option A to produce a larger income?

Exercise:

Problem: When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of $14,000 a year with a commission of 10% of your sales

Option B: Base salary of $19,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

Solution:
Equation:

$$\begin{aligned}
&\text{Option A: } y = 0.10x + 14{,}000 \text{ where } x \text{ is dollars of sales.}\\
&\text{Option B: } y = 0.04x + 19,000 \text{ where } x \text{ is dollars of sales}\\
0.10x + 14,000 &\geq 0.04x + 19,000\\
0.06x + 14,000 &\geq 19,000\\
0.06x &\geq 5,000\\
x &\geq 83,333.33\\
&\text{So, more than \$83,333.33 in sales}
\end{aligned}$$

Exercise:

Problem: When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of \$20,000 a year with a commission of 12% of your sales

Option B: Base salary of \$26,000 a year with a commission of 3% of your sales

How much electronics would you need to sell for option A to produce a larger income?

Exercise:

Problem: When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of \$10,000 a year with a commission of 9% of your sales

Option B: Base salary of \$20,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

Solution:
Equation:

$$\begin{array}{l}\text{Option A: } y = 0.09x + 10{,}000 \text{ where x is dollars of sales}\\ \text{Option B: } y = 0.04x + 20,000 \text{ where x is dollars of sales}\end{array}$$

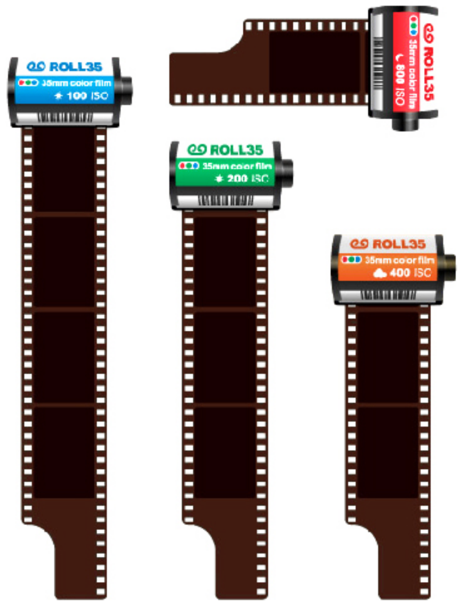
$$\begin{aligned} 0.09x + 10,000 &\geq 0.04x + 20,000 \\ 0.05x + 10,000 &\geq 20,000 \\ 0.05x &\geq 10,000 \\ x &\geq 200,000 \end{aligned}$$

So, more than \$200,000 in sales.

Introduction to Polynomial and Rational Functions

class="introduction"

35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit “film”: modification of work by Horia Varlan; credit “memory cards”: modification of work by Paul Hudson)



Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

Quadratic Functions
In this section, you will:

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function’s minimum or maximum value.
- Solve problems involving a quadratic function’s minimum or maximum value.

An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)

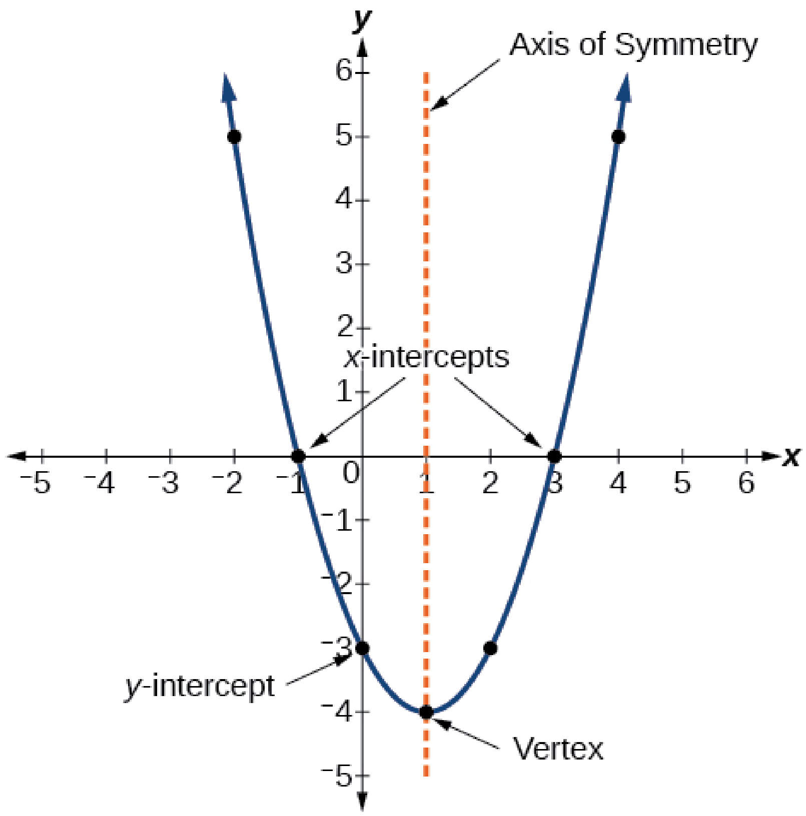
Curved antennas, such as the ones shown in [link], are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of

the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in [link].

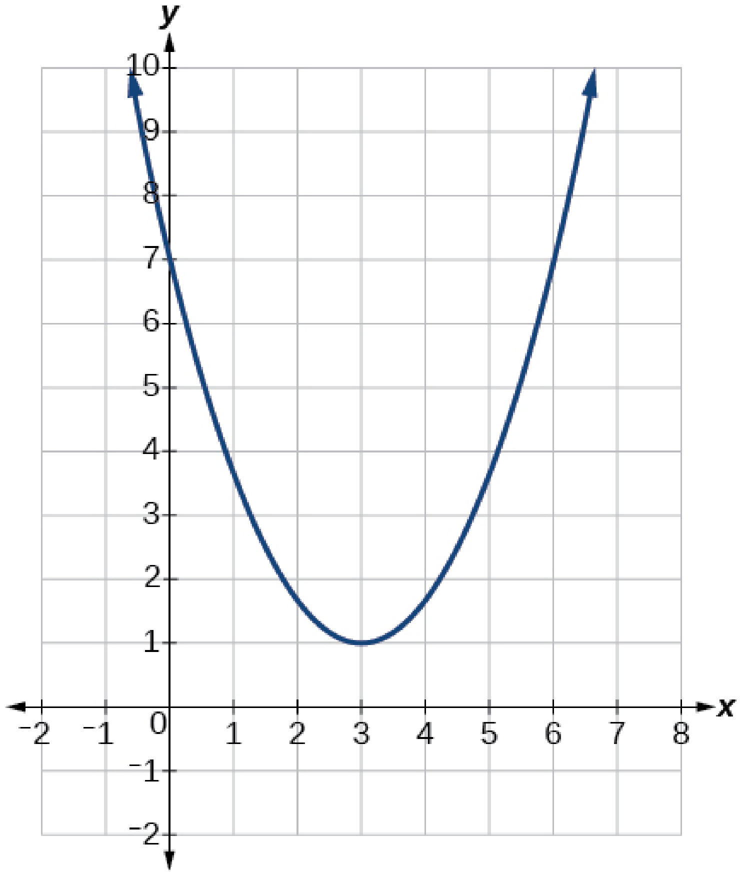


The *y*-intercept is the point at which the parabola crosses the *y*-axis. The *x*-intercepts are the points at which the parabola crosses the *x*-axis. If they exist, the *x*-intercepts represent the **zeros,** or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example:
Exercise:

Problem:
Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in [link].



Solution:

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

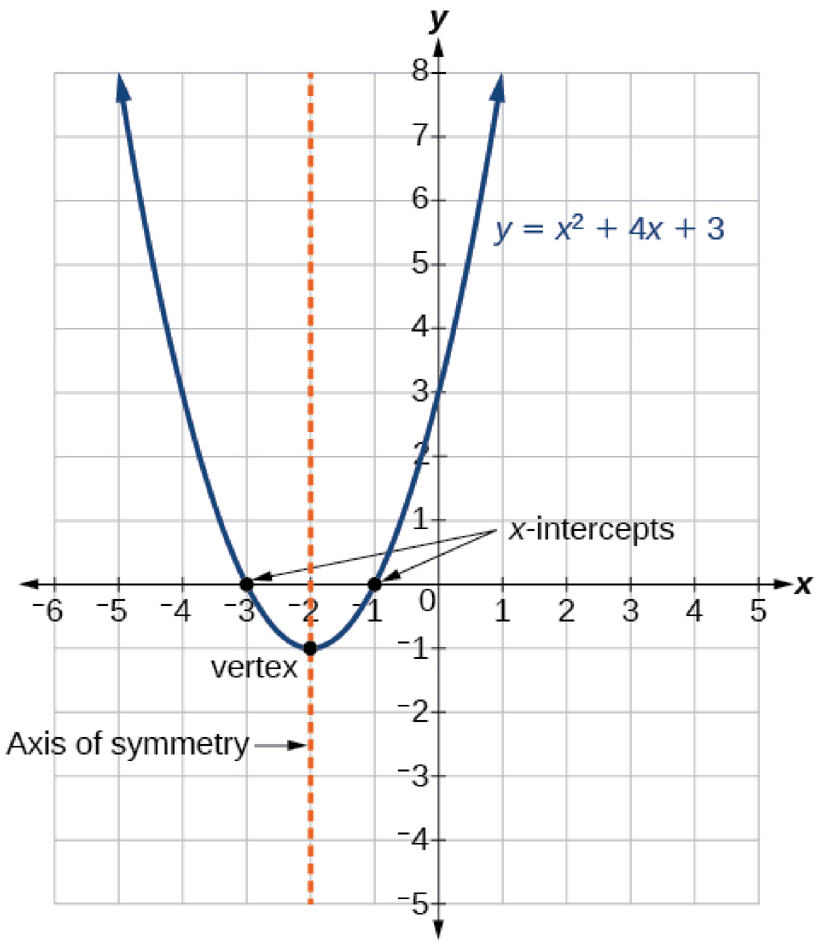
The **general form of a quadratic function** presents the function in the form
Equation:

$$f(x) = ax^2 + bx + c$$

where $a, b,$ and c are real numbers and $a \neq 0$. If $a > 0,$ the parabola opens upward. If $a < 0,$ the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

[link] represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1, b = 4,$ and $c = 3$. Because $a > 0,$ the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

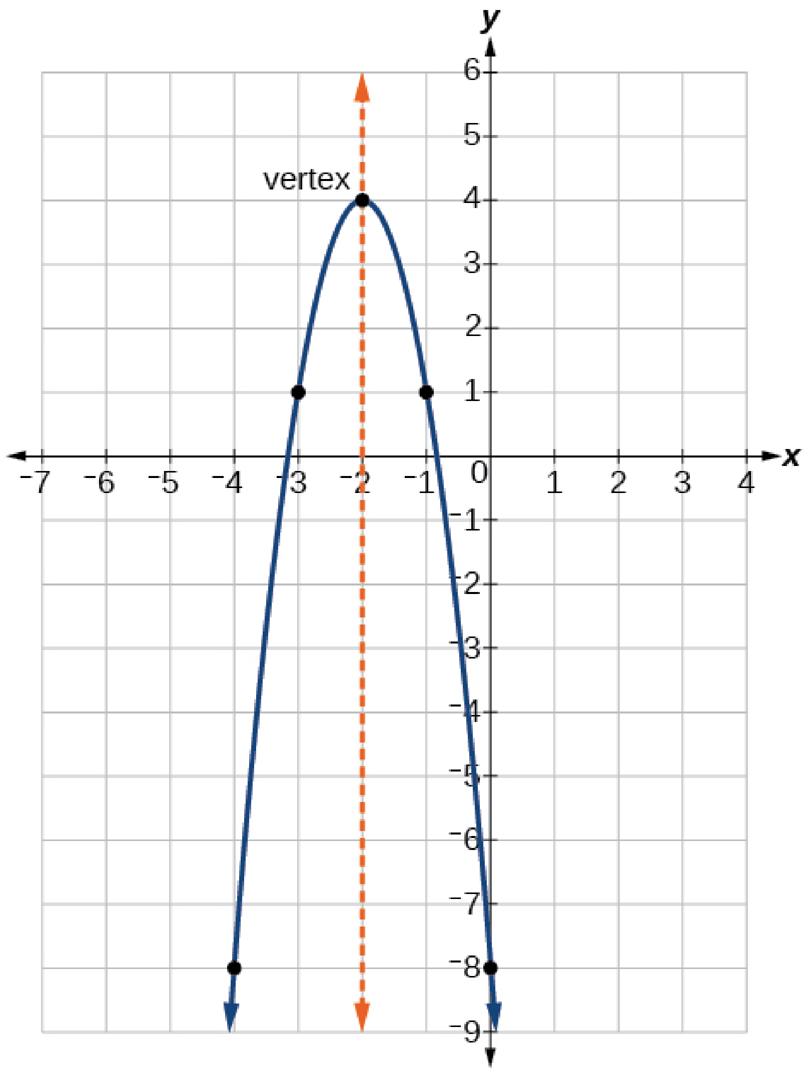


The **standard form of a quadratic function** presents the function in the form
Equation:

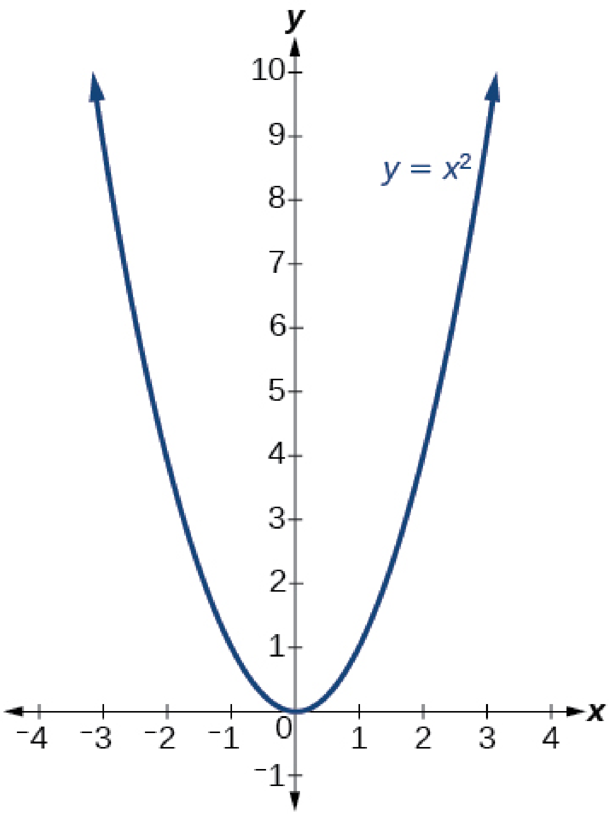
$$f(x) = a(x - h)^2 + k$$

where $(h,\ k)$ is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0,$ the parabola opens upward and the vertex is a minimum. If $a < 0,$ the parabola opens downward, and the vertex is a maximum. [link] represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4.$ Since $x - h = x + 2$ in this example, $h = -2.$ In this form, $a = -3, h = -2,$ and $k = 4.$ Because $a < 0,$ the parabola opens downward. The vertex is at $(-2,\ 4).$



The standard form is useful for determining how the graph is transformed from the graph of $y=x^2$. [link] is the graph of this basic function.



If $k>0,$ the graph shifts upward, whereas if $k<0,$ the graph shifts downward. In [link], $k>0,$ so the graph is shifted 4 units upward. If $h>0,$ the graph shifts toward the right and if $h<0,$ the graph shifts to the left. In [link], $h<0,$ so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a|>1,$ the point associated with a particular x-value shifts farther from the *x*-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a|<1,$ the point associated with a particular x-value shifts closer to the *x*-axis, so the graph appears to become wider, but in fact there is a vertical compression. In [link], $|a|>1,$ so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

Equation:

$$\begin{array}{rcl}a{(x-h)}^{2}+k & = & a{x}^{2}+bx+c\\ a{x}^{2}-2ahx+(a{h}^{2}+k) & = & a{x}^{2}+bx+c\end{array}$$

For the linear terms to be equal, the coefficients must be equal.

Equation:

$$-2ah = b, \text{ so } h = -\frac{b}{2a}$$

This is the axis of symmetry we defined earlier. Setting the constant terms equal:
Equation:

$$\begin{aligned} ah^2 + k &= c \\ k &= c - ah^2 \\ &= c - a - \left(\tfrac{b}{2a}\right)^2 \\ &= c - \tfrac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that *k* is the output value of the function when the input is h, so $f(h) = k$.

Note:
Forms of Quadratic Functions
A quadratic function is a polynomial function of degree two. The graph of a quadratic function is a parabola.
The **general form of a quadratic function** is $f(x) = ax^2 + bx + c$ where $a, b,$ and c are real numbers and $a \neq 0$.
The **standard form of a quadratic function** is $f(x) = a(x-h)^2 + k$ where $a \neq 0$.
The vertex (h, k) is located at
Equation:

$$h =- \frac{b}{2a},\ k = f(h) = f\left(\frac{-b}{2a}\right)$$

Note:
Given a graph of a quadratic function, write the equation of the function in general form.

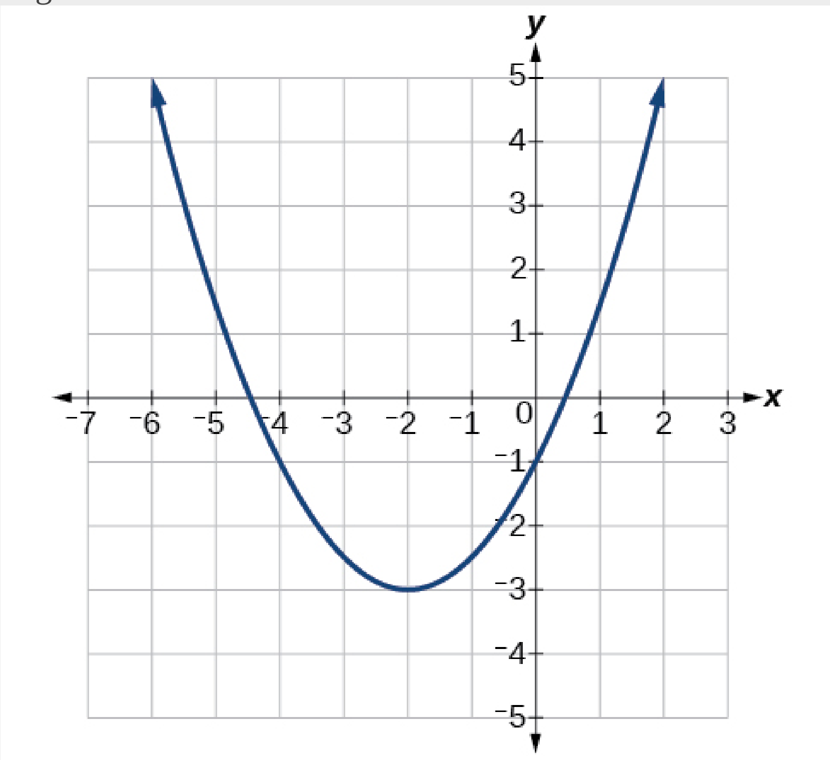
1. Identify the horizontal shift of the parabola; this value is $h.$ Identify the vertical shift of the parabola; this value is $k.$
2. Substitute the values of the horizontal and vertical shift for h and $k.$ in the function $f(x) = a(x–h)^2 + k.$

3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. Expand and simplify to write in general form.

Example:
Exercise:

Problem:
Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in [link] as a transformation of $f(x) = x^2$, and then expand the formula, and simplify terms to write the equation in general form.



Solution:

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x - (-2))^2 - 3 = a(x + 2)^2 - 3.$

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

Equation:

$$\begin{aligned} -1 &= a(0+2)^2 - 3 \\ 2 &= 4a \\ a &= \tfrac{1}{2} \end{aligned}$$

In standard form, the algebraic model for this graph is $(g)x = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

Equation:

$$\begin{aligned} g(x) &= \tfrac{1}{2}(x+2)^2 - 3 \\ &= \tfrac{1}{2}(x+2)(x+2) - 3 \\ &= \tfrac{1}{2}(x^2 + 4x + 4) - 3 \\ &= \tfrac{1}{2}x^2 + 2x + 2 - 3 \\ &= \tfrac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis

We can check our work using the table feature on a graphing utility. First enter $\mathrm{Y1} = \frac{1}{2}(x+2)^2 - 3$. Next, select TBLSET, then use $\mathrm{TblStart} = -6$ and $\Delta\mathrm{Tbl} = 2$, and select TABLE. See [link].

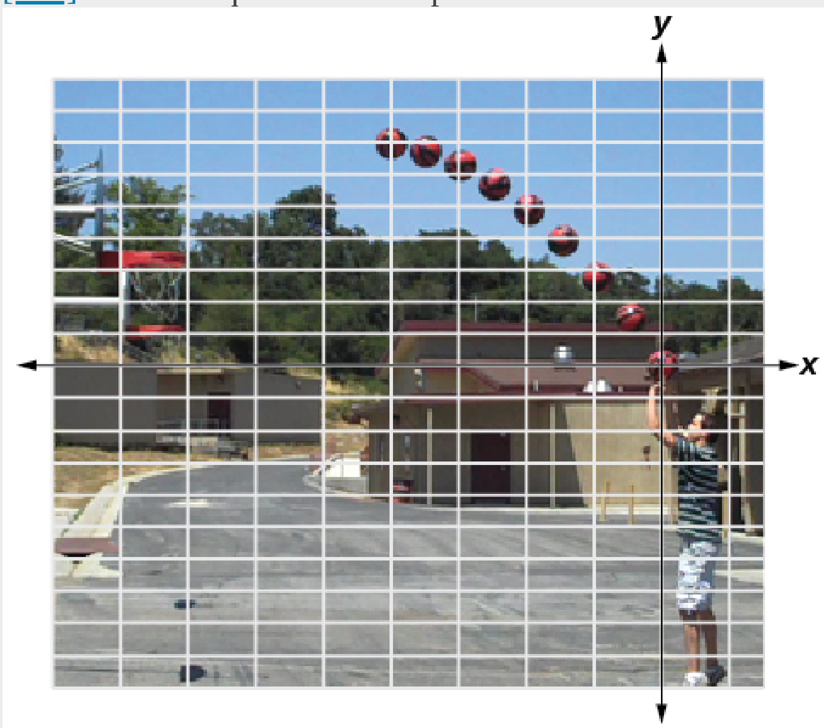
x	–6	–4	–2	0	2
y	5	–1	–3	–1	5

The ordered pairs in the table correspond to points on the graph.

Note:
Exercise:

Problem:

A coordinate grid has been superimposed over the quadratic path of a basketball in [link]. Find an equation for the path of the ball. Does the shooter make the basket?



(credit: modification of work by Dan Meyer)

Solution:

The path passes through the origin and has vertex at $(-4,\ 7)$, so $(h)x =- \frac{7}{16}(x+4)^2 + 7$. To make the shot, $h\left(-7.5\right)$ would need to be about 4 but $h(-7.5) \approx 1.64;$ he doesn’t make it.

Note:
Given a quadratic function in general form, find the vertex of the parabola.

1. Identify $a,\ b,$ and c.
2. Find $h,$ the x-coordinate of the vertex, by substituting a and b into $h =- \frac{b}{2a}$.

3. Find k, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Example:
Exercise:

Problem:
Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 – 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution:

The horizontal coordinate of the vertex will be at

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2(2)} \\ &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$

The vertical coordinate of the vertex will be at

$$\begin{aligned} k &= f(h) \\ &= f\left(\frac{3}{2}\right) \\ &= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \\ &= \frac{5}{2} \end{aligned}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. First, find the horizontal coordinate of the vertex. Then find the vertical coordinate of the vertex. Substitute the values into standard form, using the "a" from the general form.

Equation:

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ f(x) &= 2x^2 - 6x + 7 \end{aligned}$$

The standard form of a quadratic function prior to writing the function then becomes the following:

Equation:

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs, k,and where it occurs, x.

Note:
Exercise:

Problem:

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Solution:

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x-3)^2 + 4$ in standard form

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all *y*-values greater than or equal to the *y*-coordinate at the turning point or less than or equal to the *y*-coordinate at the turning point, depending on whether the parabola opens up or down.

Note:
Domain and Range of a Quadratic Function
The domain of any quadratic function is all real numbers unless the context of the function presents some restrictions.
The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $\left[f\left(-\frac{b}{2a}\right), \infty\right)$; the range of a quadratic

function written in general form with a negative a value is $f(x) \le f\left(-\frac{b}{2a}\right)$, or $\left(-\infty, f\left(-\frac{b}{2a}\right)\right]$.
The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive a value is $f(x) \ge k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \le k$.

Note:
Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \ge k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \le k$, or $(-\infty, k]$.

Example:
Exercise:

Problem:
Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution:

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.
Equation:

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{9}{2(-5)} \\ &= \frac{9}{10} \end{aligned}$$

The maximum value is given by $f(h)$.
Equation:

$$\begin{aligned} f\left(\frac{9}{10}\right) &= -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 \\ &= \frac{61}{20} \end{aligned}$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

Note:
Exercise:

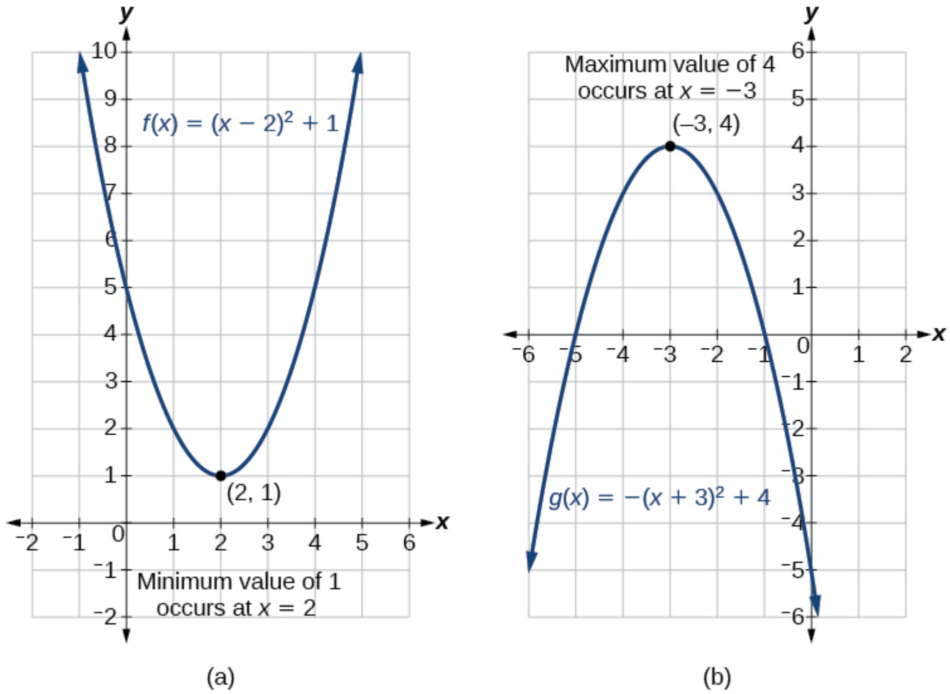
Problem: Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Solution:

The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in [link].



There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example:
Exercise:

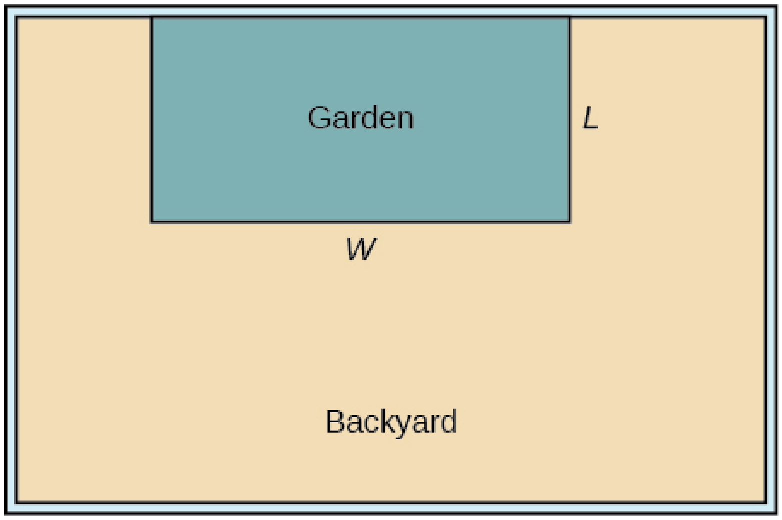
Problem:
Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solution:

Let’s use a diagram such as [link] to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.



a. We know we have only 80 feet of fence available, and $L + W + L = 80,$ or more simply, $2L + W = 80.$ This allows us to represent the width, $W,$ in terms of $L.$
Equation:

$$W = 80 - 2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so
Equation:

$$\begin{aligned} A &= LW = L(80 - 2L) \\ A(L) &= 80L - 2L^2 \end{aligned}$$

This formula represents the area of the fence in terms of the variable length $L.$ The function, written in general form, is
Equation:

$$A(L) = -2L^2 + 80L.$$

b. The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a = -2, b = 80,$ and $c = 0.$

To find the vertex:

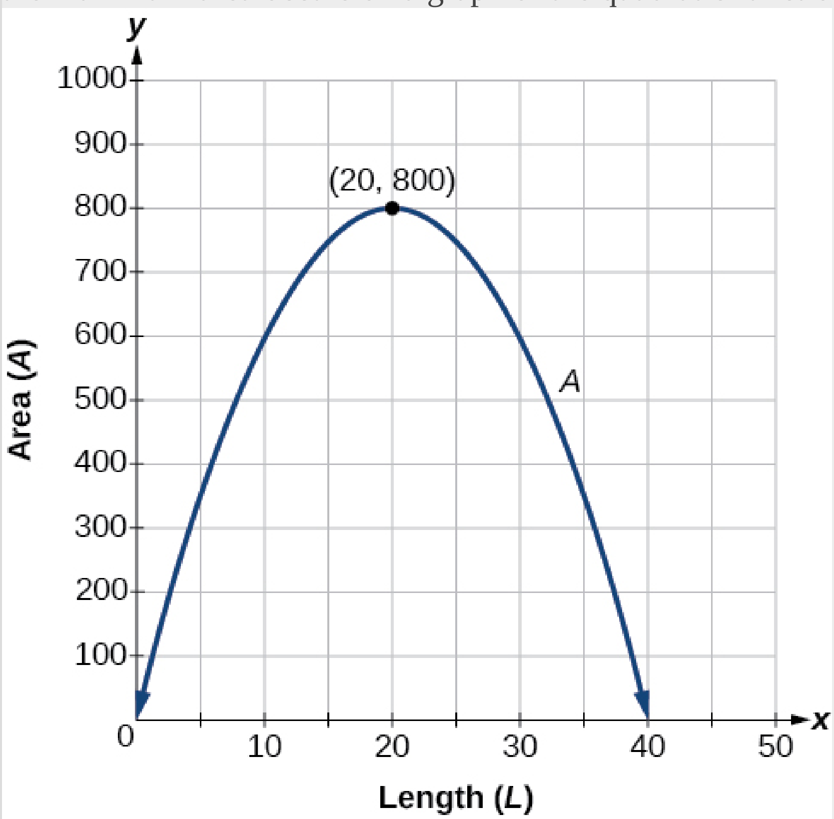
Equation:

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{80}{2(-2)} \\ &= 20 \end{aligned} \quad \text{and} \quad \begin{aligned} k &= A(20) \\ &= 80(20) - 2(20)^2 \\ &= 800 \end{aligned}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in [link].



Note:
Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for a revenue function.
2. Find the vertex of the quadratic equation.
3. Determine the *y*-value of the vertex.

Example:
Exercise:

Problem:
Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution:

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84{,}000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79{,}000$. From this we can find a linear equation relating the two quantities. The slope will be
Equation:

$$\begin{aligned} m &= \frac{79{,}000 - 84{,}000}{32 - 30} \\ &= \frac{-5{,}000}{2} \\ &= -2{,}500 \end{aligned}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the *y*-intercept.
Equation:

$$\begin{array}{rcll} Q & = & -2500p+b & \text{Substitute in the point} Q=84{,}000 \text{ and } p=30 \\ 84{,}000 & = & -2500(30)+b & \text{Solve for} b \\ b & = & 159{,}000 & \end{array}$$

This gives us the linear equation $Q=-2{,}500p+159{,}000$ relating cost and subscribers. We now return to our revenue equation.

Equation:

$$\begin{array}{rcl} \text{Revenue} & = & pQ \\ \text{Revenue} & = & p(-2{,}500p+159{,}000) \\ \text{Revenue} & = & -2{,}500p^2+159{,}000p \end{array}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

Equation:

$$\begin{array}{rcl} h & = & -\frac{159{,}000}{2(-2{,}500)} \\ & = & 31.8 \end{array}$$

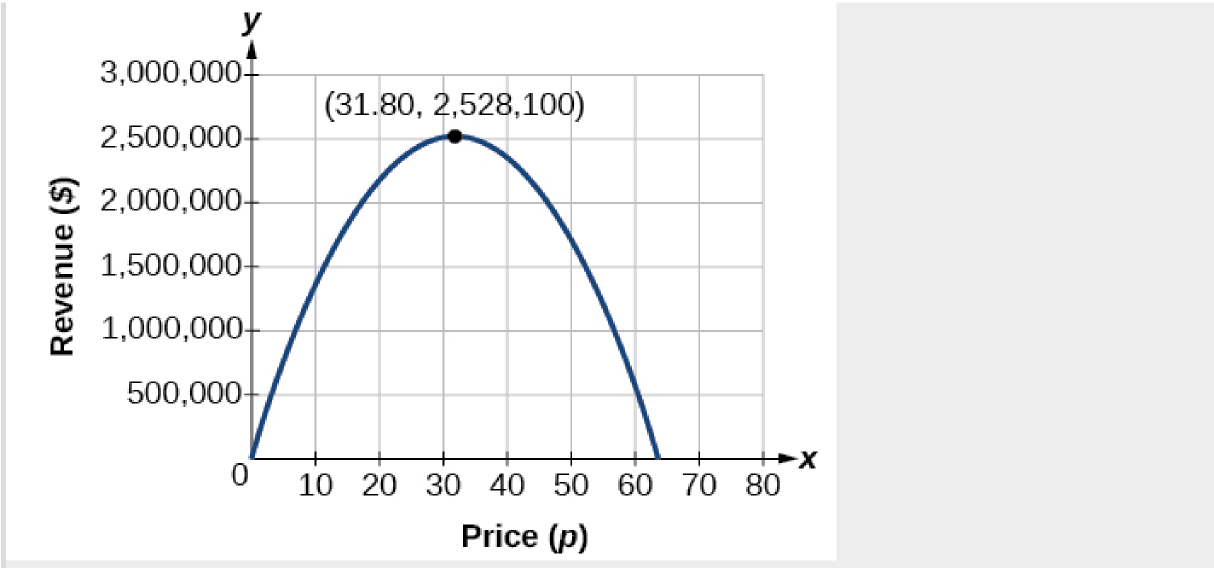
The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

Equation:

$$\begin{array}{rcl} \text{maximum revenue} & = & -2{,}500(31.8)^2+159{,}000(31.8) \\ & = & 2{,}528{,}100 \end{array}$$

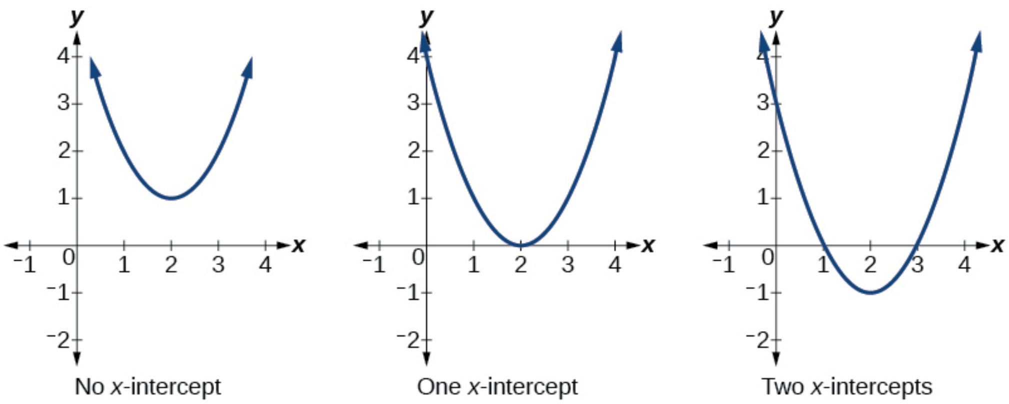
Analysis

This could also be solved by graphing the quadratic as in [link]. We can see the maximum revenue on a graph of the quadratic function.



Finding the *x*- and *y*-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in [link] that the number of x-intercepts can vary depending upon the location of the graph.



Number of *x*-intercepts of a parabola

Note:
Given a quadratic function $f(x)$, find the *y*- and *x*-intercepts.

1. Evaluate $f(0)$ to find the *y*-intercept.
2. Solve the quadratic equation $f(x) = 0$ to find the *x*-intercepts.

Example:
Exercise:

Problem:
Finding the *y*- and *x*-Intercepts of a Parabola

Find the *y*- and *x*-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution:

We find the *y*-intercept by evaluating $f(0)$.
Equation:

$$\begin{array}{rcl} f(0) & = & 3(0)^2 + 5(0) - 2 \\ & = & -2 \end{array}$$

So the *y*-intercept is at $(0, -2)$.

For the *x*-intercepts, we find all solutions of $f(x) = 0$.
Equation:

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.
Equation:

$$0 = (3x - 1)(x + 2)$$

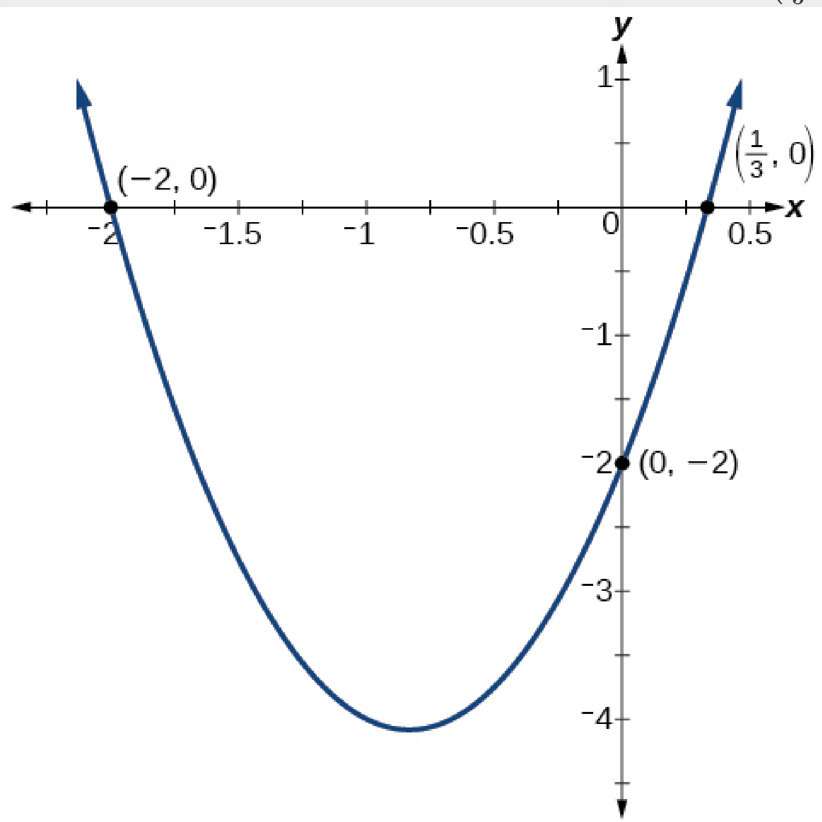
Equation:

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So the *x*-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the *y*-axis at $(0, -2)$. We can also confirm that the graph crosses the *x*-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.See [link]



Rewriting Quadratics in Standard Form

In [link], the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

Note:

Given a quadratic function, find the x-intercepts by rewriting in standard form.

1. Substitute a and b into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.
4. Solve for when the output of the function will be zero to find the x-intercepts.

Example:
Exercise:

Problem:
Finding the *x*-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution:

We begin by solving for when the output will be zero.
Equation:

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.
Equation:

$$f\left(x\right) = a(x-h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.
Equation:

$$\begin{array}{rcl} h & = & -\frac{b}{2a} \\ & = & -\frac{4}{2(2)} \\ & = & -1 \end{array} \qquad \begin{array}{rcl} k & = & f(-1) \\ & = & 2(-1)^2 + 4(-1) - 4 \\ & = & -6 \end{array}$$

So now we can rewrite in standard form.
Equation:

$$f(x) = 2(x+1)^2 - 6$$

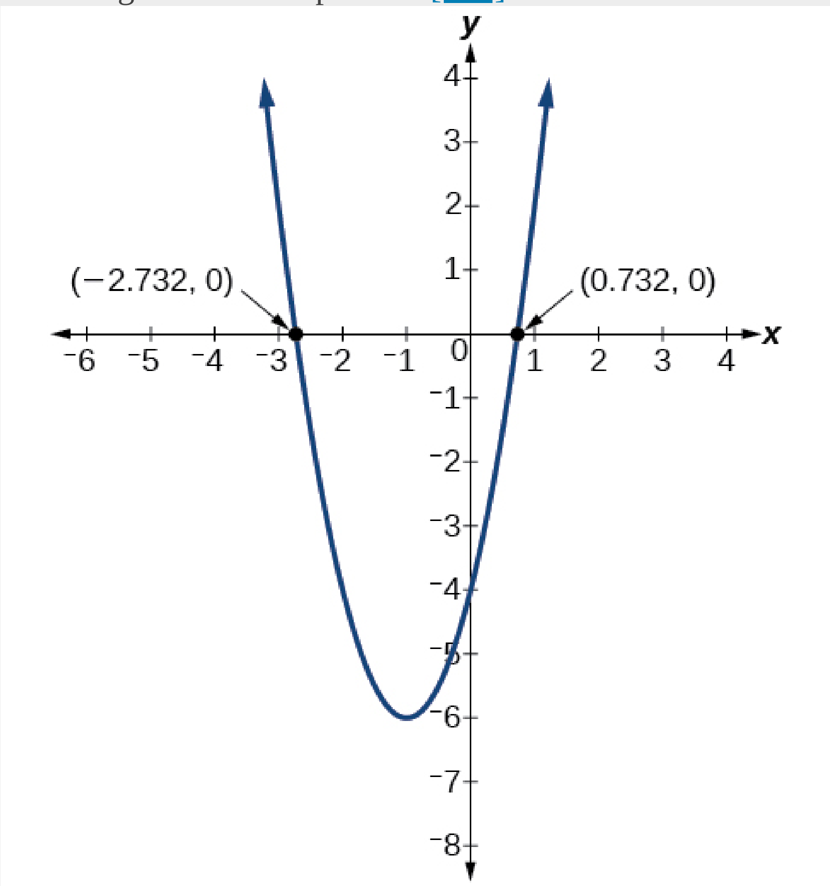
We can now solve for when the output will be zero.

Equation:

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has *x*-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$.

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See [link].



Analysis

We could have achieved the same results using the quadratic formula. Identify $a=2, b=4$ and $c=-4.$

Equation:

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-4\pm\sqrt{4^2-4(2)(-4)}}{2(2)} \\ &= \frac{-4\pm\sqrt{48}}{4} \\ &= \frac{-4\pm\sqrt{3(16)}}{4} \\ &= -1\pm\sqrt{3} \end{aligned}$$

So the *x*-intercepts occur at $\left(-1-\sqrt{3},0\right)$ and $\left(-1+\sqrt{3},0\right)$.

Note:
Exercise:

Problem:

In a Try It, we found the standard and general form for the function $g(x)=13+x^2-6x.$ Now find the *y*- and *x*-intercepts (if any).

Solution:

y-intercept at (0, 13), No x-intercepts

Example:
Exercise:

Problem:
Applying the Vertex and *x*-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t)=-16t^2+80t+40.$

a. When does the ball reach the maximum height?
b. What is the maximum height of the ball?

c. When does the ball hit the ground?

Solution:

a. The ball reaches the maximum height at the vertex of the parabola.
Equation:

$$\begin{aligned} h &= -\frac{80}{2(-16)} \\ &= \frac{80}{32} \\ &= \frac{5}{2} \\ &= 2.5 \end{aligned}$$

The ball reaches a maximum height after 2.5 seconds.

b. To find the maximum height, find the y-coordinate of the vertex of the parabola.
Equation:

$$\begin{aligned} k &= H\left(-\frac{b}{2a}\right) \\ &= H\left(2.5\right) \\ &= -16(2.5)^2 + 80\left(2.5\right) + 40 \\ &= 140 \end{aligned}$$

The ball reaches a maximum height of 140 feet.

c. To find when the ball hits the ground, we need to determine when the height is zero, $H\left(t\right) = 0$.

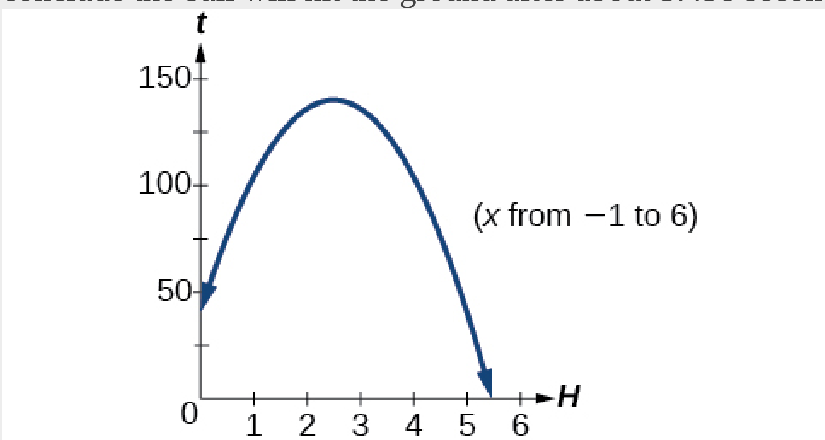
We use the quadratic formula.
Equation:

$$\begin{aligned} t &= \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \\ &= \frac{-80 \pm \sqrt{8960}}{-32} \end{aligned}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.
Equation:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See [link].



Note that the graph does not represent the physical path of the ball upward and downward. Keep the quantities on each axis in mind while interpreting the graph.

Note:
Exercise:

Problem:

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Solution:

3 seconds 256 feet 7 seconds

Note:
Access these online resources for additional instruction and practice with quadratic equations.

- Graphing Quadratic Functions in General Form
- Graphing Quadratic Functions in Standard Form
- Quadratic Function Review
- Characteristics of a Quadratic Function

Key Equations

general form of a quadratic function	$f(x) = ax^2 + bx + c$
standard form of a quadratic function	$f(x) = a(x - h)^2 + k$

Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis. See [link], [link], and [link].
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph. See [link].
- The vertex can be found from an equation representing a quadratic function. See [link].
- The domain of a quadratic function is all real numbers. The range varies with the function. See [link].

- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue. See [link] and [link].
- The vertex and the intercepts can be identified and interpreted to solve real-world problems. See [link].

Section Exercises

Verbal

Exercise:

Problem: Explain the advantage of writing a quadratic function in standard form.

Solution:

When written in that form, the vertex can be easily identified.

Exercise:

Problem: How can the vertex of a parabola be used in solving real-world problems?

Exercise:

Problem:

Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

Solution:

If $a = 0$ then the function becomes a linear function.

Exercise:

Problem: What is another name for the standard form of a quadratic function?

Exercise:

Problem:

What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Solution:

If possible, we can use factoring. Otherwise, we can use the quadratic formula.

Algebraic

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

Exercise:

Problem: $f\left(x\right)={x}^{2}-12x+32$

Exercise:

Problem: $g\left(x\right)={x}^{2}+2x-3$

Solution:

$f(x)={\left(x+1\right)}^{2}-2,\text{ Vertex }\left(-1,-4\right)$

Exercise:

Problem: $f(x)={x}^{2}-x$

Exercise:

Problem: $f(x)={x}^{2}+5x-2$

Solution:

$f(x)={\left(x+\frac{5}{2}\right)}^{2}-\frac{33}{4},\text{ Vertex }\left(-\frac{5}{2},-\frac{33}{4}\right)$

Exercise:

Problem: $h\left(x\right)=2{x}^{2}+8x-10$

Exercise:

Problem: $k\left(x\right)=3{x}^{2}-6x-9$

Solution:

$f(x) = 3(x-1)^2 - 12,$ Vertex $(1, -12)$

Exercise:

Problem: $f(x) = 2x^2 - 6x$

Exercise:

Problem: $f(x) = 3x^2 - 5x - 1$

Solution:

$f(x) = 3\left(x - \frac{5}{6}\right)^2 - \frac{37}{12},$ Vertex $\left(\frac{5}{6}, -\frac{37}{12}\right)$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

Exercise:

Problem: $y\left(x\right) = 2x^2 + 10x + 12$

Exercise:

Problem: $f\left(x\right) = 2x^2 - 10x + 4$

Solution:

Minimum is $-\frac{17}{2}$ and occurs at $\frac{5}{2}$. Axis of symmetry is $x = \frac{5}{2}$.

Exercise:

Problem: $f(x) = -x^2 + 4x + 3$

Exercise:

Problem: $f(x) = 4x^2 + x - 1$

Solution:

Minimum is $-\frac{17}{16}$ and occurs at $-\frac{1}{8}$. Axis of symmetry is $x = -\frac{1}{8}$.

Exercise:

Problem: $h\left(t\right) = -4t^2 + 6t - 1$

Exercise:

Problem: $f(x) = \frac{1}{2}x^2 + 3x + 1$

Solution:

Minimum is $-\frac{7}{2}$ and occurs at -3. Axis of symmetry is $x = -3$.

Exercise:

Problem: $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the following exercises, determine the domain and range of the quadratic function.

Exercise:

Problem: $f(x) = (x-3)^2 + 2$

Solution:

Domain is $(-\infty, \infty)$. Range is $[2, \infty)$.

Exercise:

Problem: $f(x) = -2(x+3)^2 - 6$

Exercise:

Problem: $f(x) = x^2 + 6x + 4$

Solution:

Domain is $(-\infty, \infty)$. Range is $[-5, \infty)$.

Exercise:

Problem: $f(x) = 2x^2 - 4x + 2$

Exercise:

Problem: $k\left(x\right) = 3x^2 - 6x - 9$

Solution:

Domain is $(-\infty, \infty)$. Range is $[-12, \infty)$.

For the following exercises, use the vertex (h, k) and a point on the graph (x, y) to find the general form of the equation of the quadratic function.

Exercise:

Problem: $(h, k) = (2, 0), (x, y) = (4, 4)$

Solution:

$f(x) = x^2 - 4x + 4$

Exercise:

Problem: $(h, k) = (-2, -1), (x, y) = (-4, 3)$

Exercise:

Problem: $(h, k) = (0, 1), (x, y) = (2, 5)$

Solution:

$f(x) = x^2 + 1$

Exercise:

Problem: $(h, k) = (2, 3), (x, y) = (5, 12)$

Exercise:

Problem: $(h, k) = (-5, 3), (x, y) = (2, 9)$

Solution:

$f(x) = \frac{6}{49}x^2 + \frac{60}{49}x + \frac{297}{49}$

Exercise:

Problem: $(h, k) = (3, 2), (x, y) = (10, 1)$

Exercise:

Problem: $(h, k) = (0, 1), (x, y) = (1, 0)$

Solution:

$f(x) = -x^2 + 1$

Exercise:

Problem: $(h, k) = (1, 0), (x, y) = (0, 1)$

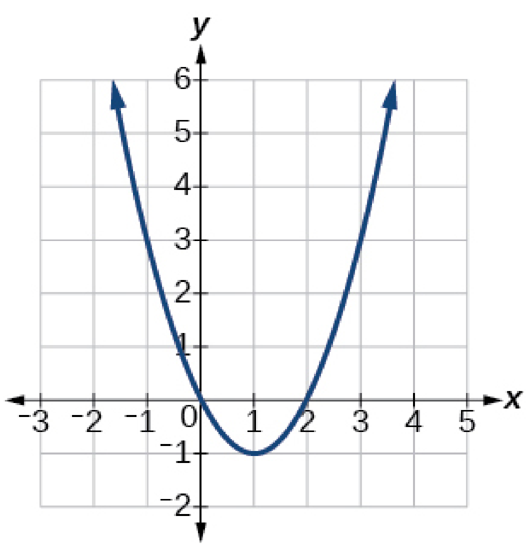
Graphical

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

Exercise:

Problem: $f(x) = x^2 - 2x$

Solution:



Vertex $(1,\ -1)$, Axis of symmetry is $x = 1$. Intercepts are $(0, 0),\ (2, 0)$.

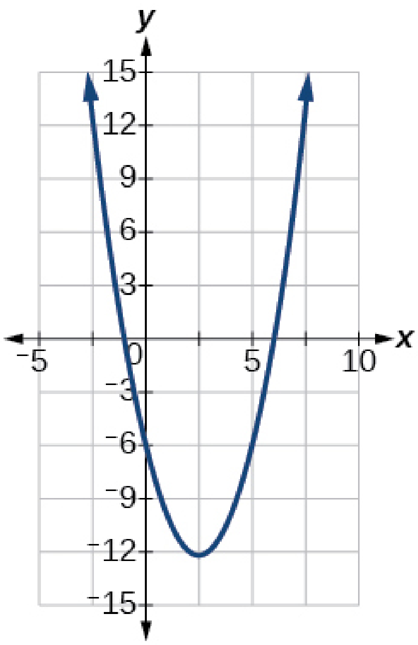
Exercise:

Problem: $f(x) = x^2 - 6x - 1$

Exercise:

Problem: $f(x) = x^2 - 5x - 6$

Solution:



Vertex $\left(\frac{5}{2}, \frac{-49}{4}\right)$, Axis of symmetry is $(0, -6), (-1, 0), (6, 0)$.

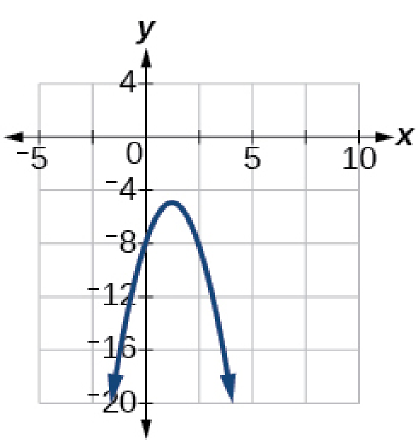
Exercise:

Problem: $f(x) = x^2 - 7x + 3$

Exercise:

Problem: $f(x) = -2x^2 + 5x - 8$

Solution:

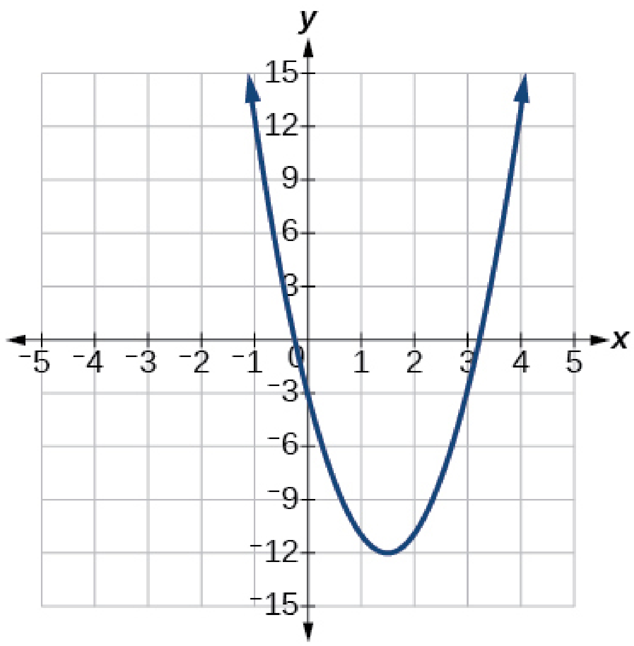


Vertex $\left(\frac{5}{4},\ -\frac{39}{8}\right)$, Axis of symmetry is $x = \frac{5}{4}$. Intercepts are $(0,\ -8)$.

Exercise:

Problem: $f(x) = 4x^2 - 12x - 3$

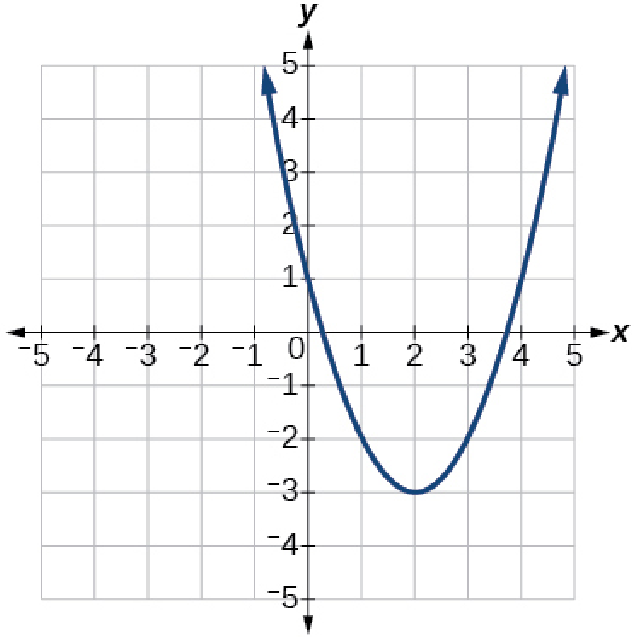
Solution:



For the following exercises, write the equation for the graphed quadratic function.

Exercise:

Problem:

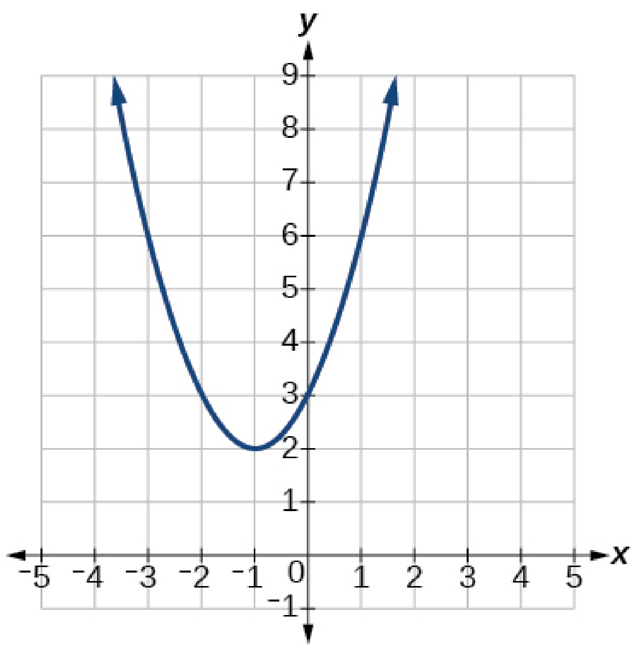


Solution:

$f(x) = x^2 - 4x + 1$

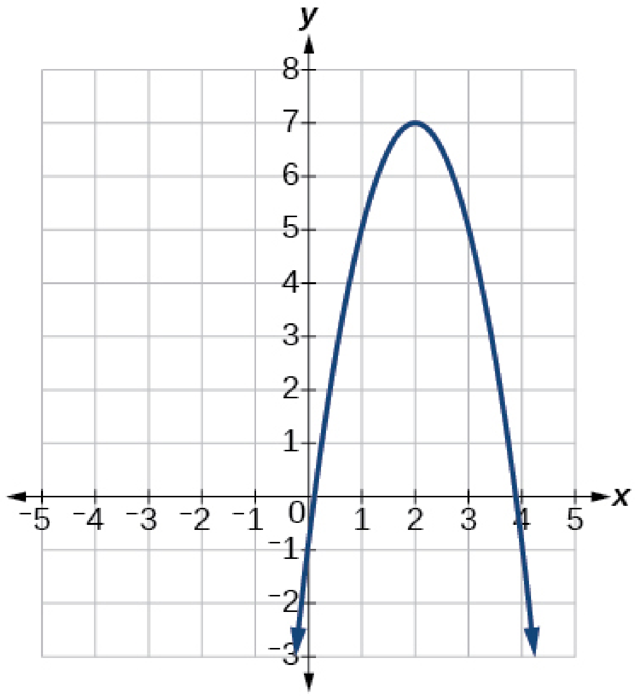
Exercise:

Problem:



Exercise:

Problem:

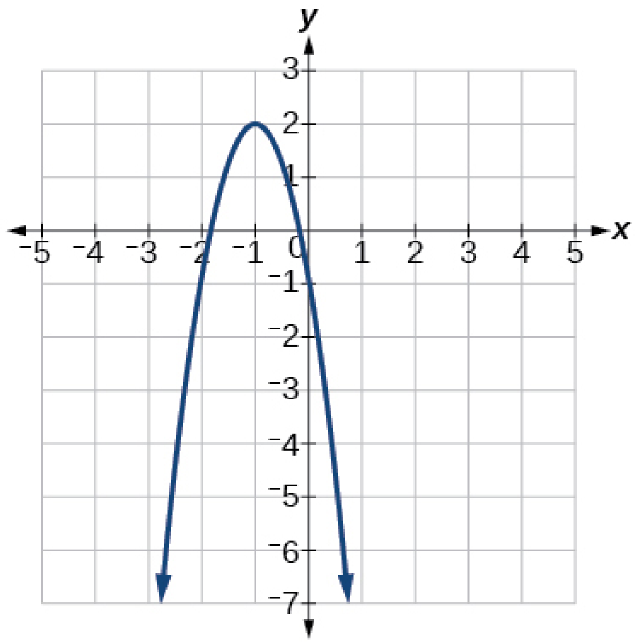


Solution:

$f(x) = -2x^2 + 8x - 1$

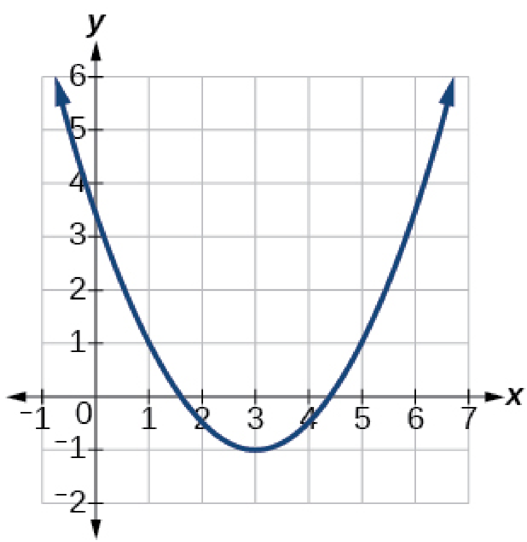
Exercise:

Problem:



Exercise:

Problem:

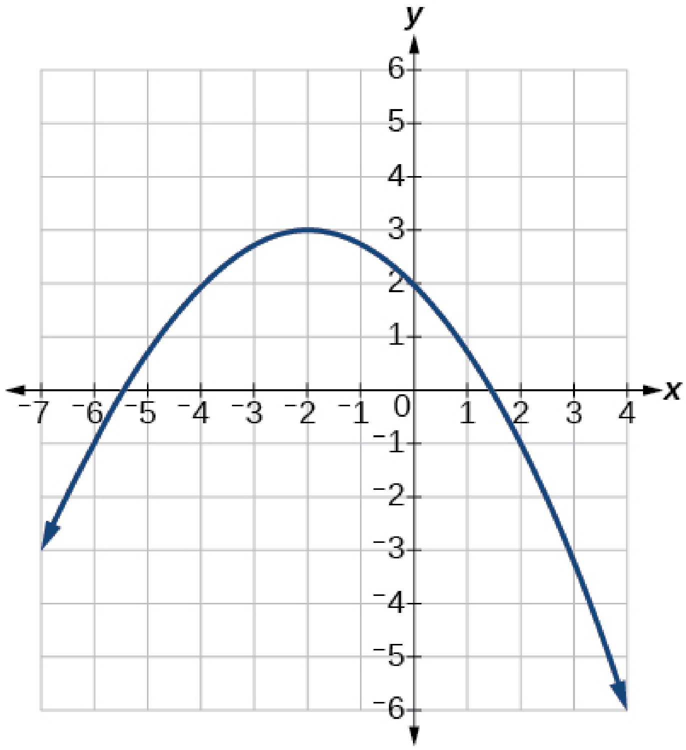


Solution:

$f(x) = \frac{1}{2}x^2 - 3x + \frac{7}{2}$

Exercise:

Problem:



Numeric

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

Exercise:

Problem:

x	–2	–1	0	1	2
y	5	2	1	2	5

Solution:

$f(x) = x^2 + 1$

Exercise:

Problem:

x	–2	–1	0	1	2
y	1	0	1	4	9

Exercise:

Problem:

x	–2	–1	0	1	2
y	–2	1	2	1	–2

Solution:

$f(x) = 2 - x^2$

Exercise:

Problem:

x	–2	–1	0	1	2
y	–8	–3	0	1	0

Exercise:

Problem:

x	–2	–1	0	1	2
y	8	2	0	2	8

Solution:

$f(x) = 2x^2$

Technology

For the following exercises, use a calculator to find the answer.

Exercise:

Problem:

Graph on the same set of axes the functions $f(x) = x^2, f(x) = 2x^2,$ and $f(x) = \frac{1}{3}x^2.$

What appears to be the effect of changing the coefficient?

Exercise:

Problem:

Graph on the same set of axes $f(x) = x^2, f(x) = x^2 + 2$ and $f(x) = x^2, f(x) = x^2 + 5$ and $f(x) = x^2 - 3.$ What appears to be the effect of adding a constant?

Solution:

The graph is shifted up or down (a vertical shift).

Exercise:

Problem:

Graph on the same set of axes $f(x)=x^2, f(x)=(x-2)^2, f(x-3)^2,$ and $f(x)=(x+4)^2.$

What appears to be the effect of adding or subtracting those numbers?

Exercise:

Problem:

The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x)=\frac{-32}{(80)^2}x^2+x$ where x is the horizontal distance traveled and $h\left(x\right)$ is the height in feet. Use the TRACE feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

Solution:

50 feet

Exercise:

Problem:

A suspension bridge can be modeled by the quadratic function $h(x)=.0001x^2$ with $-2000\le x\le 2000$ where $|x|$ is the number of feet from the center and $h\left(x\right)$ is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

Extensions

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

Exercise:

Problem: Vertex $(1,-2),$ opens up.

Solution:

Domain is $(-\infty,\infty).$ Range is $[-2,\infty).$

Exercise:

Problem: Vertex $(-1, 2)$ opens down.

Exercise:

Problem: Vertex $(-5, 11),$ opens down.

Solution:

Domain is $(-\infty, \infty)$ Range is $(-\infty, 11]$.

Exercise:

Problem: Vertex $(-100, 100),$ opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

Exercise:

Problem: Contains $(1, 1)$ and has shape of $f(x) = 2x^2.$ Vertex is on the y-axis.

Solution:

$f(x) = 2x^2 - 1$

Exercise:

Problem: Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2.$ Vertex is on the y-axis.

Exercise:

Problem: Contains $(2, 3)$ and has the shape of $f(x) = 3x^2.$ Vertex is on the y-axis.

Solution:

$f(x) = 3x^2 - 9$

Exercise:

Problem:

Contains $(1, -3)$ and has the shape of $f(x) = -x^2.$ Vertex is on the y-axis.

Exercise:

Problem: Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the y-axis.

Solution:

$f(x) = 5x^2 - 77$

Exercise:

Problem:

Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has x-coordinate of -1.

Real-World Applications

Exercise:

Problem:

Find the dimensions of the rectangular corral producing the greatest enclosed area given 200 feet of fencing.

Solution:

50 feet by 50 feet. Maximize $f(x) = -x^2 + 100x$.

Exercise:

Problem:

Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

Exercise:

Problem:

Find the dimensions of the rectangular corral producing the greatest enclosed area split into 3 pens of the same size given 500 feet of fencing.

Solution:

125 feet by 62.5 feet. Maximize $f(x) = -2x^2 + 250x$.

Exercise:

Problem:

Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

Exercise:

Problem:

Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?

Solution:

6 and $-6;$ product is –36; maximize $f(x) = x^2 + 12x.$

Exercise:

Problem:

Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x,$ where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p.$ Find the production level that will maximize revenue.

Exercise:

Problem:

A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h\left(t\right) = -4.9t^2 + 229t + 234.$ Find the maximum height the rocket attains.

Solution:

2909.56 meters

Exercise:

Problem:

A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h\left(t\right) = -4.9t^2 + 24t + 8.$ How long does it take to reach maximum height?

Exercise:

Problem:

A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

Solution:

$10.70

Exercise:

Problem:

A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

Glossary

axis of symmetry
: a vertical line drawn through the vertex of a parabola, that opens up or down, around which the parabola is symmetric; it is defined by $x=-\frac{b}{2a}$.

general form of a quadratic function
: the function that describes a parabola, written in the form $f(x)=ax^2+bx+c$, where $a,b,$ and c are real numbers and $a\neq 0$.

roots
: in a given function, the values of x at which $y=0$, also called zeros

standard form of a quadratic function
: the function that describes a parabola, written in the form $f(x)=a(x-h)^2+k$, where $(h,\ k)$ is the vertex

vertex
: the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

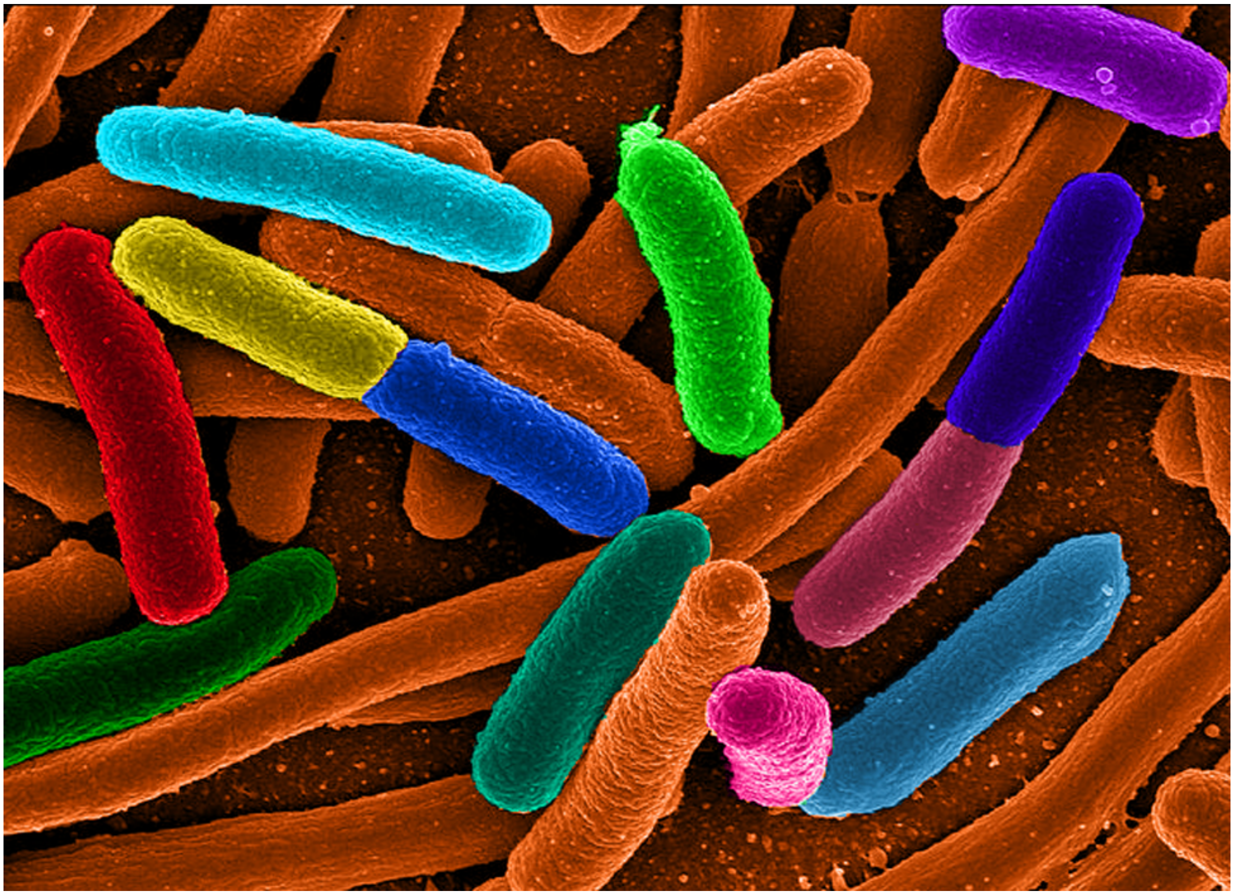
vertex form of a quadratic function
: another name for the standard form of a quadratic function

zeros

in a given function, the values of x at which $y = 0$, also called roots

Introduction to Exponential and Logarithmic Functions
class="introduction"

Electron micrograph of *E.Coli* bacteria (credit: “Mattosaurus, ” Wikimedia Commons)



Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years.[footnote]

Todar, PhD, Kenneth. Todar's Online Textbook of Bacteriology. http://textbookofbacteriology.net/growth_3.html.

For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. [link] shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

Exponential Functions
In this section, you will:

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about 1.2% each year[footnote]. If this rate continues, the population of India will exceed China's population by the year 2031.When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.
http://www.worldometers.info/world-population/. Accessed February 24, 2014.

Identifying Exponential Functions

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$,the slope tells us the output increases by 3 each time the input increases by 1. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2021.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See [link].

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

From [link] we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Apparently, the difference between "the same percentage" and "the same amount" is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

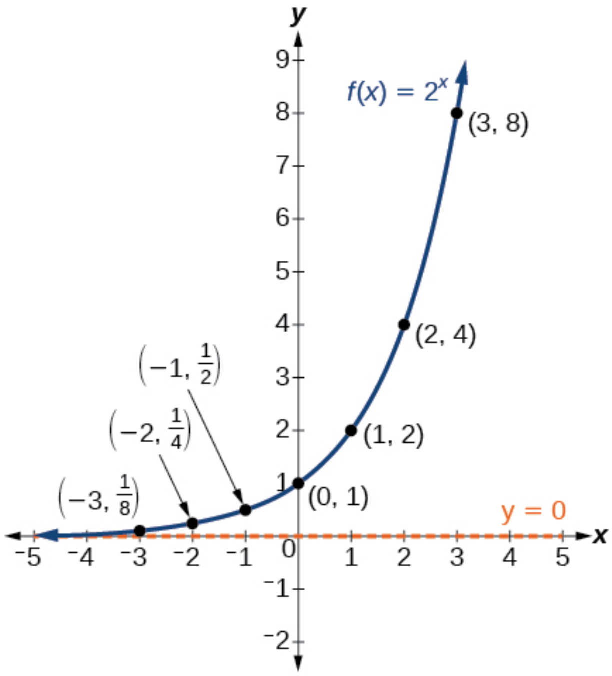
The general form of the exponential function is $f(x) = ab^x,$ where a is any nonzero number, b is a positive real number not equal to 1.

- If $b > 1$,the function grows at a rate proportional to its size.
- If $0 < b < 1$, the function decays at a rate proportional to its size.

Let's look at the function $f(x) = 2^x$ from our example. We will create a table ([link]) to determine the corresponding outputs over an interval in the domain from -3 to 3.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Let us examine the graph of f by plotting the ordered pairs we observe on the table in [link], and then make a few observations.



Let's define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty$, $f(x) \to \infty$,
- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the *y*-intercept is 1.

Note:
Exponential Function
For any real number x,an exponential function is a function with the form
Equation:

$$f(x) = ab^x$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The *y*-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

Example:
Exercise:

Problem:
Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution:

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Note:
Exercise:

Problem: Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Solution:

$g(x) = 0.875^x$ and$j(x) = 1095.6^{-2x}$ represent exponential functions.

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1.Why do we limit the base b to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$,which is not a real number.

Why do we limit the base to positive values other than 1?Because base 1 results in the constant function. Observe what happens if the base is 1 :

- Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x.

To evaluate an exponential function with the form $f(x) = b^x$,we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x) = 2^x$. What is $f(3)$?
Equation:

$$\begin{aligned} f\left(x\right) &= 2^x & \\ f\left(3\right) &= 2^3 & \text{Substitute } x = 3. \\ &= 8 & \text{Evaluate the power.} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x) = 30(2)^x$. What is $f(3)$?
Equation:

$$\begin{aligned} f\left(x\right) &= 30(2)^x & \\ f\left(3\right) &= 30(2)^3 & \text{Substitute } x = 3. \\ &= 30\left(8\right) & \text{Simplify the power first.} \\ &= 240 & \text{Multiply.} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect:
Equation:

$$f(3) = 30(2)^3 \neq 60^3 = 216{,}000$$

Example:
Exercise:

Problem:
Evaluating Exponential Functions

Let $f\left(x\right) = 5(3)^{x+1}$. Evaluate $f\left(2\right)$ without using a calculator.

Solution:

Follow the order of operations. Be sure to pay attention to the parentheses.
Equation:

$$\begin{aligned} f\left(x\right) &= 5(3)^{x+1} & \\ f\left(2\right) &= 5(3)^{2+1} & \text{Substitute } x = 2. \\ &= 5(3)^3 & \text{Add the exponents.} \\ &= 5\left(27\right) & \text{Simplify the power.} \\ &= 135 & \text{Multiply.} \end{aligned}$$

Note:
Exercise:

Problem: Let $f\left(x\right)=8{\left(1.2\right)}^{x-5}.$ Evaluate $f\left(3\right)$ using a calculator. Round to four decimal places.

Solution:

5.5556

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Note:
Exponential Growth
A function that models **exponential growth** grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b\ne 1$, an exponential growth function has the form
Equation:

$$f(x)=a{b}^{x}$$

where

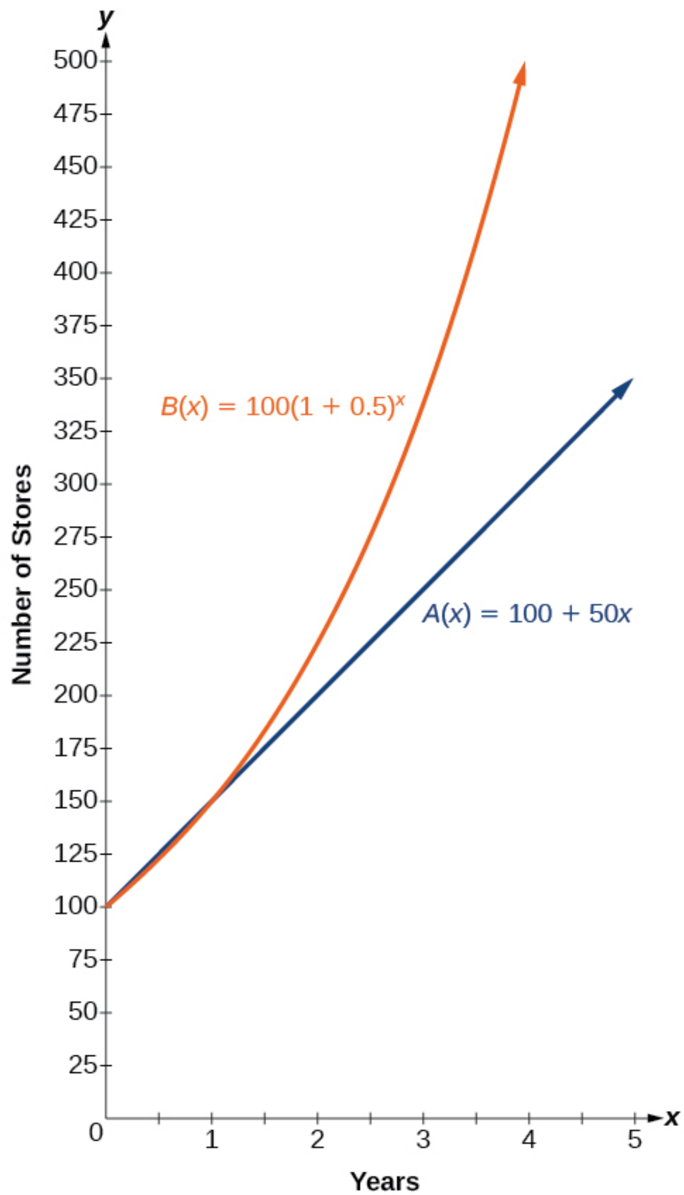
- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A\left(x\right)=100+50x.$ Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x)=100{(1+0.5)}^{x}.$

A few years of growth for these companies are illustrated in [link].

Year, x	Stores, Company A	Stores, Company B
0	$100+50(0)=100$	$100(1+0.5)^{0}=100$
1	$100+50(1)=150$	$100(1+0.5)^{1}=150$
2	$100+50(2)=200$	$100(1+0.5)^{2}=225$
3	$100+50(3)=250$	$100(1+0.5)^{3}=337.5$
x	$A(x)=100+50x$	$B(x)=100(1+0.5)^{x}$

The graphs comparing the number of stores for each company over a five-year period are shown in [link]. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.



The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$,and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$,where 100 is the initial value, 1.5 is called the *base*, and x is called the *exponent*.

Example:
Exercise:

Problem:
Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about $1.2\%.$ This situation is represented by the growth function $P(t)=1.25{(1.012)}^{t},$ where t is the number of years since $2013.$ To the nearest thousandth, what will the population of India be in $2031?$

Solution:

To estimate the population in 2031, we evaluate the models for $t=18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,
Equation:

$$P(18)=1.25{(1.012)}^{18}\approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

Note:
Exercise:

Problem:

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about $0.6\%.$ This situation is represented by the growth function $P(t)=1.39{(1.006)}^{t},$ where t is the number of years since $2013.$To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in [link]?

Solution:

About 1.548 billion people; by the year 2031, India’s population will exceed China’s by about 0.001 billion, or 1 million people.

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b,and evaluate the function.

Note:
Given two data points, write an exponential model.

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example:
Exercise:

Problem:
Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population (N) of deer over time t.

Solution:

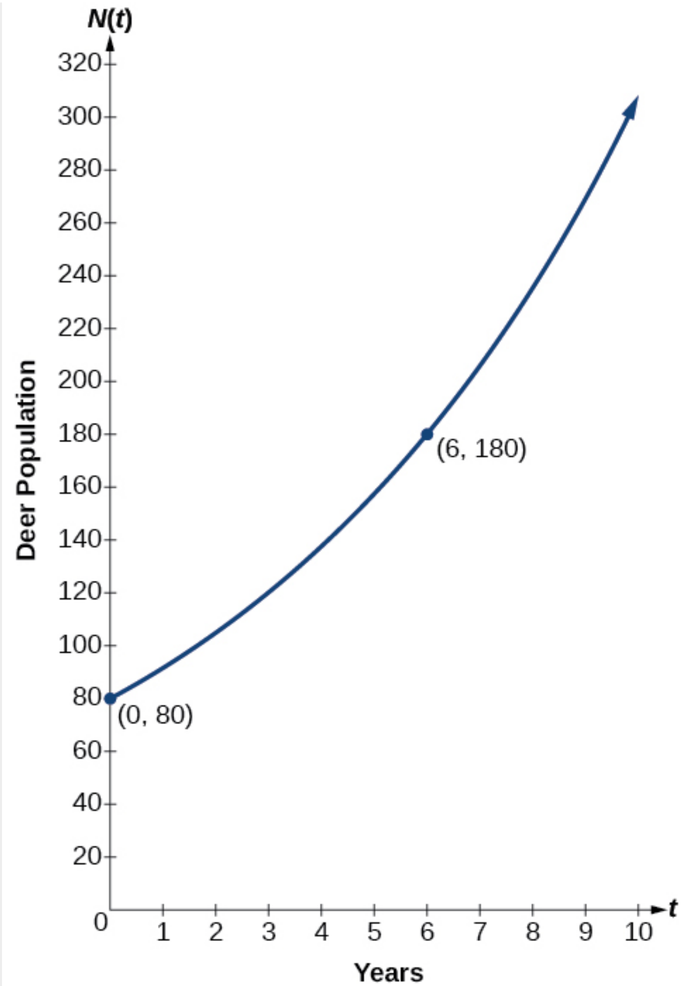
We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b :
Equation:

$$\begin{aligned} N(t) &= 80b^t && \\ 180 &= 80b^6 && \text{Substitute using point } (6,\ 180). \\ \tfrac{9}{4} &= b^6 && \text{Divide and write in lowest terms.} \\ b &= \left(\tfrac{9}{4}\right)^{\frac{1}{6}} && \text{Isolate } b \text{ using properties of exponents.} \\ b &\approx 1.1447 && \text{Round to 4 decimal places.} \end{aligned}$$

NOTE: *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in [link] passes through the initial points given in the problem, $(0,\ 80)$ and $(6,\ 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.



Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, t years after 2006

Note:
Exercise:

Problem:

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Solution:

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

Example:
Exercise:

Problem:
Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2,6)$ and $(2,1)$.

Solution:

Because we don't have the initial value, we substitute both points into an equation of the form $f(x)=ab^x$, and then solve the system for a and b.

- Substituting $(-2,6)$ gives $6=ab^{-2}$
- Substituting $(2,1)$ gives $1=ab^2$

Use the first equation to solve for a in terms of b :

$$\begin{array}{ll} 6=ab^{-2} & \\ \frac{6}{b^{-2}}=a & \text{Divide.} \\ a=6b^2 & \text{Use properties of exponents to rewrite the denominator.} \end{array}$$

Substitute a in the second equation, and solve for b :

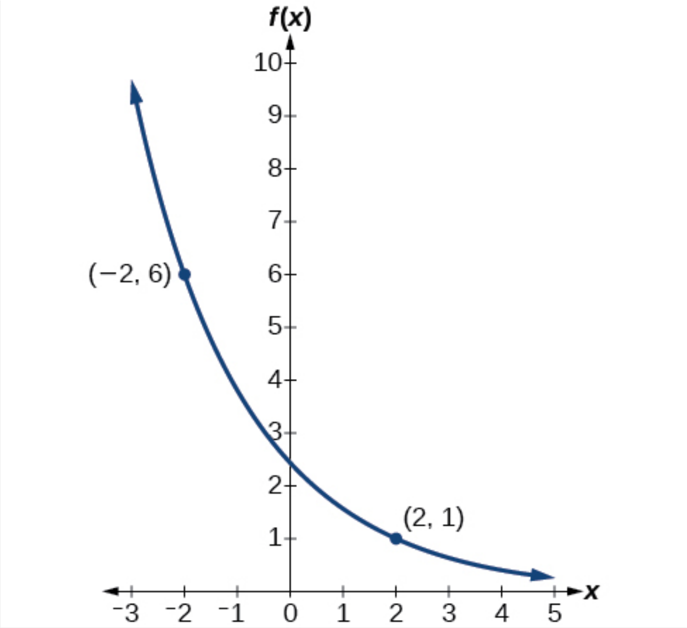
$$\begin{array}{ll} 1=ab^2 & \\ 1=6b^2b^2=6b^4 & \text{Substitute } a. \\ b=\left(\frac{1}{6}\right)^{\frac{1}{4}} & \text{Use properties of exponents to isolate } b. \\ b\approx 0.6389 & \text{Round 4 decimal places.} \end{array}$$

Use the value of b in the first equation to solve for the value of a :

$$a=6b^2\approx 6(0.6389)^2\approx 2.4492$$

Thus, the equation is $f(x)=2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in [link] passes through the initial points given in the problem, $(-2,\ 6)$ and $(2,\ 1)$. The graph is an example of an exponential decay function.



The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

Note:
Exercise:

Problem:

Given the two points $(1, 3)$ and $(2, 4.5)$,find the equation of the exponential function that passes through these two points.

Solution:

$f(x) = 2(1.5)^x$

Note:
Do two points always determine a unique exponential function?
Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x*, which in many real world cases involves time.*

Note:

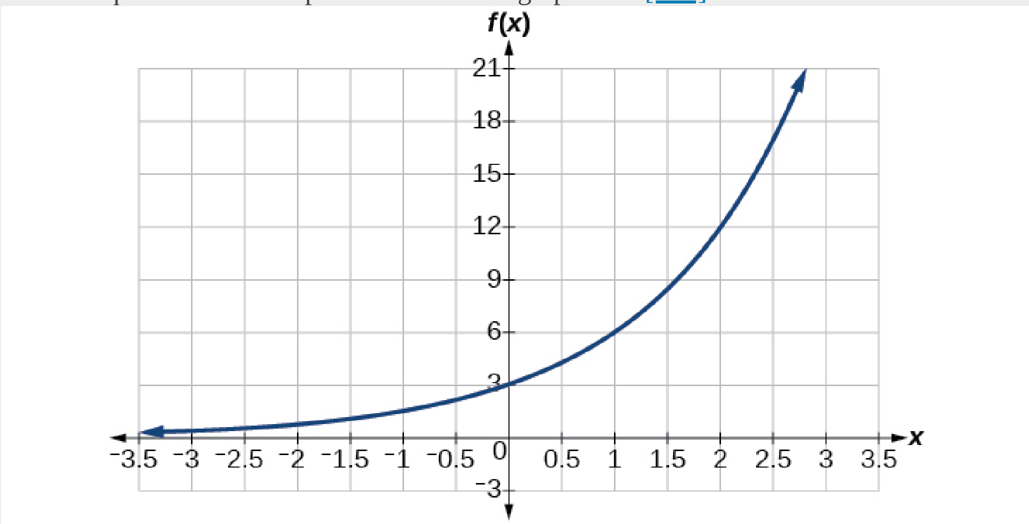
Given the graph of an exponential function, write its equation.

1. First, identify two points on the graph. Choose the *y*-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the *y*-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x) = a(b)^x$.

Example:
Exercise:

Problem:
Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in [link].



Solution:

We can choose the *y*-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

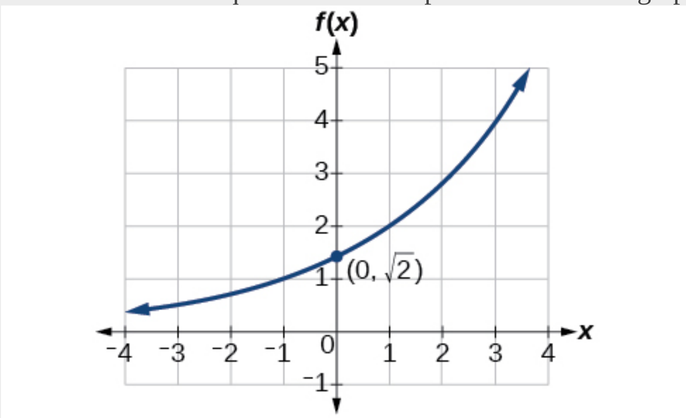
Equation:

$$\begin{array}{ll} y = ab^x & \text{Write the general form of an exponential equation.} \\ y = 3b^x & \text{Substitute the initial value 3 for } a. \\ 12 = 3b^2 & \text{Substitute in 12 for } y \text{ and 2 for } x. \\ 4 = b^2 & \text{Divide by 3.} \\ b = \pm 2 & \text{Take the square root.} \end{array}$$

Because we restrict ourselves to positive values of b, we will use $b = 2$. Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

Note:
Exercise:

Problem: Find an equation for the exponential function graphed in [link].



Solution:

$f(x) = \sqrt{2}\left(\sqrt{2}\right)^x$. Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

Note:
Given two points on the curve of an exponential function, use a graphing calculator to find the equation.

1. Press **[STAT].**
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
6. The screen displays the values of a and b in the exponential equation $y = a \cdot b^x$.

Example:
Exercise:

Problem:
Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2, 24.8)$ and $(5, 198.4)$.

Solution:

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit…],** and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the *x*-coordinates, 2 and 5. Do the same in the **L2** column for the *y*-coordinates, 24.8 and 198.4.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a = 6.2$ and $b = 2$ will be displayed. The exponential equation is $y = 6.2 \cdot 2^x$.

Note:
Exercise:

Problem:

Use a graphing calculator to find the exponential equation that includes the points (3, 75.98) and (6, 481.07).

Solution:

$y \approx 12 \cdot 1.85^x$

Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, APR r, and number of compounding periods in a year n :
Equation:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe [link], which shows the result of investing $1,000 at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Note:
The Compound Interest Formula
Compound interest can be calculated using the formula
Equation:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Example:
Exercise:

Problem:
Calculating Compound Interest

If we invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution:

Because we are starting with $3,000, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$,the value when $t = 10$.
Equation:

$$\begin{aligned} A(t) &= P\left(1 + \tfrac{r}{n}\right)^{nt} && \text{Use the compound interest formula.} \\ A(10) &= 3000\left(1 + \tfrac{0.03}{4}\right)^{4 \cdot 10} && \text{Substitute using given values.} \\ &\approx \$4045.05 && \text{Round to two decimal places.} \end{aligned}$$

The account will be worth about $4,045.05 in 10 years.

Note:
Exercise:

Problem:

An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Solution:

about $3,644,675.88

Example:
Exercise:

Problem:
Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution:

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth $40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.
Equation:

$$\begin{array}{rll} A(t) & = P\left(1+\frac{r}{n}\right)^{nt} & \text{Use the compound interest formula.} \\ 40{,}000 & = P\left(1+\frac{0.06}{2}\right)^{2(18)} & \text{Substitute using given values } A,\ r,\ n, \text{ and } t. \\ 40{,}000 & = P(1.03)^{36} & \text{Simplify.} \\ \frac{40{,}000}{(1.03)^{36}} & = P & \text{Isolate } P. \\ P & \approx \$13,801 & \text{Divide and round to the nearest dollar.} \end{array}$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

Note:
Exercise:

Problem:

Refer to [link]. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Solution:

$13,693

Evaluating Functions with Base *e*

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. [link] shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $1 invested at 100% interest for 1 year, compounded at various frequencies, listed in [link].

Frequency	$A(t) = \left(1+\frac{1}{n}\right)^n$	**Value**
Annually	$\left(1+\frac{1}{1}\right)^1$	$2
Semiannually	$\left(1+\frac{1}{2}\right)^2$	$2.25
Quarterly	$\left(1+\frac{1}{4}\right)^4$	$2.441406
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	$2.613035
Daily	$\left(1+\frac{1}{365}\right)^{365}$	$2.714567
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	$2.718127
Once per minute	$\left(1+\frac{1}{525600}\right)^{525600}$	$2.718279
Once per second	$\left(1+\frac{1}{31536000}\right)^{31536000}$	$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

Note:
The Number e
The letter e represents the irrational number
Equation:

$$\left(1+\frac{1}{n}\right)^n, \text{as } n \text{ increases without bound}$$

The letter e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example:
Exercise:

Problem:
Using a Calculator to Find Powers of *e*

Calculate $e^{3.14}$. Round to five decimal places.

Solution:

On a calculator, press the button labeled $[e^x]$. The window shows $[e\ \hat{}\ (\ ]$. Type 3.14 and then close parenthesis, $[)]$. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an “Exp” button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Note:
Exercise:

Problem: Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Solution:

$e^{-0.5} \approx 0.60653$

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called *continuous growth or decay models*. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Note:
The Continuous Growth/Decay Formula

For all real numbers t,and all positive numbers a and r,continuous growth or decay is represented by the formula

Equation:

$$A(t) = ae^{rt}$$

where

- a is the initial value,
- r is the continuous growth rate per unit time,
- and t is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

Equation:

$$A(t) = Pe^{rt}$$

where

- P is the principal or the initial invested,
- r is the growth or interest rate per unit time,
- and t is the period or term of the investment.

Note:

Given the initial value, rate of growth or decay, and time t, solve a continuous growth or decay function.

1. Use the information in the problem to determine a, the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
    a. If the problem refers to continuous growth, then $r > 0$.
    b. If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

Example:

Exercise:

Problem:

Calculating Continuous Growth

A person invested $1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution:

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $1,000, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

Equation:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula.} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P,\ r,\ \text{and } t. \\ &\approx 1105.17 && \text{Use a calculator to approximate.} \end{aligned}$$

The account is worth $1,105.17 after one year.

Note:
Exercise:

Problem:

A person invests $100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Solution:

$3,659,823.44

Example:
Exercise:

Problem:
Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Solution:

Since the substance is decaying, the rate, 17.3%, is negative. So, $r = -0.173$. The initial amount of radon-222 was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

Equation:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula.} \\ &= 100e^{-0.173(3)} && \text{Substitute known values for } a,\ r,\ \text{and } t. \\ &\approx 59.5115 && \text{Use a calculator to approximate.} \end{aligned}$$

So 59.5115 mg of radon-222 will remain.

Note:
Exercise:

Problem: Using the data in [link], how much radon-222 will remain after one year?

Solution:

3.77E-26 (This is calculator notation for the number written as 3.77×10^{-26} in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

Note:
Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

Key Equations

definition of the exponential function	$f(x) = b^x,\ \text{where}\ \ b > 0,\ b \neq 1$
definition of exponential growth	$f(x) = ab^x,\ \text{where}\ a > 0, b > 0, b \neq 1$
compound interest formula	$A(t) = P\left(1 + \frac{r}{n}\right)^{nt},\ \text{where}$ $A(t)$ is the account value at time t t is the number of years P is the initial investment, often called the principal r is the annual percentage rate (APR), or nominal rate n is the number of compounding periods in one year
continuous growth formula	$A(t) = ae^{rt},\ \text{where}$ $t$is the number of unit time periods of growth $a$is the starting amount (in the continuous compounding formula a is replaced with P, the principal) $e$is the mathematical constant, $e \approx 2.718282$

Key Concepts

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent. See [link].
- A function is evaluated by solving at a specific value. See [link] and [link].
- An exponential model can be found when the growth rate and initial value are known. See [link].
- An exponential model can be found when the two data points from the model are known. See [link].
- An exponential model can be found using two data points from the graph of the model. See [link].
- An exponential model can be found using two data points from the graph and a calculator. See [link].
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See [link].
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See [link].
- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282.$
- Scientific and graphing calculators have the key $[e^x]$ or $[\exp(x)]$ for calculating powers of $e.$ See [link].
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

Solution:

Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

Exercise:

Problem:

Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

Exercise:

Problem:

The Oxford Dictionary defines the word *nominal* as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.”[footnote] Develop a reasonable argument for why the term *nominal rate* is used to describe the annual percentage rate of an investment account that compounds interest.
Oxford Dictionary. http://oxforddictionaries.com/us/definition/american_english/nomina.

Solution:

When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not

necessarily correspond to the real interest earned, which is the very definition of *nominal*.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.
Exercise:

Problem: The average annual population increase of a pack of wolves is 25.

Exercise:

Problem: A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.

Solution:

exponential; the population decreases by a proportional rate. .

Exercise:

Problem: The value of a coin collection has increased by 3.25% annually over the last 20 years.

Exercise:

Problem:

For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

Solution:

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function. .

Exercise:

Problem: The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$.

For the following exercises, consider this scenario: For each year t,the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)
Exercise:

Problem: Which forest's population is growing at a faster rate?

Solution:

The forest represented by the function $B(t) = 82(1.029)^t$.

Exercise:

Problem: Which forest had a greater number of trees initially? By how many?

Exercise:

Problem:

Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

Solution:

After $t = 20$ years, forest A will have 43 more trees than forest B.

Exercise:

Problem:

Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?

Exercise:

Problem:

Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

Solution:

Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

Exercise:

Problem: $y = 300(1 - t)^5$

Exercise:

Problem: $y = 220(1.06)^x$

Solution:

exponential growth; The growth factor, 1.06, is greater than 1.

Exercise:

Problem: $y = 16.5(1.025)^{\frac{1}{x}}$

Exercise:

Problem: $y = 11,701(0.97)^t$

Solution:

exponential decay; The decay factor, 0.97, is between 0 and 1.

For the following exercises, find the formula for an exponential function that passes through the two points given.

Exercise:

Problem: $(0,6)$ and $(3,750)$

Exercise:

Problem: $(0,2000)$ and $(2,20)$

Solution:

$f(x)=2000(0.1)^x$

Exercise:

Problem: $\left(-1,\frac{3}{2}\right)$ and $(3,24)$

Exercise:

Problem: $(-2,6)$ and $(3,1)$

Solution:

$f(x)=\left(\frac{1}{6}\right)^{-\frac{3}{5}}\left(\frac{1}{6}\right)^{\frac{x}{5}}\approx 2.93(0.699)^x$

Exercise:

Problem: $(3,1)$ and $(5,4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

Exercise:

Problem:

x	1	2	3	4
$f(x)$	70	40	10	-20

Solution:

Linear

Exercise:

Problem:

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

Exercise:

Problem:

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

Solution:

Neither

Exercise:

Problem:

x	1	2	3	4
$f(x)$	10	20	40	80

Exercise:

Problem:

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

Solution:

Linear

For the following exercises, use the compound interest formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

Exercise:

Problem:

After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?

Exercise:

Problem: What was the initial deposit made to the account in the previous exercise?

Solution:

$\$10,250$

Exercise:

Problem: How many years had the account from the previous exercise been accumulating interest?

Exercise:

Problem:

An account is opened with an initial deposit of $6,500 and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?

Solution:

$\$13,268.58$

Exercise:

Problem:

How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

Exercise:

Problem: Solve the compound interest formula for the principal, P.

Solution:

$P = A(t) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$

Exercise:

Problem:

Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)

Exercise:

Problem:

How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

Solution:

$\$4{,}572.56$

Exercise:

Problem:

Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

Exercise:

Problem:

Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $9,000 and was worth $13,373.53 after 10 years.

Solution:

4%

Exercise:

Problem:

Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $5,500, and was worth $38,455 after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

Exercise:

Problem: $y = 3742(e)^{0.75t}$

Solution:

continuous growth; the growth rate is greater than 0.

Exercise:

Problem: $y = 150(e)^{\frac{3.25}{t}}$

Exercise:

Problem: $y = 2.25(e)^{-2t}$

Solution:

continuous decay; the growth rate is less than 0.

Exercise:

Problem:

Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?

Exercise:

Problem:

How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

Solution:

$\$669.42$

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

Exercise:

Problem: $f(x) = 2(5)^x$, for $f(-3)$

Exercise:

Problem: $f(x) = -4^{2x+3}$, for $f(-1)$

Solution:

$f(-1) = -4$

Exercise:

Problem: $f(x) = e^x$, for $f(3)$

Exercise:

Problem: $f(x) = -2e^{x-1}$, for $f(-1)$

Solution:

$f(-1) \approx -0.2707$

Exercise:

Problem: $f(x)=2.7(4)^{-x+1}+1.5$, for $f(-2)$

Exercise:

Problem: $f(x)=1.2e^{2x}-0.3$, for $f(3)$

Solution:

$f(3)\approx 483.8146$

Exercise:

Problem: $f(x)=-\frac{3}{2}(3)^{-x}+\frac{3}{2}$, for $f(2)$

Technology

For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.

Exercise:

Problem: $(0,3)$ and $(3,375)$

Solution:

$y=3\cdot 5^x$

Exercise:

Problem: $(3,222.62)$ and $(10,77.456)$

Exercise:

Problem: $(20,29.495)$ and $(150,730.89)$

Solution:

$y\approx 18\cdot 1.025^x$

Exercise:

Problem: $(5,2.909)$ and $(13,0.005)$

Exercise:

Problem: $(11{,}310.035)$ and $(25{,}356.3652)$

Solution:

$y\approx 0.2\cdot 1.95^x$

Extensions

Exercise:

Problem:

The *annual percentage yield* (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $\text{APY} = \left(1 + \frac{r}{12}\right)^{12} - 1.$

Exercise:

Problem:

Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.

Solution:

$\text{APY} = \frac{A(t)-a}{a} = \frac{a\left(1+\frac{r}{365}\right)^{365(1)}-a}{a} = \frac{a\left[\left(1+\frac{r}{365}\right)^{365}-1\right]}{a} = \left(1 + \frac{r}{365}\right)^{365} - 1; I(n) = \left(1 + \frac{r}{n}\right)^{n} - 1$

Exercise:

Problem:

Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1$. Any positive number b can be written as $b = e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.

Exercise:

Problem:

In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = a(e)^{-nx}$ for some positive number n .

Solution:

Let f be the exponential decay function $f(x) = a \cdot \left(\frac{1}{b}\right)^x$ such that $b > 1$. Then for some number $n > 0$, $f(x) = a \cdot \left(\frac{1}{b}\right)^x = a\left(b^{-1}\right)^x = a\left(\left(e^n\right)^{-1}\right)^x = a(e^{-n})^x = a(e)^{-nx}$.

Exercise:

Problem:

The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t) = a(e)^{rt}$,where a is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time t can be calculated with the formula $I(t) = e^{rt} - 1$.

Real-World Applications

Exercise:

Problem:

The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were 23,900 fox counted in the area. What is the fox population predicted to be in the year 2020?

Solution:

$47,622$ fox

Exercise:

Problem:

A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

Exercise:

Problem:

In the year 1985, a house was valued at $110,000. By the year 2005, the value had appreciated to $145,000. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

Solution:

1.39%; $\$155,368.09$

Exercise:

Problem:

A car was valued at $38,000 in the year 2007. By 2013, the value had depreciated to $11,000 If the car's value continues to drop by the same percentage, what will it be worth by 2017?

Exercise:

Problem:

Jamal wants to save $54,000 for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

Solution:

$\$35,838.76$

Exercise:

Problem:

Kyoko has $10,000 that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $15,000 by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (*Hint*: solve the compound interest formula for the interest rate.)

Exercise:

Problem:

Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

Solution:

$\$82,247.78;\ \$449.75$

Exercise:

Problem:

An investment account with an annual interest rate of 7% was opened with an initial deposit of $4,000 Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

Glossary

annual percentage rate (APR)
: the yearly interest rate earned by an investment account, also called *nominal rate*

compound interest
: interest earned on the total balance, not just the principal

exponential growth
: a model that grows by a rate proportional to the amount present

nominal rate
: the yearly interest rate earned by an investment account, also called *annual percentage rate*

Graphs of Exponential Functions

- Graph exponential functions.
- Graph exponential functions using transformations.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x)=b^x$ whose base is greater than one. We'll use the function $f(x)=2^x$. Observe how the output values in [link] change as the input increases by 1.

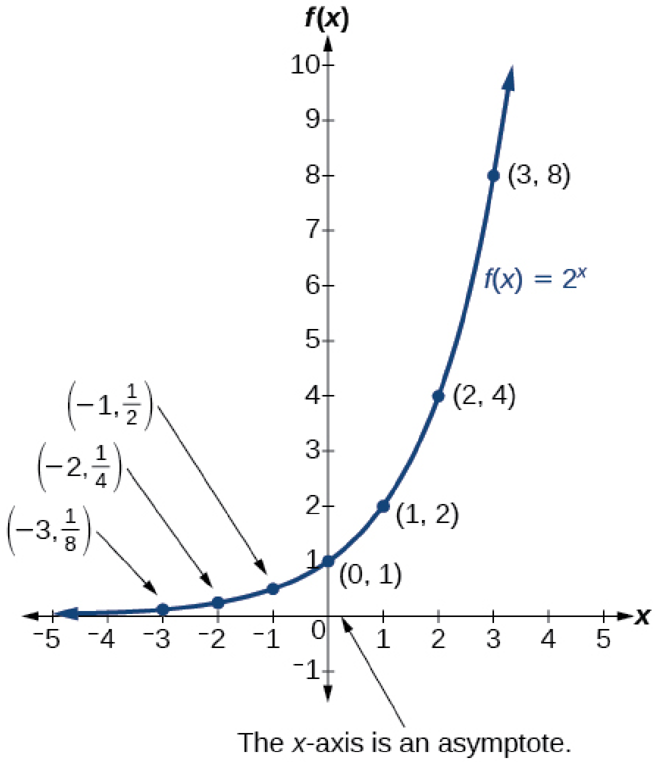
x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x)=ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

[link] shows the exponential growth function $f(x)=2^x$.



Notice that the graph gets close to the *x*-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in [link] change as the input increases by 1.

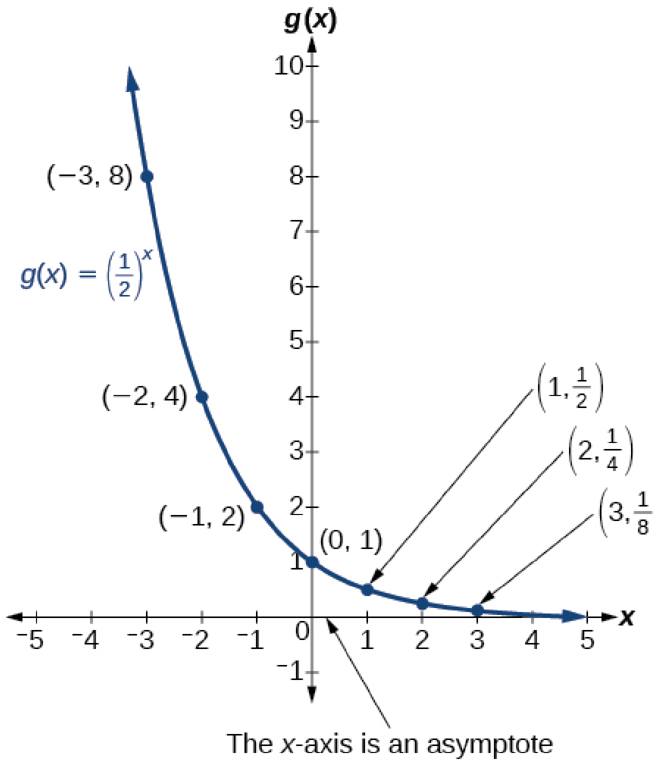
x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

[link] shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

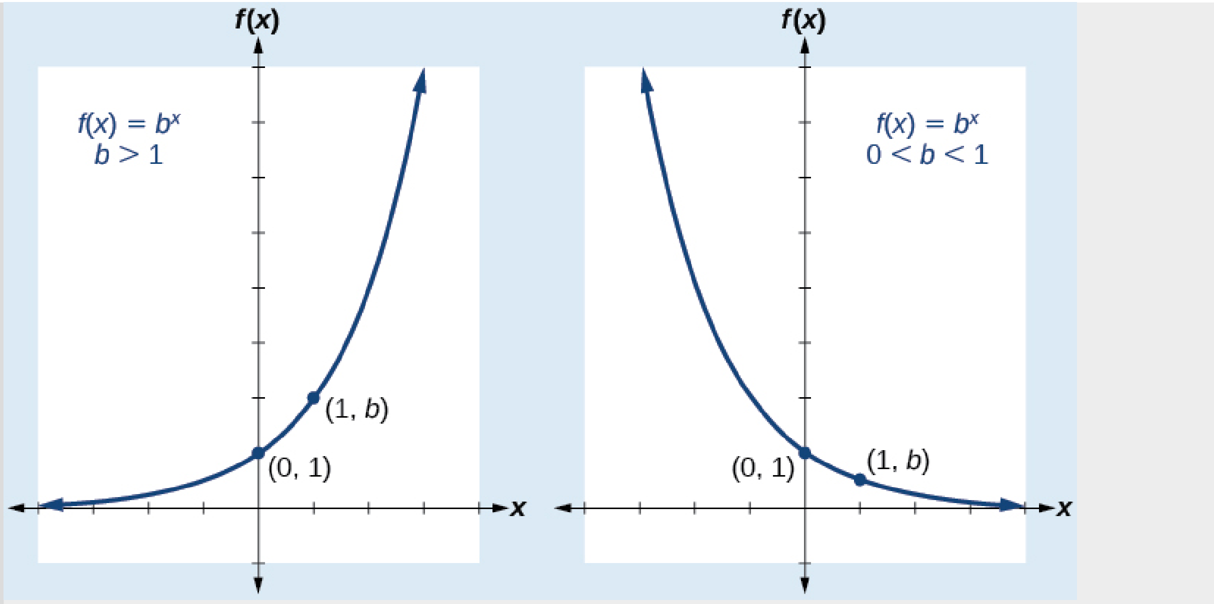


The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$,and the horizontal asymptote is $y = 0$.

Note:
Characteristics of the Graph of the Parent Function $f(x) = b^x$
An exponential function with the form $f(x) = b^x, b > 0, b \neq 1$,has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \ \infty)$
- range: $(0, \infty)$
- *x*-intercept: none
- *y*-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

[link] compares the graphs of exponential growth and decay functions.



Note:
Given an exponential function of the form $f(x) = b^x$,graph the function.

1. Create a table of points.
2. Plot at least 3 point from the table, including the *y*-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $(-\infty, \infty)$,the range, $(0, \infty)$,and the horizontal asymptote, $y = 0$.

Example:
Exercise:

Problem:
Sketching the Graph of an Exponential Function of the Form *f*(*x*) = *b*^*x*

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Solution:

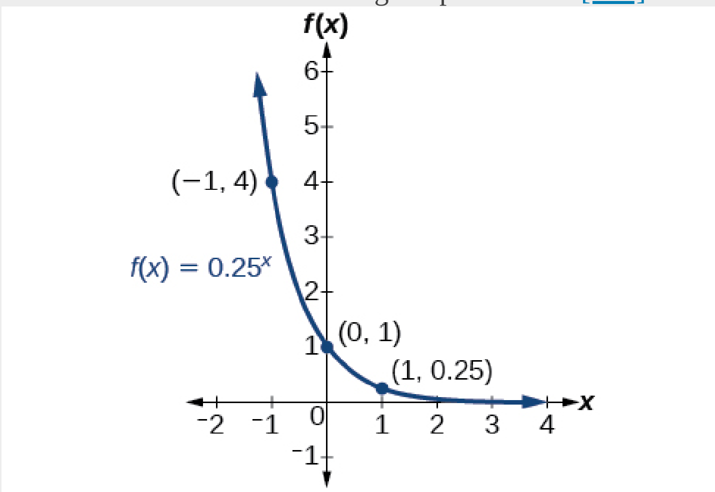
Before graphing, identify the behavior and create a table of points for the graph.

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in [link].

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	0.0625	0.015625

- Plot the *y*-intercept, $(0, 1)$,along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in [link].



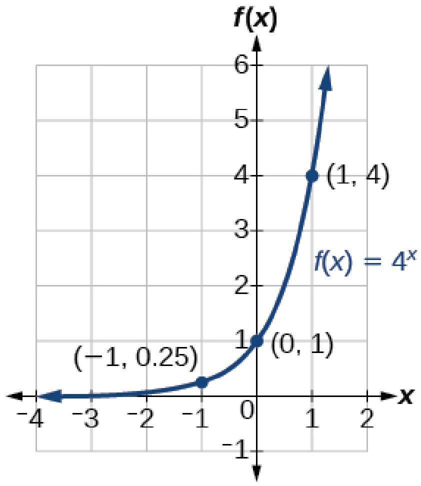
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Note:
Exercise:

Problem: Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Solution:

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

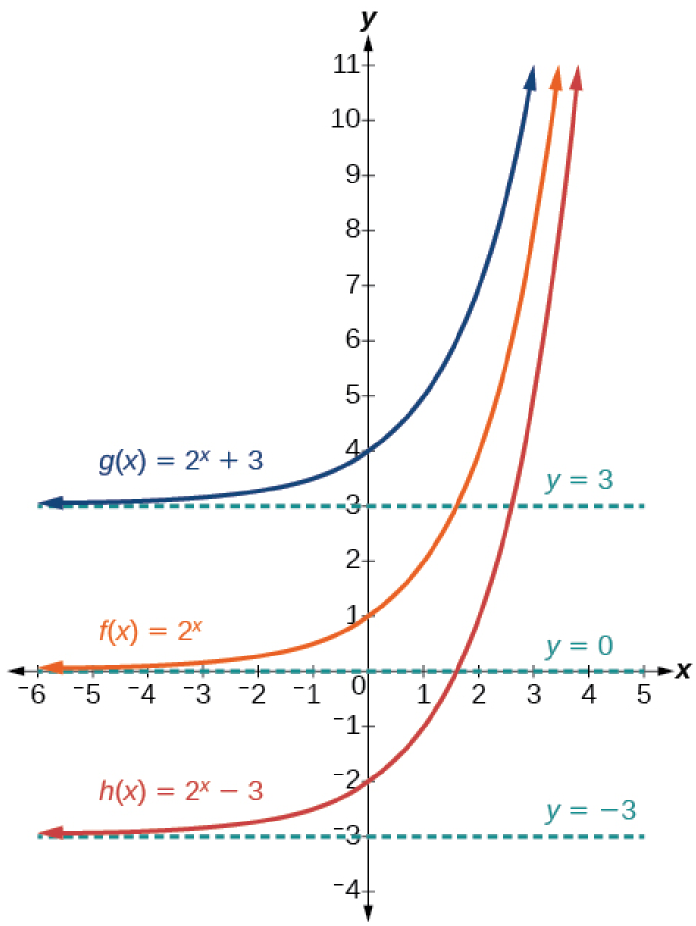


Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x)=b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the parent function $f(x)=b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x)=2^x$, we can then graph two vertical shifts alongside it, using $d=3:$ the upward shift, $g(x)=2^x+3$ and the downward shift, $h(x)=2^x-3$. Both vertical shifts are shown in [link].



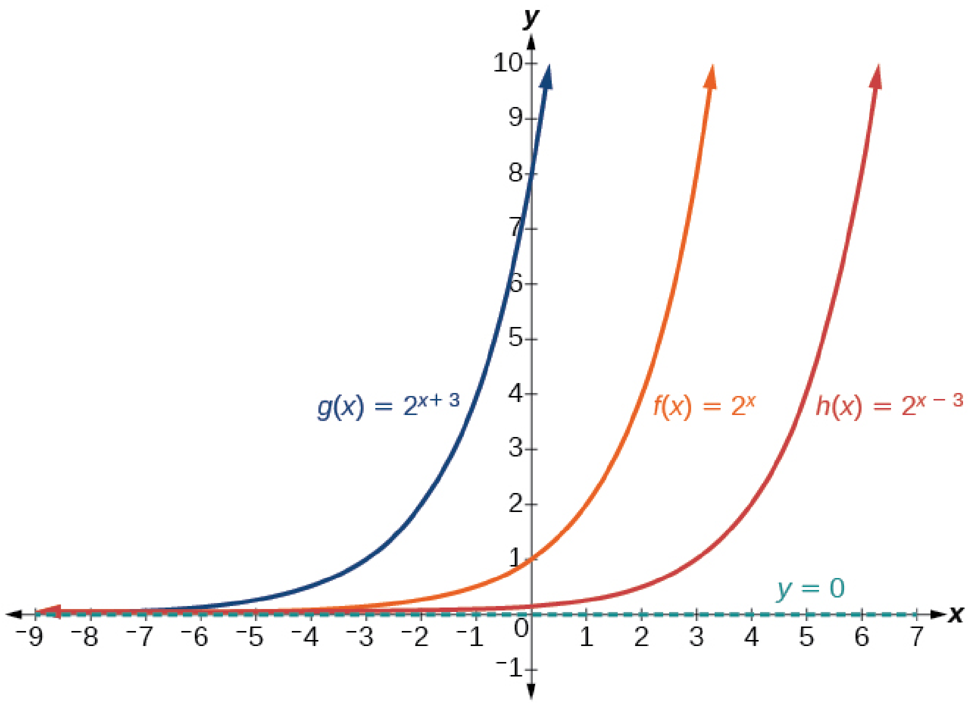
Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
    - The *y*-intercept shifts up 3 units to $(0, 4)$.
    - The asymptote shifts up 3 units to $y = 3$.
    - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
    - The *y*-intercept shifts down 3 units to $(0, -2)$.
    - The asymptote also shifts down 3 units to $y = -3$.
    - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the parent function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using

$c=3:$ the shift left, $g(x)=2^{x+3}$, and the shift right, $h(x)=2^{x-3}$. Both horizontal shifts are shown in [link].



Observe the results of shifting $f(x)=2^x$ horizontally:

- The domain, $(-\infty,\infty)$,remains unchanged.
- The asymptote, $y=0$,remains unchanged.
- The *y*-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x)=2^{x+3}$,the *y*-intercept becomes $(0,8)$. This is because $2^{x+3}=(8)2^x$,so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x)=2^{x-3}$,the *y*-intercept becomes $\left(0,\frac{1}{8}\right)$. Again, see that $2^{x-3}=\left(\frac{1}{8}\right)2^x$,so the initial value of the function is $\frac{1}{8}$.

Note:
Shifts of the Parent Function *f*(*x*) = *b*^{x}
For any constants c and d,the function $f(x)=b^{x+c}+d$ shifts the parent function $f(x)=b^x$

- vertically d units, in the *same* direction of the sign of d.
- horizontally c units, in the *opposite* direction of the sign of c.
- The *y*-intercept becomes $(0,b^c+d)$.
- The horizontal asymptote becomes $y=d$.
- The range becomes (d,∞).
- The domain, $(-\infty,\infty)$,remains unchanged.

Note:
Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation.

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $(-\infty, \infty)$, the range, (d, ∞), and the horizontal asymptote $y = d$.

Example:
Exercise:

Problem:
Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

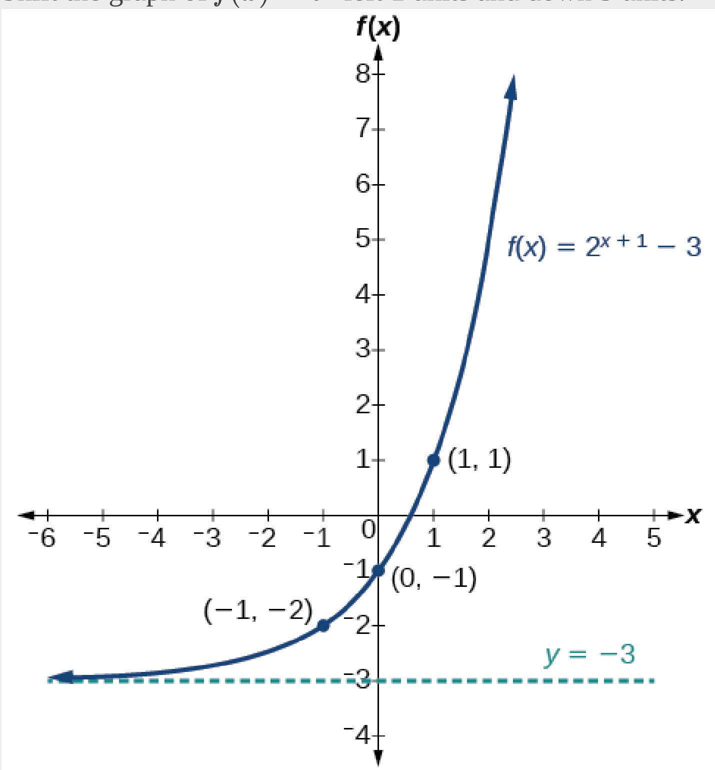
Solution:

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b^x$ left 1 units and down 3 units.



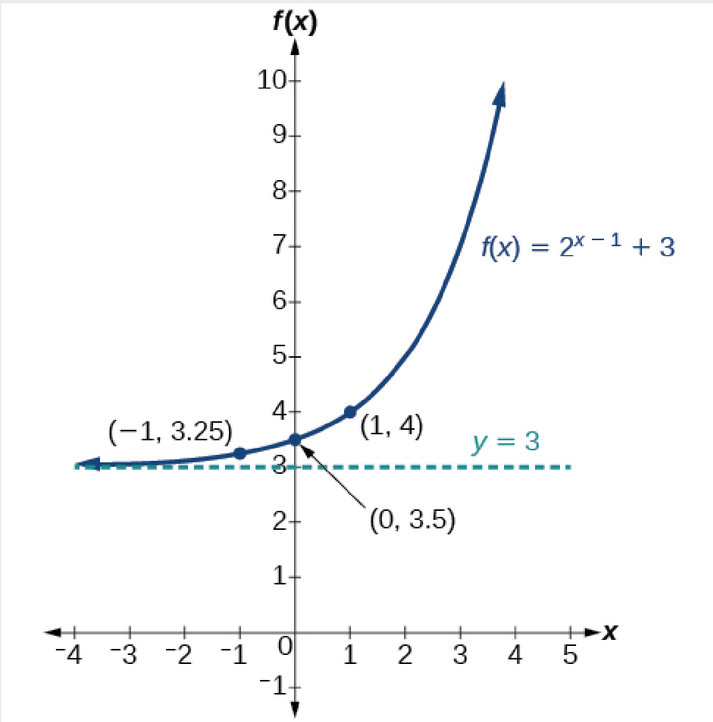
The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

Note:
Exercise:

Problem: Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Solution:

The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.



Note:
Given an equation of the form $f(x) = b^{x+c} + d$ for x, use a graphing calculator to approximate the solution.

- Press **[Y=]**. Enter the given exponential equation in the line headed "**Y_1=**".
- Enter the given value for $f(x)$ in the line headed "**Y_2=**".
- Press **[WINDOW]**. Adjust the *y*-axis so that it includes the value entered for "**Y_2=**".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value of $f(x)$.
- To find the value of x,we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of *x* for the indicated value of the function.

Example:
Exercise:

Problem:
Approximating the Solution of an Exponential Equation

Solve $42 = 1.2(5)^x + 2.8$ graphically. Round to the nearest thousandth.

Solution:

Press **[Y=]** and enter $1.2(5)^x + 2.8$ next to **Y$_1$**=. Then enter 42 next to **Y2=**. For a window, use the values –3 to 3 for x and –5 to 55 for y. Press **[GRAPH]**. The graphs should intersect somewhere near $x = 2.$

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 2.1661943. (Your answer may be different if you use a different window or use a different value for **Guess?**) To the nearest thousandth, $x \approx 2.166.$

Note:
Exercise:

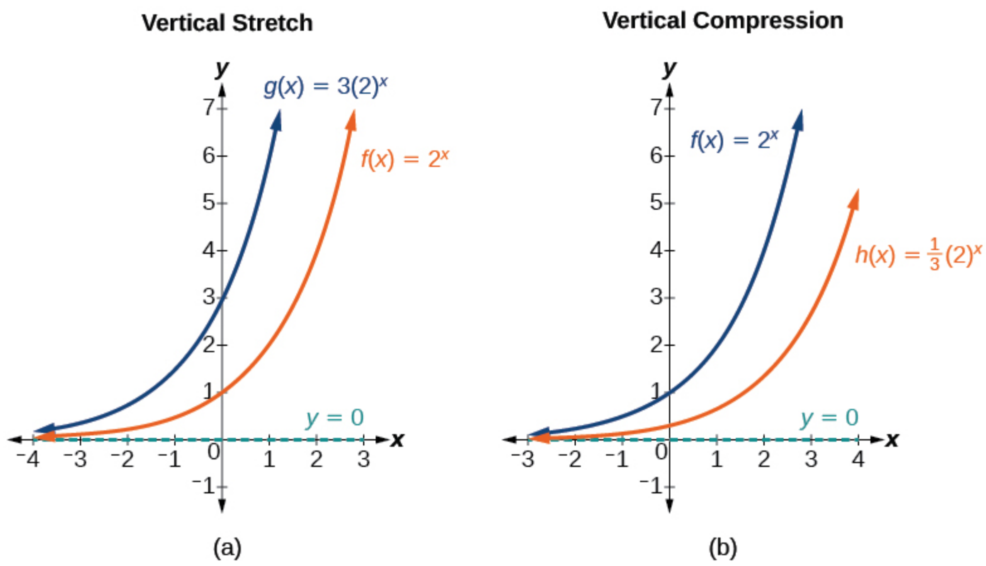
Problem: Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

Solution:

$x \approx -1.608$

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the parent function $f(x) = b^x$ by a constant $|a| > 0.$ For example, if we begin by graphing the parent function $f(x) = 2^x$,we can then graph the stretch, using $a = 3$,to get $g(x) = 3(2)^x$ as shown on the left in [link], and the compression, using $a = \frac{1}{3}$,to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in [link].



(a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

Note:
Stretches and Compressions of the Parent Function $f(x) = b^x$
For any factor $a > 0$,the function $f(x) = a(b)^x$

- is stretched vertically by a factor of a if $|a| > 1$.
- is compressed vertically by a factor of a if $|a| < 1$.
- has a y-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$,which are unchanged from the parent function.

Example:
Exercise:

Problem: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution:

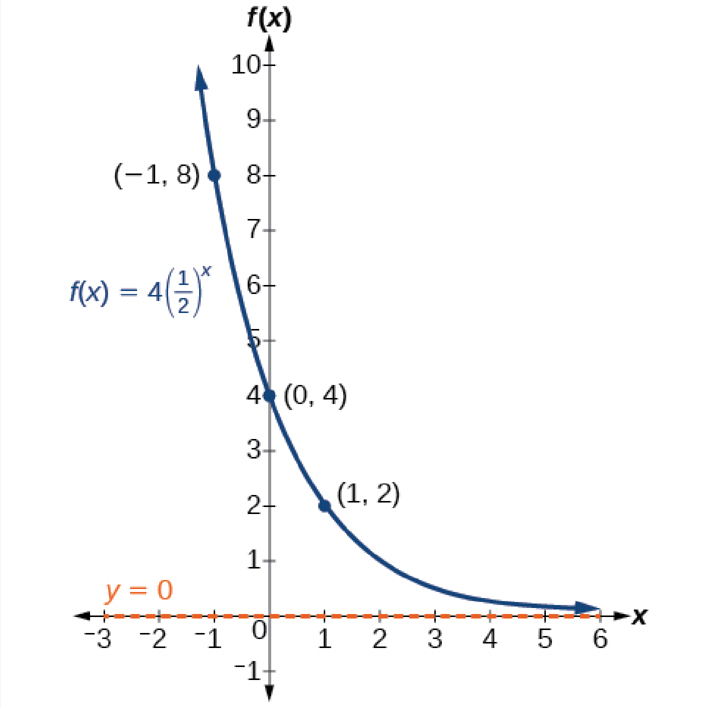
Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the *x*-axis as x increases.
- Since $a = 4$,the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in [link].

x	-3	-2	-1	0	1	2	3
$f(x) = 4\left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

- Plot the *y*-intercept, $(0, 4)$,along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in [link].



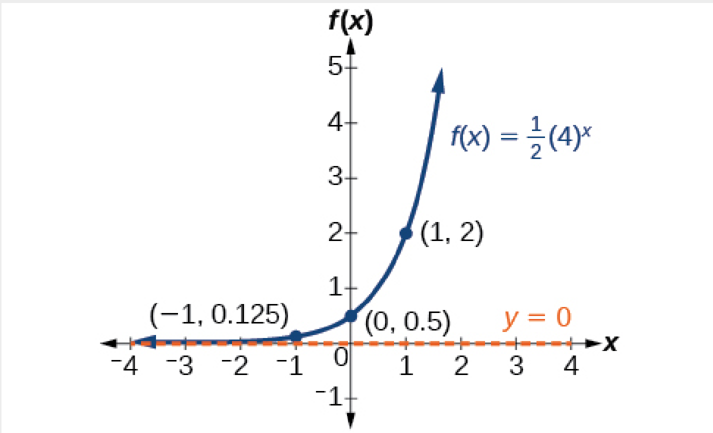
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Note:
Exercise:

Problem: Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

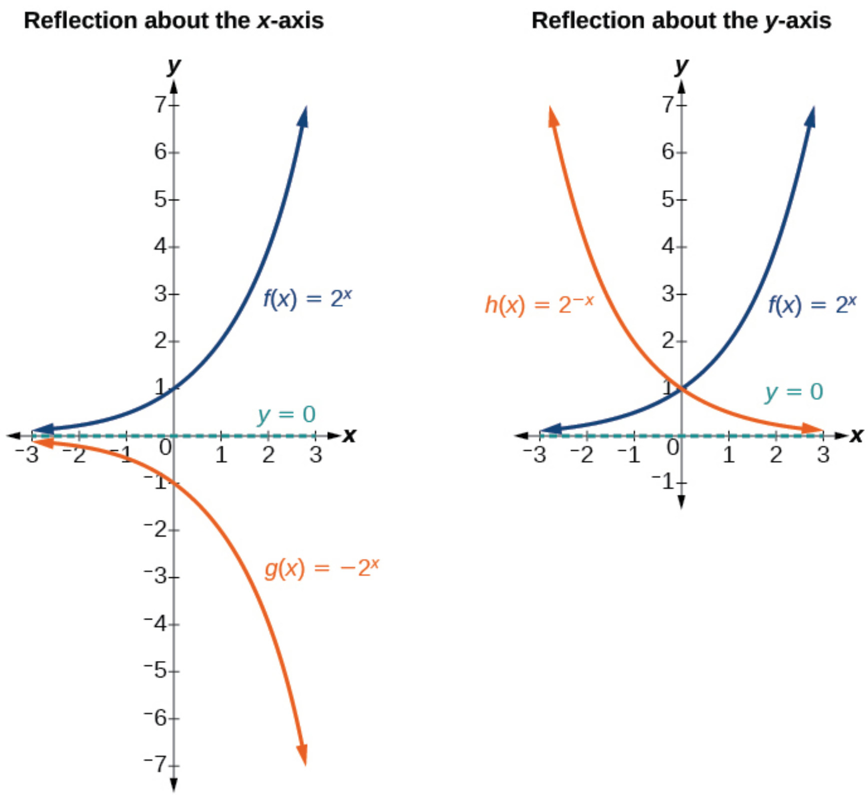
Solution:

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the *y*-axis. When we multiply the parent function $f(x) = b^x$ by -1,we get a reflection about the *x*-axis. When we multiply the input by -1,we get a reflection about the *y*-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the *x*-axis, $g(x) = -2^x$,is shown on the left side of [link], and the reflection about the *y*-axis $h(x) = 2^{-x}$, is shown on the right side of [link].



(a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the *y*-axis.

Note:
Reflections of the Parent Function *f*(*x*) = b^x
The function $f(x) = -b^x$

- reflects the parent function $f(x) = b^x$ about the *x*-axis.
- has a *y*-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$.
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$,which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the *y*-axis.
- has a *y*-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

Example:
Exercise:

Problem:
Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$,that reflects $f(x)=\left(\frac{1}{4}\right)^x$ about the *x*-axis. State its domain, range, and asymptote.

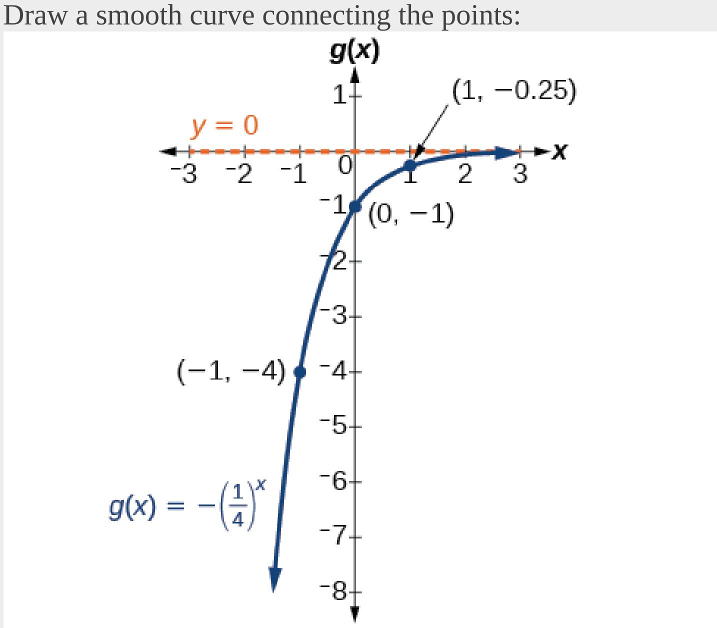
Solution:

Since we want to reflect the parent function $f(x)=\left(\frac{1}{4}\right)^x$ about the *x*-axis, we multiply $f(x)$ by -1 to get, $g(x)=-\left(\frac{1}{4}\right)^x$. Next we create a table of points as in [link].

x	-3	-2	-1	0	1	2	3
$g(x)=-\left(\frac{1}{4}\right)^x$	-64	-16	-4	-1	-0.25	-0.0625	-0.0156

Plot the *y*-intercept, $(0,-1)$,along with two other points. We can use $(-1,-4)$ and $(1,-0.25)$.

Draw a smooth curve connecting the points:



The domain is $(-\infty,\infty)$; the range is $(-\infty,0)$; the horizontal asymptote is $y=0$.

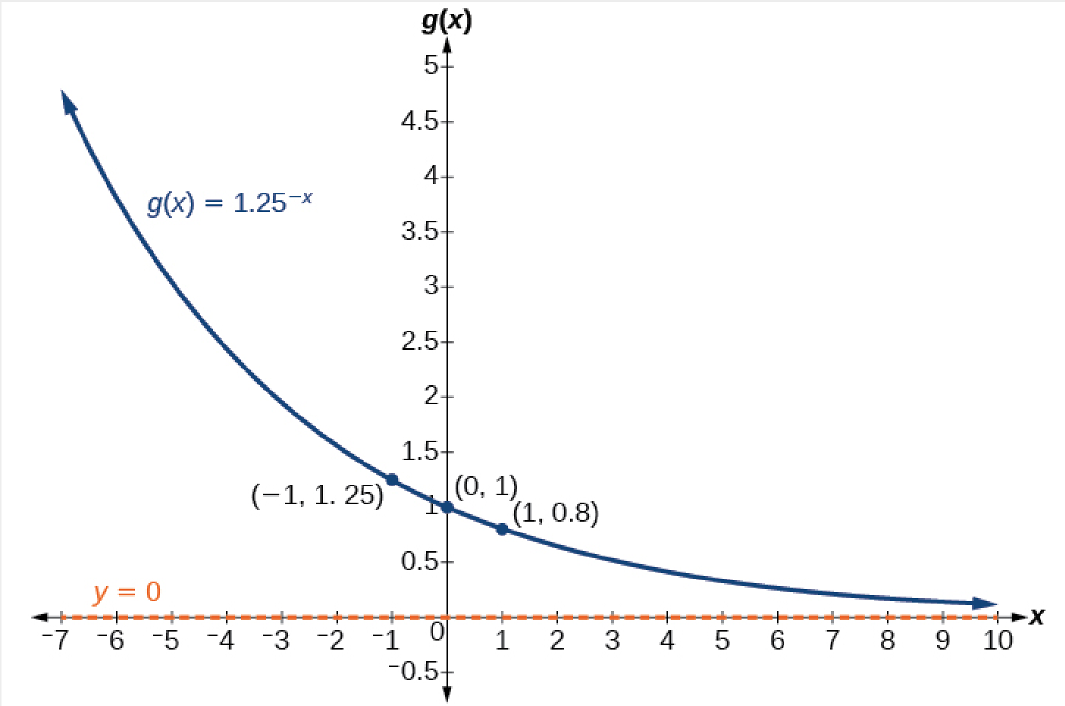
Note:
Exercise:

Problem:

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the *y*-axis. State its domain, range, and asymptote.

Solution:

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in [link] to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $f(x) = b^x$	
Translation	**Form**

Translations of the Parent Function $f(x) = b^x$	
Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$f(x) = b^{x+c} + d$
Stretch and Compress • Stretch if $\lvert a\rvert > 1$ • Compression if $0 < \lvert a\rvert < 1$	$f(x) = ab^x$
Reflect about the *x*-axis	$f(x) = -b^x$
Reflect about the *y*-axis	$f(x) = b^{-x} = \left(\frac{1}{b}\right)^x$
General equation for all translations	$f(x) = ab^{x+c} + d$

Note:
Translations of Exponential Functions
A translation of an exponential function has the form
Equation:

$$f(x) = ab^{x+c} + d$$

Where the parent function, $y = b^x$, $b > 1$,is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- shifted vertically d units.

- reflected about the *x*-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

Example:
Exercise:

Problem: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the *y*-axis, and then shifted up 4 units.

Solution:

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2, so $a = 2$.
- The function is reflected about the *y*-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,
Equation:

$$\begin{aligned} f(x) &= ab^{x+c} + d \\ &= 2e^{-x+0} + 4 \\ &= 2e^{-x} + 4 \end{aligned}$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

Note:
Exercise:

Problem:

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down 2 units.

Solution:

$f(x)=-\frac{1}{3}e^{x}-2$; the domain is $(-\infty,\infty)$; the range is $(-\infty,2)$; the horizontal asymptote is $y=2$.

Note:
Access this online resource for additional instruction and practice with graphing exponential functions.

- [Graph Exponential Functions]

Key Equations

General Form for the Translation of the Parent Function $f(x)=b^x$	$f(x)=ab^{x+c}+d$

Key Concepts

- The graph of the function $f(x)=b^x$ has a y-intercept at $(0,\ 1)$,domain $(-\infty,\ \infty)$,range $(0,\ \infty)$, and horizontal asymptote $y=0$. See [link].
- If $b>1$,the function is increasing. The left tail of the graph will approach the asymptote $y=0$, and the right tail will increase without bound.
- If $0<b<1$, the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y=0$.
- The equation $f(x)=b^x+d$ represents a vertical shift of the parent function $f(x)=b^x$.
- The equation $f(x)=b^{x+c}$ represents a horizontal shift of the parent function $f(x)=b^x$. See [link].
- Approximate solutions of the equation $f(x)=b^{x+c}+d$ can be found using a graphing calculator. See [link].
- The equation $f(x)=ab^x$, where $a>0$, represents a vertical stretch if $|a|>1$ or compression if $0<|a|<1$ of the parent function $f(x)=b^x$. See [link].
- When the parent function $f(x)=b^x$ is multiplied by -1,the result, $f(x)=-b^x$, is a reflection about the x-axis. When the input is multiplied by -1,the result, $f(x)=b^{-x}$, is a reflection about the y-axis. See [link].
- All translations of the exponential function can be summarized by the general equation $f(x)=ab^{x+c}+d$. See [link].
- Using the general equation $f(x)=ab^{x+c}+d$, we can write the equation of a function given its description. See [link].

Section Exercises

Verbal

Exercise:

Problem:

What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

Solution:

An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

Exercise:

Problem:

What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Algebraic

Exercise:

Problem:

The graph of $f(x)=3^x$ is reflected about the *y*-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

Solution:

$g(x)=4(3)^{-x}$; *y*-intercept: $(0,4)$; Domain: all real numbers; Range: all real numbers greater than 0.

Exercise:

Problem:

The graph of $f(x)=\left(\frac{1}{2}\right)^{-x}$ is reflected about the *y*-axis and compressed vertically by a factor of $\frac{1}{5}$. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

Exercise:

Problem:

The graph of $f(x)=10^x$ is reflected about the *x*-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

Solution:

$g(x)=-10^x+7$; *y*-intercept: $(0,6)$; Domain: all real numbers; Range: all real numbers less than 7.

Exercise:

Problem:

The graph of $f(x) = (1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2,reflected about the *x*-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its *y*-intercept (to the nearest thousandth), domain, and range.

Exercise:

Problem:

The graph of $f(x) = 2\left(\frac{1}{4}\right)^{x-20}$ is shifted left 2 units, stretched vertically by a factor of 4,reflected about the *x*-axis, and then shifted downward 4 units. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

Solution:

$g(x) = 2\left(\frac{1}{4}\right)^x;$ *y*-intercept: $(0,\ 2);$ Domain: all real numbers; Range: all real numbers greater than 0.

Graphical

For the following exercises, graph the function and its reflection about the *y*-axis on the same axes, and give the *y*-intercept.

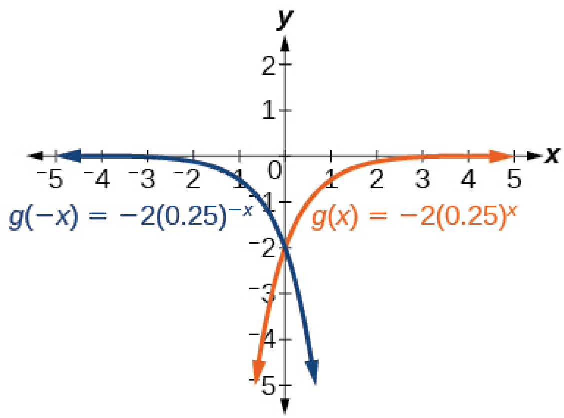
Exercise:

Problem: $f(x) = 3\left(\frac{1}{2}\right)^x$

Exercise:

Problem: $g(x) = -2(0.25)^x$

Solution:



y-intercept: $(0, -2)$

Exercise:

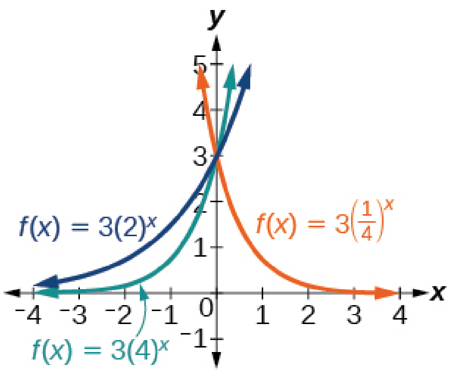
Problem: $h(x)=6(1.75)^{-x}$

For the following exercises, graph each set of functions on the same axes.

Exercise:

Problem: $f(x)=3\left(\frac{1}{4}\right)^{x}$,$g(x)=3(2)^{x}$,and $h(x)=3(4)^{x}$

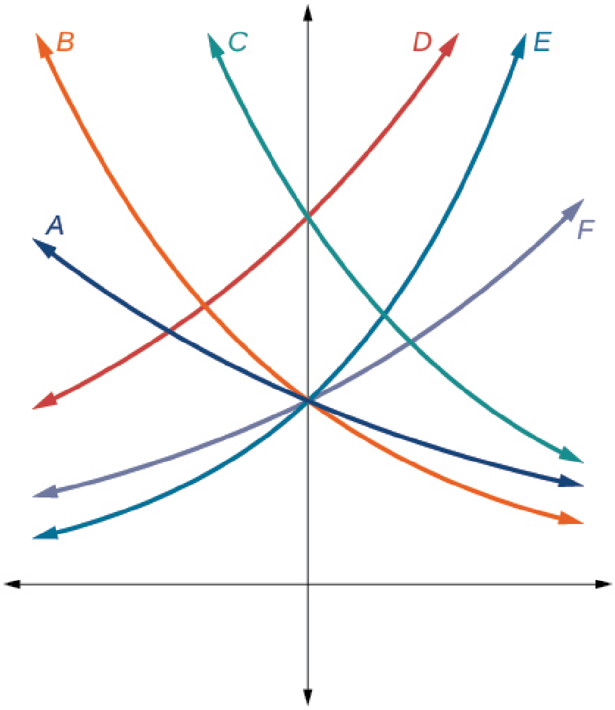
Solution:



Exercise:

Problem: $f(x)=\frac{1}{4}(3)^{x}$,$g(x)=2(3)^{x}$,and $h(x)=4(3)^{x}$

For the following exercises, match each function with one of the graphs in [link].



Exercise:

Problem: $f(x) = 2(0.69)^x$

Solution:

B

Exercise:

Problem: $f(x) = 2(1.28)^x$

Exercise:

Problem: $f(x) = 2(0.81)^x$

Solution:

A

Exercise:

Problem: $f(x) = 4(1.28)^x$

Exercise:

Problem: $f(x) = 2(1.59)^x$

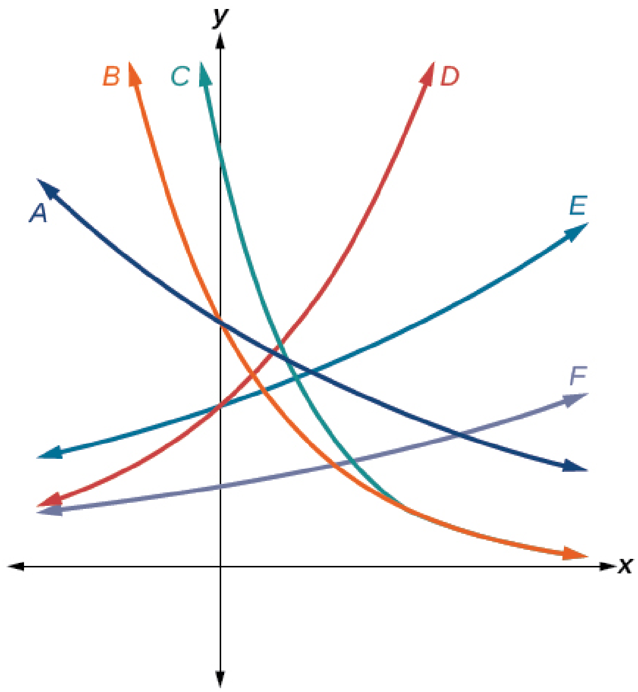
Solution:

E

Exercise:

Problem: $f\left(x\right)=4{\left(0.69\right)}^{x}$

For the following exercises, use the graphs shown in [link]. All have the form $f\left(x\right)=a{b}^{x}$.



Exercise:

Problem: Which graph has the largest value for b?

Solution:

D

Exercise:

Problem: Which graph has the smallest value for b?

Exercise:

Problem: Which graph has the largest value for a?

Solution:

C

Exercise:

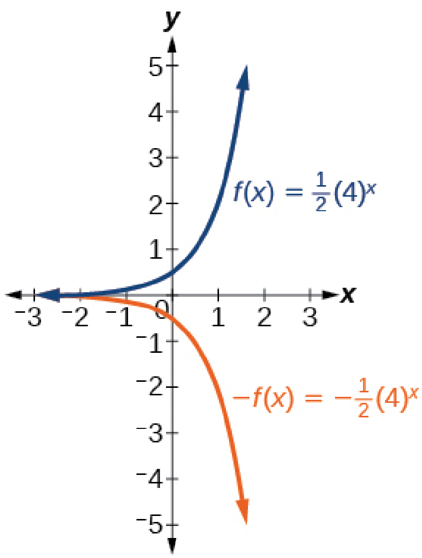
Problem: Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the *x*-axis on the same axes.

Exercise:

Problem: $f(x) = \frac{1}{2}(4)^x$

Solution:



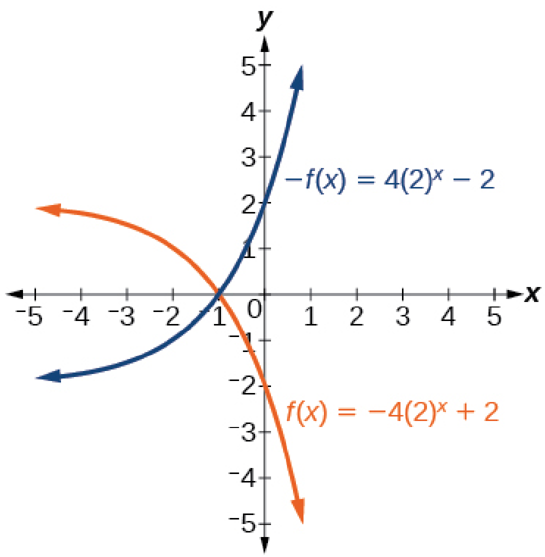
Exercise:

Problem: $f(x) = 3(0.75)^x - 1$

Exercise:

Problem: $f(x) = -4(2)^x + 2$

Solution:



For the following exercises, graph the transformation of $f(x) = 2^x$. Give the horizontal asymptote, the domain, and the range.

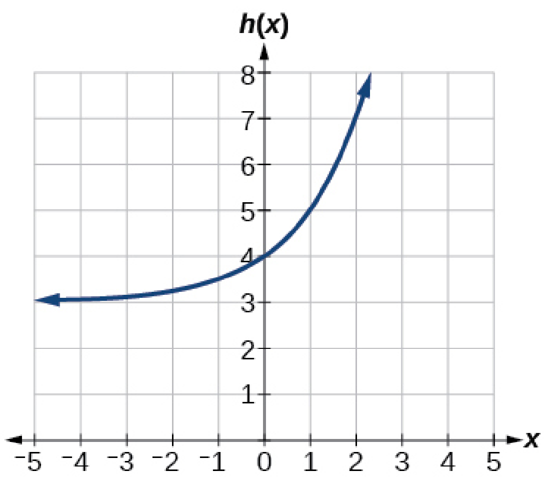
Exercise:

Problem: $f\left(x\right) = 2^{-x}$

Exercise:

Problem: $h\left(x\right) = 2^x + 3$

Solution:



Horizontal asymptote: $h(x) = 3;$ Domain: all real numbers; Range: all real numbers strictly greater than $3.$

Exercise:

Problem: $f\left(x\right)=2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.
Exercise:

Problem: $f\left(x\right)=-5(4)^{x}-1$

Solution:

As $x\to\infty$, $f\left(x\right)\to-\infty$;
As $x\to-\infty$, $f\left(x\right)\to-1$

Exercise:

Problem: $f\left(x\right)=3\left(\frac{1}{2}\right)^{x}-2$

Exercise:

Problem: $f\left(x\right)=3(4)^{-x}+2$

Solution:

As $x\to\infty$, $f\left(x\right)\to2$;
As $x\to-\infty$, $f\left(x\right)\to\infty$

For the following exercises, start with the graph of $f\left(x\right)=4^{x}$. Then write a function that results from the given transformation.
Exercise:

Problem: Shift $f(x)$ 4 units upward

Exercise:

Problem: Shift $f(x)$ 3 units downward

Solution:

$f\left(x\right)=4^{x}-3$

Exercise:

Problem: Shift $f(x)$ 2 units left

Exercise:

Problem: Shift $f(x)$ 5 units right

Solution:

$f(x)=4^{x-5}$

Exercise:

Problem: Reflect $f(x)$ about the *x*-axis

Exercise:

Problem: Reflect $f(x)$ about the *y*-axis

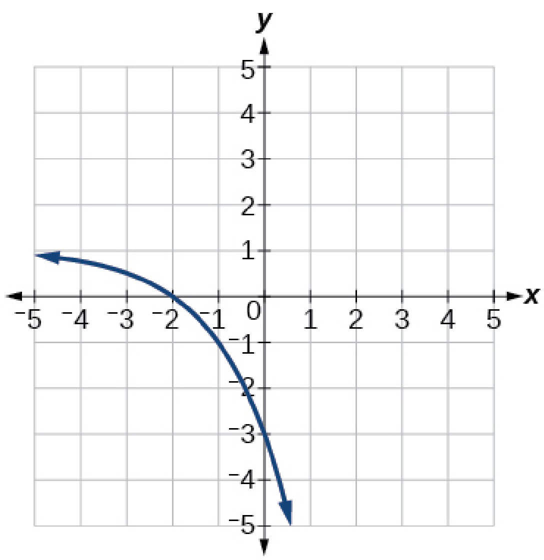
Solution:

$f\left(x\right)=4^{-x}$

For the following exercises, each graph is a transformation of $y=2^{x}.$ Write an equation describing the transformation.

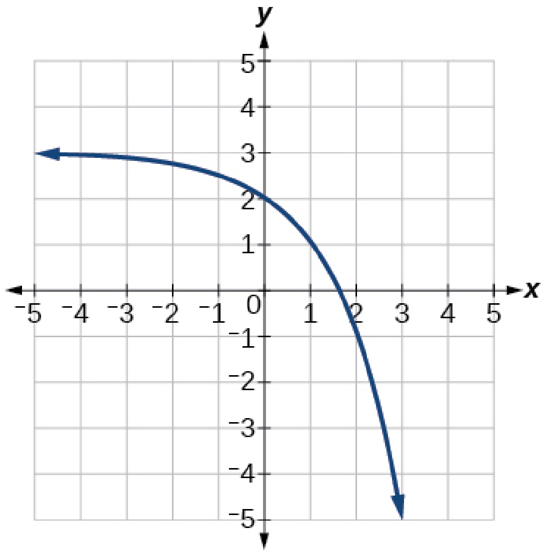
Exercise:

Problem:



Exercise:

Problem:

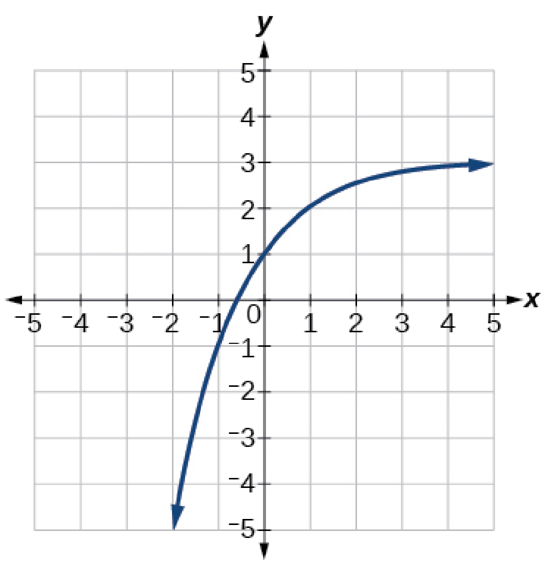


Solution:

$y = -2^x + 3$

Exercise:

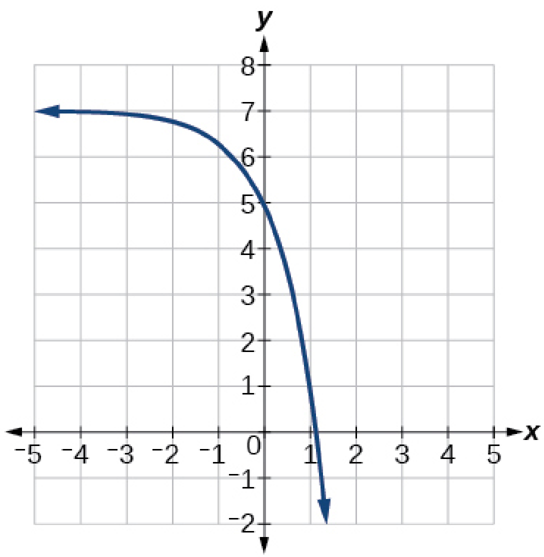
Problem:



For the following exercises, find an exponential equation for the graph.

Exercise:

Problem:

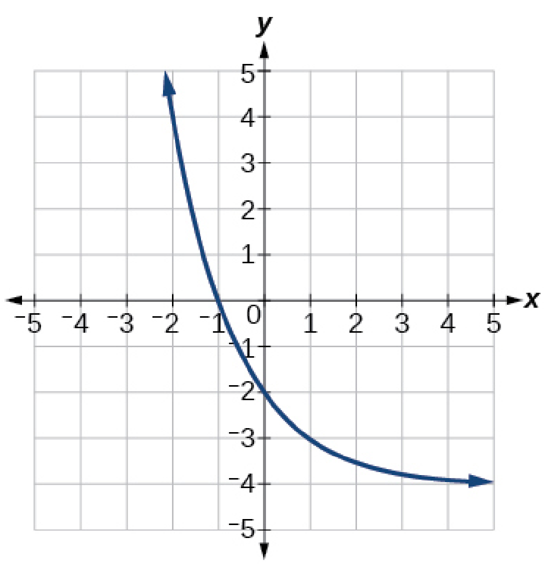


Solution:

$y = -2(3)^x + 7$

Exercise:

Problem:



Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

Exercise:

Problem: $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.

Solution:

$g(6) = 800 + \frac{1}{3} \approx 800.3333$

Exercise:

Problem: $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.

Exercise:

Problem: $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^{x} + 6$ for $h(-7)$.

Solution:

$h(-7) = -58$

Technology

For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.

Exercise:

Problem: $-50 = -\left(\frac{1}{2}\right)^{-x}$

Exercise:

Problem: $116 = \frac{1}{4}\left(\frac{1}{8}\right)^{x}$

Solution:

$x \approx -2.953$

Exercise:

Problem: $12 = 2(3)^{x} + 1$

Exercise:

Problem: $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

Solution:

$x \approx -0.222$

Exercise:

Problem: $-30 = -4(2)^{x+2} + 2$

Extensions

Exercise:

Problem:

Explore and discuss the graphs of $F(x) = (b)^x$ and $G(x) = \left(\frac{1}{b}\right)^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b}\right)^x$ for any real number $b > 0$.

Solution:

The graph of $G(x) = \left(\frac{1}{b}\right)^x$ is the refelction about the *y*-axis of the graph of $F(x) = b^x$; For any real number $b > 0$ and function $f(x) = b^x$,the graph of $\left(\frac{1}{b}\right)^x$ is the the reflection about the *y*-axis, $F(-x)$.

Exercise:

Problem: Prove the conjecture made in the previous exercise.

Exercise:

Problem:

Explore and discuss the graphs of $f(x) = 4^x$, $g(x) = 4^{x-2}$,and $h(x) = \left(\frac{1}{16}\right)4^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b^n}\right)b^x$ for any real number *n* and real number $b > 0$.

Solution:

The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$; For any real number *n*, real number $b > 0$, and function $f(x) = b^x$, the graph of $\left(\frac{1}{b^n}\right)b^x$ is the horizontal shift $f(x - n)$.

Exercise:

Problem: Prove the conjecture made in the previous exercise.

Logarithmic Functions

In this section, you will:

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes[footnote]. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings,[footnote] like those shown in [link]. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a

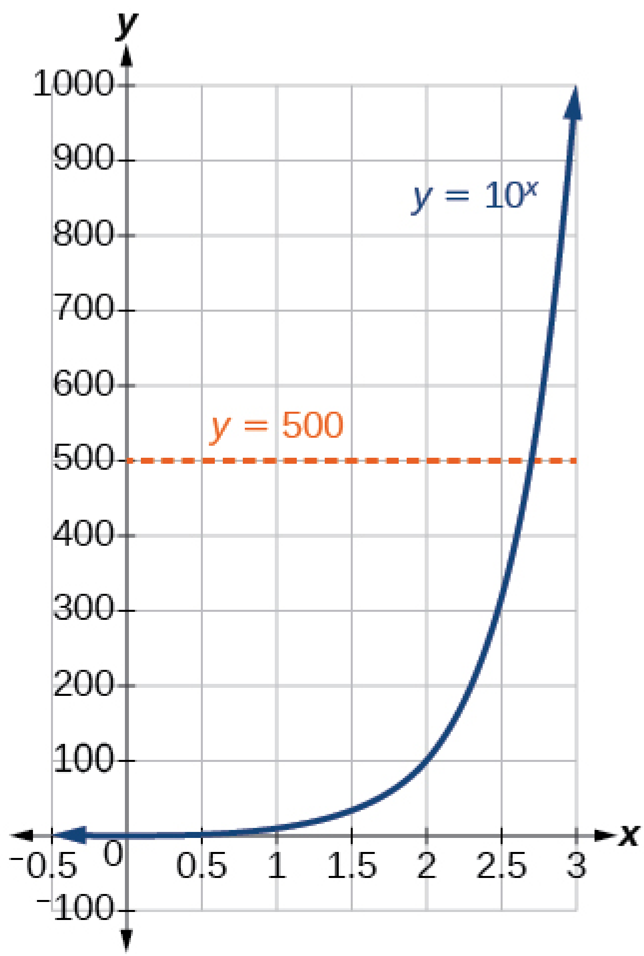
7.0 on the Richter Scale[footnote] whereas the Japanese earthquake registered a 9.0.[footnote]
http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/#summary. Accessed 3/4/2013.
http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#summary. Accessed 3/4/2013.
http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/. Accessed 3/4/2013.
http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#details. Accessed 3/4/2013.

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4}=10^4=10{,}000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x=500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x=500$. We know that $10^2=100$ and $10^3=1000$, so it is clear that x must be some value between 2 and 3, since $y=10^x$ is increasing. We can examine a graph, as in [link], to better estimate the solution.



Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in [link] passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

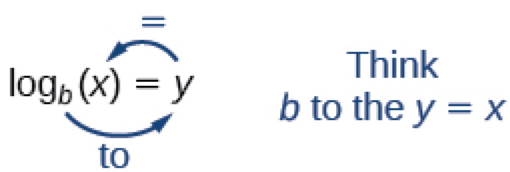
We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

Equation:

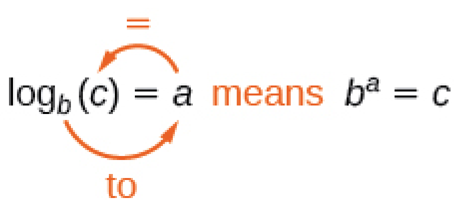
$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions.

Note:
Definition of the Logarithmic Function
A **logarithm** base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1$,
Equation:

$$y = \log_b(x) \text{ is equivalent to } b^y = x$$

where,

- we read $\log_b(x)$ as, "the logarithm with base b of x" or the "log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Note:
Can we take the logarithm of a negative number?
No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

Note:
Given an equation in logarithmic form $\log_b(x) = y$, **convert it to exponential form.**

1. Examine the equation $y = \log_b x$ and identify $b, y,$ and x.
2. Rewrite $\log_b x = y$ as $b^y = x$.

Example:
Exercise:

Problem:
Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solution:

First, identify the values of $b, y, \text{and} x.$ Then, write the equation in the form $b^y = x$.

a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$

Here, $b = 6, y = \frac{1}{2},$ and $x = \sqrt{6}.$ Therefore, the equation $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}.$$

b. $\log_3(9) = 2$

Here, $b = 3, y = 2,$ and $x = 9.$ Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9.$

Note:
Exercise:

Problem:

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1{,}000{,}000) = 6$
b. $\log_5(25) = 2$

Solution:

a. $\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$
b. $\log_5(25) = 2$ is equivalent to $5^2 = 25$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b,exponent x,and output y. Then we write $x = \log_b(y)$.

Example:
Exercise:

Problem:
Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \frac{1}{10{,}000}$

Solution:

First, identify the values of $b, y,$ and x. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$, and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$

Here, $b = 5$, $x = 2$, and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \frac{1}{10{,}000}$

Here, $b = 10$, $x = -4$, and $y = \frac{1}{10{,}000}$. Therefore, the equation $10^{-4} = \frac{1}{10{,}000}$ is equivalent to $\log_{10}\left(\frac{1}{10{,}000}\right) = -4$.

Note:
Exercise:

Problem:

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \frac{1}{2}$

Solution:

a. $3^2 = 9$ is equivalent to $\log_3(9) = 2$
b. $5^3 = 125$ is equivalent to $\log_5(125) = 3$
c. $2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, “To what exponent must 2 be raised in order to get 8?” Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, “To what exponent must 7 be raised in order to get 49?” We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, “To what exponent must 3 be raised in order to get 27?” We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let’s evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, “To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? ” We know $2^2 = 4$ and $3^2 = 9$,so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{\frac{2}{3}} \left(\frac{4}{9}\right) = 2$.

Note:
Given a logarithm of the form $y = \log_b(x)$,evaluate it mentally.

1. Rewrite the argument x as a power of b : $b^y = x$.
2. Use previous knowledge of powers of b identify y by asking, “To what exponent should b be raised in order to get x?”

Example:
Exercise:

Problem:
Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution:

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, “To what exponent must 4 be raised in order to get 64?”

We know
Equation:

$$4^3 = 64$$

Therefore,
Equation:

$$\log_4(64) = 3$$

Note:
Exercise:

Problem: Solve $y = \log_{121}(11)$ without using a calculator.

Solution:

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

Example:

Exercise:

Problem:
Evaluating the Logarithm of a Reciprocal

Evaluate $y=\log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution:

First we rewrite the logarithm in exponential form: $3^y=\frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3=27$,but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a}=\frac{1}{b^a}$. We use this information to write

Equation:

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right)=-3$.

Note:
Exercise:

Problem: Evaluate $y=\log_2\left(\frac{1}{32}\right)$ without using a calculator.

Solution:

$\log_2\left(\frac{1}{32}\right)=-5$

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x).$ We call a base-10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

Note:
Definition of the Common Logarithm
A **common logarithm** is a logarithm with base $10.$ We write $\log_{10}(x)$ simply as $\log(x).$ The common logarithm of a positive number x satisfies the following definition.
For $x > 0,$
Equation:

$$y = \log(x) \text{ is equivalent to } 10^y = x$$

We read $\log(x)$ as, "the logarithm with base 10 of x" or "log base 10 of $x.$"
The logarithm y is the exponent to which 10 must be raised to get $x.$

Note:
Given a common logarithm of the form $y = \log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of $10:\ 10^y = x.$
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x?"

Example:
Exercise:

Problem:
Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution:

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, “To what exponent must 10 be raised in order to get 1000?” We know

Equation:

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

Note:
Exercise:

Problem: Evaluate $y = \log(1{,}000{,}000)$.

Solution:

$\log(1,000,000) = 6$

Note:
Given a common logarithm with the form $y = \log(x)$, evaluate it using a calculator.

1. Press **[LOG]**.
2. Enter the value given for x,followed by **[ )]**.
3. Press **[ENTER]**.

Example:
Exercise:

Problem:
Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution:

- Press **[LOG]**.
- Enter 321, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

Equation:

$$\begin{array}{ccccc} 100 & < & 321 & < & 1000 \\ 2 & < & 2.5065 & < & 3 \end{array}$$

Note:
Exercise:

Problem:

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Solution:

$\log(123) \approx 2.0899$

Example:
Exercise:

Problem:
Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution:

We begin by rewriting the exponential equation in logarithmic form.
Equation:

$$\begin{aligned} 10^x &= 500 \\ \log(500) &= x \qquad \text{Use the definition of the common log.} \end{aligned}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500,followed by **[ )]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about 2.699.

Note:
Exercise:

Problem:

The amount of energy released from one earthquake was $8{,}500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution:

The difference in magnitudes was about 3.929.

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$.

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Note:
Definition of the Natural Logarithm

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.
For $x > 0$,
Equation:

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."
The logarithm y is the exponent to which e must be raised to get x.
Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

Note:
Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator.

1. Press **[LN]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Example:
Exercise:

Problem:
Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution:

- Press **[LN]**.

- Enter 500, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

Note:
Exercise:

Problem: Evaluate $\ln(-500)$.

Solution:

It is not possible to take the logarithm of a negative number in the set of real numbers.

Note:
Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms

Key Equations

Definition of the logarithmic function	For $x > 0, b > 0, b \neq 1$, $y = \log_b(x)$ if and only if $b^y = x$.

Definition of the common logarithm	For $x > 0, y = \log(x)$ if and only if $10^y = x$.
Definition of the natural logarithm	For $x > 0, y = \ln(x)$ if and only if $e^y = x$.

Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See [link].
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See [link].
- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of b. See [link] and [link].
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10. See [link].
- When common logarithms cannot be evaluated mentally, a calculator can be used. See [link].
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See [link].
- Natural logarithms can be evaluated using a calculator [link].

Section Exercises

Verbal

Exercise:

Problem:

What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$.

Solution:

A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, y,in the expression b^y can also be written as the logarithm, $\log_b x$,and the value of x is the result of raising b to the power of y.

Exercise:

Problem:

How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

Exercise:

Problem:

How can the logarithmic equation $\log_b x = y$ be solved for x using the properties of exponents?

Solution:

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$, and then properties of exponents can be applied to solve for x.

Exercise:

Problem:

Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Exercise:

Problem:

Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Solution:

The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$,the notation used is $\ln(x)$.

Algebraic

For the following exercises, rewrite each equation in exponential form.

Exercise:

Problem: $\log_4(q) = m$

Exercise:

Problem: $\log_a(b) = c$

Solution:

$a^c = b$

Exercise:

Problem: $\log_{16}(y) = x$

Exercise:

Problem: $\log_x(64) = y$

Solution:

$x^y = 64$

Exercise:

Problem: $\log_y(x) = -11$

Exercise:

Problem: $\log_{15}(a) = b$

Solution:

$15^b = a$

Exercise:

Problem: $\log_y(137) = x$

Exercise:

Problem: $\log_{13}(142) = a$

Solution:

$13^a = 142$

Exercise:

Problem: $\log(v) = t$

Exercise:

Problem: $\ln(w) = n$

Solution:

$e^n = w$

For the following exercises, rewrite each equation in logarithmic form.
Exercise:

Problem: $4^x = y$

Exercise:

Problem: $c^d = k$

Solution:

$\log_c(k) = d$

Exercise:

Problem: $m^{-7} = n$

Exercise:

Problem: $19^x = y$

Solution:

$\log_{19} y = x$

Exercise:

Problem: $x^{-\frac{10}{13}} = y$

Exercise:

Problem: $n^4 = 103$

Solution:

$\log_n (103) = 4$

Exercise:

Problem: $\left(\frac{7}{5}\right)^m = n$

Exercise:

Problem: $y^x = \frac{39}{100}$

Solution:

$\log_y\left(\frac{39}{100}\right) = x$

Exercise:

Problem: $10^a = b$

Exercise:

Problem: $e^k = h$

Solution:

$\ln(h) = k$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

Exercise:

Problem: $\log_3(x) = 2$

Exercise:

Problem: $\log_2(x) = -3$

Solution:

$x = 2^{-3} = \frac{1}{8}$

Exercise:

Problem: $\log_5(x) = 2$

Exercise:

Problem: $\log_3{(x)} = 3$

Solution:

$x = 3^3 = 27$

Exercise:

Problem: $\log_2(x) = 6$

Exercise:

Problem: $\log_9(x) = \frac{1}{2}$

Solution:

$x = 9^{\frac{1}{2}} = 3$

Exercise:

Problem: $\log_{18}(x) = 2$

Exercise:

Problem: $\log_6{(x)} = -3$

Solution:

$x = 6^{-3} = \frac{1}{216}$

Exercise:

Problem: $\log(x) = 3$

Exercise:

Problem: $\ln(x) = 2$

Solution:

$x = e^2$

For the following exercises, use the definition of common and natural logarithms to simplify.

Exercise:

Problem: $\log(100^8)$

Exercise:

Problem: $10^{\log(32)}$

Solution:

32

Exercise:

Problem: $2\log(.0001)$

Exercise:

Problem: $e^{\ln(1.06)}$

Solution:

1.06

Exercise:

Problem: $\ln\left(e^{-5.03}\right)$

Exercise:

Problem: $e^{\ln(10.125)} + 4$

Solution:

14.125

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

Exercise:

Problem: $\log_3\left(\frac{1}{27}\right)$

Exercise:

Problem: $\log_6(\sqrt{6})$

Solution:

$\frac{1}{2}$

Exercise:

Problem: $\log_2\left(\frac{1}{8}\right) + 4$

Exercise:

Problem: $6\log_8(4)$

Solution:

4

For the following exercises, evaluate the common logarithmic expression without using a calculator.

Exercise:

Problem: $\log(10,000)$

Exercise:

Problem: $\log(0.001)$

Solution:

-3

Exercise:

Problem: $\log(1)+7$

Exercise:

Problem: $2\log(100^{-3})$

Solution:

-12

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

Exercise:

Problem: $\ln(e^{\frac{1}{3}})$

Exercise:

Problem: $\ln(1)$

Solution:

0

Exercise:

Problem: $\ln(e^{-0.225})-3$

Exercise:

Problem: $25\ln(e^{\frac{2}{5}})$

Solution:

10

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

Exercise:

Problem: $\log(0.04)$

Exercise:

Problem: $\ln(15)$

Solution:

2.708

Exercise:

Problem: $\ln\left(\frac{4}{5}\right)$

Exercise:

Problem: $\log(\sqrt{2})$

Solution:

0.151

Exercise:

Problem: $\ln(\sqrt{2})$

Extensions

Exercise:

Problem:

Is $x=0$ in the domain of the function $f(x)=\log(x)$? If so, what is the value of the function when $x=0$? Verify the result.

Solution:

No, the function has no defined value for $x=0$. To verify, suppose $x=0$ is in the domain of the function $f(x)=\log(x)$. Then there is some number n such that $n=\log(0)$. Rewriting as an exponential equation gives: $10^n=0$, which is impossible since no such real number n exists. Therefore, $x=0$ is *not* the domain of the function $f(x)=\log(x)$.

Exercise:

Problem:

Is $f(x)=0$ in the range of the function $f(x)=\log(x)$? If so, for what value of x? Verify the result.

Exercise:

Problem:

Is there a number x such that $\ln x = 2$? If so, what is that number? Verify the result.

Solution:

Yes. Suppose there exists a real number x such that $\ln x = 2$. Rewriting as an exponential equation gives $x = e^2$, which is a real number. To verify, let $x = e^2$. Then, by definition, $\ln(x) = \ln(e^2) = 2$.

Exercise:

Problem: Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$? Verify the result.

Exercise:

Problem: Is the following true: $\frac{\ln\left(e^{1.725}\right)}{\ln(1)} = 1.725$? Verify the result.

Solution:

No; $\ln(1) = 0$, so $\frac{\ln\left(e^{1.725}\right)}{\ln(1)}$ is undefined.

Real-World Applications

Exercise:

Problem:

The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI=\log_2\left(\frac{f^2}{t}\right)$, where f is the “f-stop” setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

Exercise:

Problem:

Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?

Solution:

2

Exercise:

Problem:

The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula $\log\frac{I_1}{I_2}=M_1-M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0.[footnote] How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number. http://earthquake.usgs.gov/earthquakes/world/historical.php. Accessed 3/4/2014.

Glossary

common logarithm

the exponent to which 10 must be raised to get x; $\log_{10}(x)$ is written simply as $\log(x)$.

logarithm
: the exponent to which b must be raised to get x; written $y = \log_b(x)$

natural logarithm
: the exponent to which the number e must be raised to get x; $\log_e(x)$ is written as $\ln(x)$.

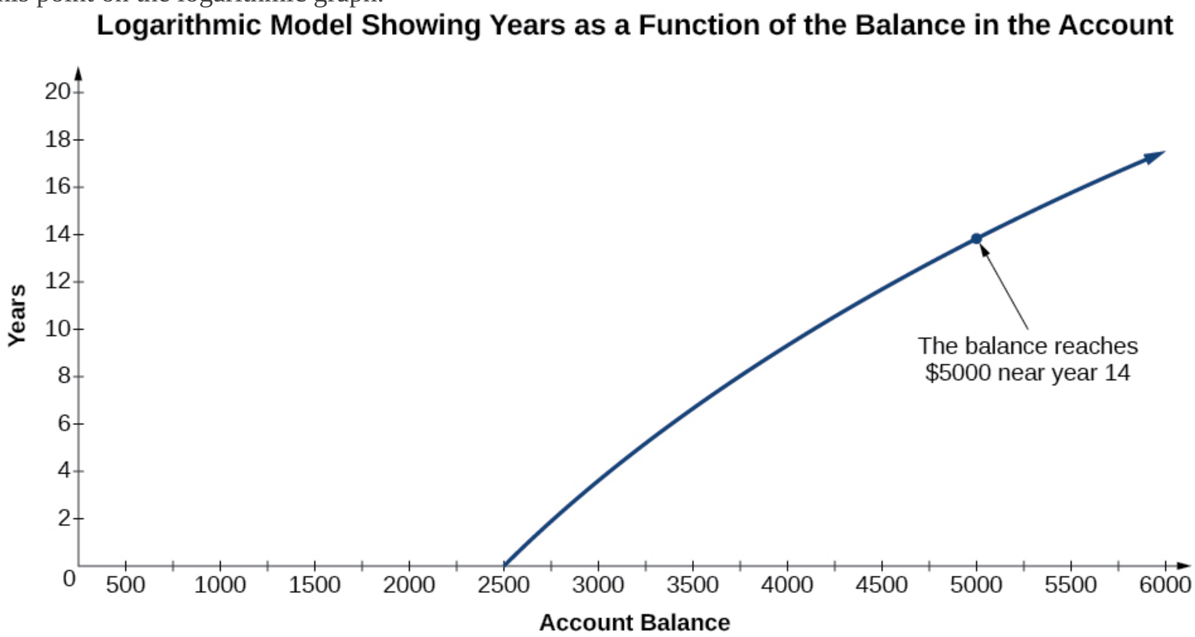
Graphs of Logarithmic Functions
In this section, you will:

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

In Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%,compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? [link] shows this point on the logarithmic graph.



In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.

- The range of y is $(0,\infty)$.

In the last section we learned that the logarithmic function $y=\log_b(x)$ is the inverse of the exponential function $y=b^x$. So, as inverse functions:

- The domain of $y=\log_b(x)$ is the range of $y=b^x$: $(0,\infty)$.
- The range of $y=\log_b(x)$ is the domain of $y=b^x$: $(-\infty,\infty)$.

Transformations of the parent function $y=\log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y=b^x$. Similarly, applying transformations to the parent function $y=\log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x)=\log_4(2x-3)$. This function is defined for any values of x such that the argument, in this case $2x-3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

Equation:

$$\begin{array}{rl} 2x-3>0 & \text{Show the argument greater than zero.} \\ 2x>3 & \text{Add 3.} \\ x>1.5 & \text{Divide by 2.} \end{array}$$

In interval notation, the domain of $f(x)=\log_4(2x-3)$ is $(1.5,\infty)$.

Note:
Given a logarithmic function, identify the domain.

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Example:
Exercise:

Problem:
Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x)=\log_2(x+3)$?

Solution:

The logarithmic function is defined only when the input is positive, so this function is defined when $x+3>0$. Solving this inequality,

Equation:

$$\begin{array}{rl} x+3>0 & \text{The input must be positive.} \\ x>-3 & \text{Subtract 3.} \end{array}$$

The domain of $f(x)=\log_2(x+3)$ is $(-3,\infty)$.

Note:
Exercise:

Problem: What is the domain of $f(x)=\log_5(x-2)+1$?

Solution:

$(2,\infty)$

Example:
Exercise:

Problem:
Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x)=\log(5-2x)$?

Solution:

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x>0$. Solving this inequality,
Equation:

$$\begin{array}{rl} 5-2x>0 & \text{The input must be positive.} \\ -2x>-5 & \text{Subtract 5.} \\ x<\frac{5}{2} & \text{Divide by }-2\text{ and switch the inequality.} \end{array}$$

The domain of $f(x)=\log(5-2x)$ is $\left(-\infty,\frac{5}{2}\right)$.

Note:
Exercise:

Problem: What is the domain of $f(x)=\log(x-5)+2$?

Solution:

$(5,\infty)$

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y=\log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

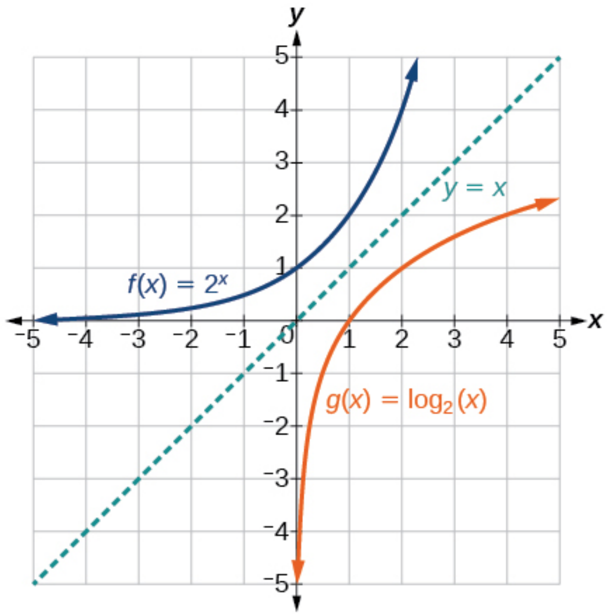
We begin with the parent function $y=\log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y=b^x$, their graphs will be reflections of each other across the line $y=x$. To illustrate this, we can observe the relationship between the input and output values of $y=2^x$ and its equivalent $x=\log_2(y)$ in [link].

x	-3	-2	-1	0	1	2	3
$2^x=y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y)=x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from [link], we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x)=2^x$ and $g(x)=\log_2(x)$. See [link].

$f(x)=2^x$	$\left(-3,\frac{1}{8}\right)$	$\left(-2,\frac{1}{4}\right)$	$\left(-1,\frac{1}{2}\right)$	$(0,1)$	$(1,2)$	$(2,4)$	$(3,8)$
$g(x)=\log_2(x)$	$\left(\frac{1}{8},-3\right)$	$\left(\frac{1}{4},-2\right)$	$\left(\frac{1}{2},-1\right)$	$(1,0)$	$(2,1)$	$(4,2)$	$(8,3)$

As we’d expect, the *x*- and *y*-coordinates are reversed for the inverse functions. [link] shows the graph of *f* and *g*.



Notice that the graphs of $f\left(x\right)={2}^{x}$ and $g\left(x\right)={\mathrm{log}}_{2}\left(x\right)$ are reflections about the line $y=x$.

Observe the following from the graph:

- $f(x)={2}^{x}$ has a *y*-intercept at $(0,1)$ and $g(x)={\mathrm{log}}_{2}\left(x\right)$ has an *x*- intercept at $(1,0)$.
- The domain of $f(x)={2}^{x}$, $(-\infty ,\infty)$, is the same as the range of $g(x)={\mathrm{log}}_{2}\left(x\right)$.
- The range of $f(x)={2}^{x}$, $(0,\infty)$, is the same as the domain of $g(x)={\mathrm{log}}_{2}\left(x\right)$.

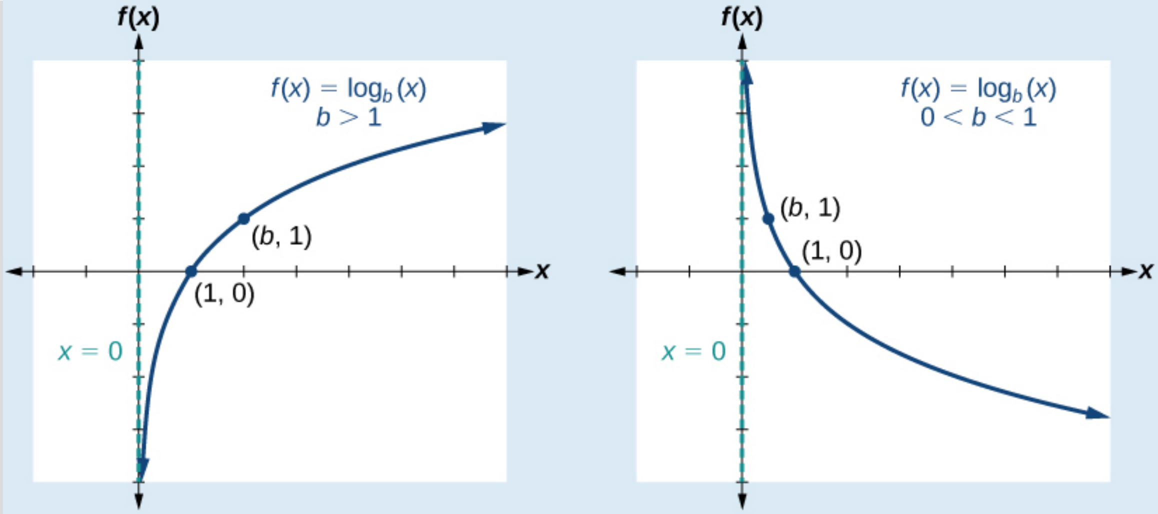
Note:

Characteristics of the Graph of the Parent Function, $f(x) = \log_b(x)$

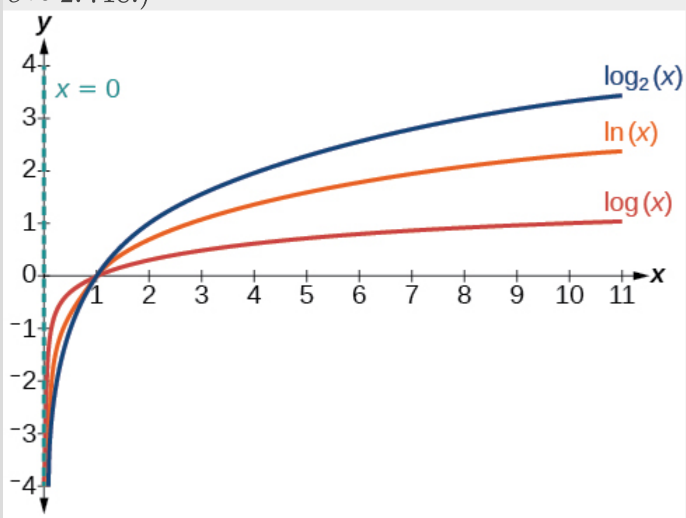
For any real number x and constant $b>0$,$b\ne 1$, we can see the following characteristics in the graph of $f\left(x\right)={\mathrm{log}}_{b}\left(x\right)$:

- one-to-one function
- vertical asymptote: $x=0$
- domain: $(0,\infty)$
- range: $(-\infty ,\infty)$
- *x*-intercept: $(1,0)$ and key point $(b,1)$
- *y*-intercept: none
- increasing if $b>1$
- decreasing if $0<b<1$

See [link].



[link] shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note:* recall that the function $\ln(x)$ has base $e \approx 2.718$.)



The graphs of three logarithmic functions with different bases, all greater than 1.

Note:
Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$,the range, $(-\infty, \infty)$,and the vertical asymptote, $x = 0$.

Example:
Exercise:

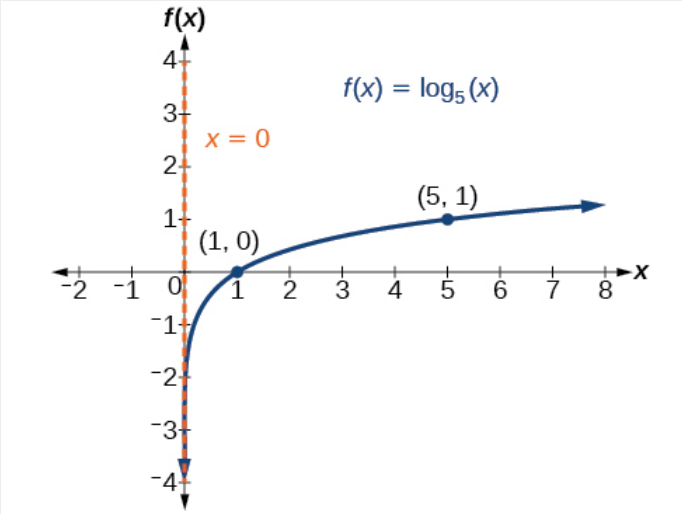
Problem:
Graphing a Logarithmic Function with the Form *f*(*x*) = log*b*(*x*).

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution:

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The *x*-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see [link]).

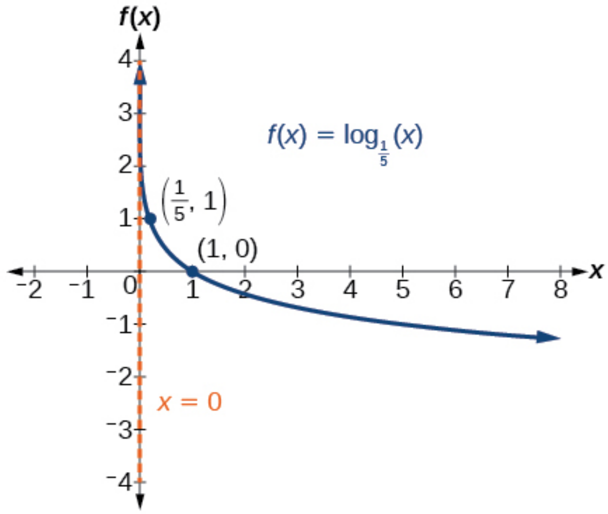


The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem: Graph $f(x) = \log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

Solution:



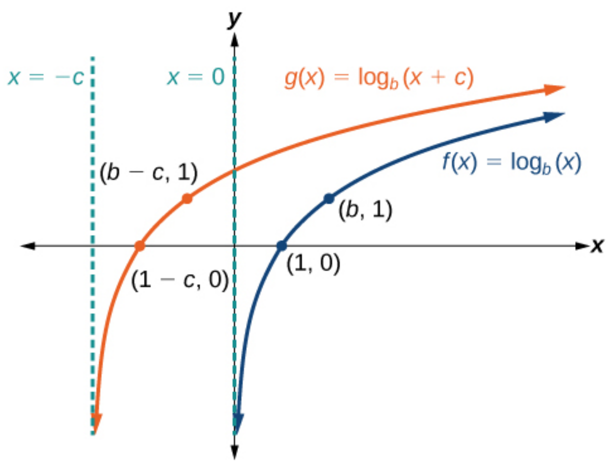
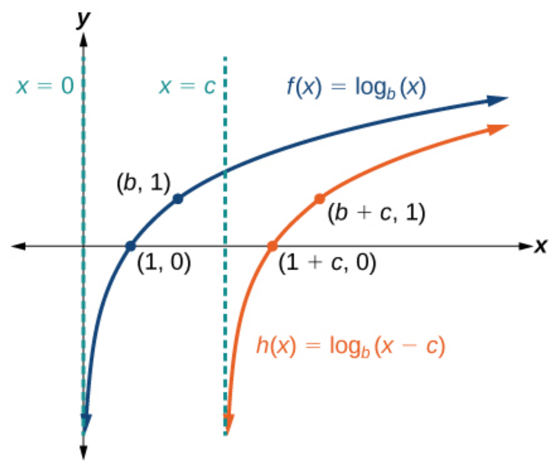
The domain is $(0,\infty)$,the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y=\log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of *f*(*x*) = log$_b$(*x*)

When a constant c is added to the input of the parent function $f(x)=log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x)=\log_b(x)$ and for $c>0$ alongside the shift left, $g(x)=\log_b(x+c)$, and the shift right, $h(x)=\log_b(x-c)$. See [link].

Shift left $g(x) = \log_b(x + c)$	Shift right $h(x) = \log_b(x - c)$
 •The asymptote changes to $x = -c$. •The domain changes to $(-c, \infty)$. •The range remains $(-\infty, \infty)$.	 •The asymptote changes to $x = c$. •The domain changes to (c, ∞). •The range remains $(-\infty, \infty)$.

Note:
Horizontal Shifts of the Parent Function $y = \log_b(x)$
For any constant c,the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right c units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

Note:
Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:

   a. If $c > 0$,shift the graph of $f(x) = \log_b(x)$ left c units.
   b. If $c < 0$,shift the graph of $f(x) = \log_b(x)$ right c units.

2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

Example:
Exercise:

Problem:
Graphing a Horizontal Shift of the Parent Function *y* = log*b*(*x*)

Sketch the horizontal shift $f(x) = \log_3(x-2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution:

Since the function is $f(x) = \log_3(x-2)$, we notice $x + (-2) = x - 2$.

Thus $c = -2$,so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

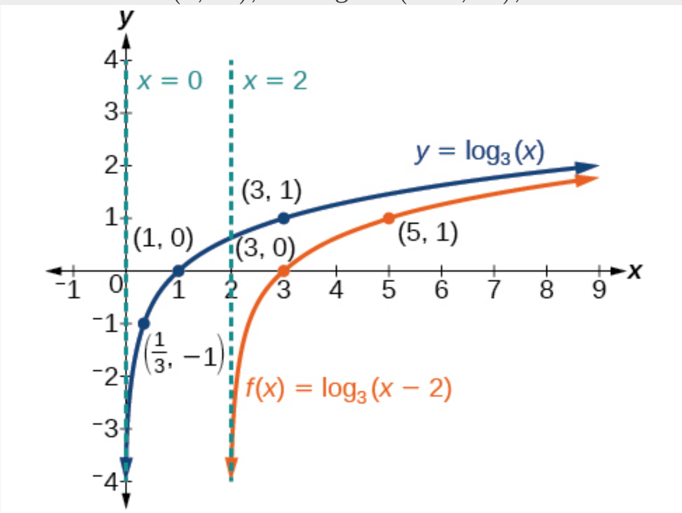
The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$,$(1, 0)$,and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$,$(3, 0)$,and $(5, 1)$.

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

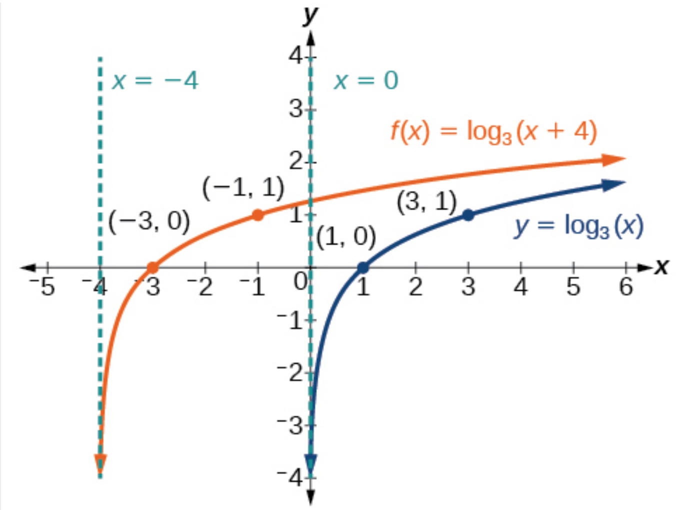


Note:
Exercise:

Problem:

Sketch a graph of $f(x) = \log_3(x+4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

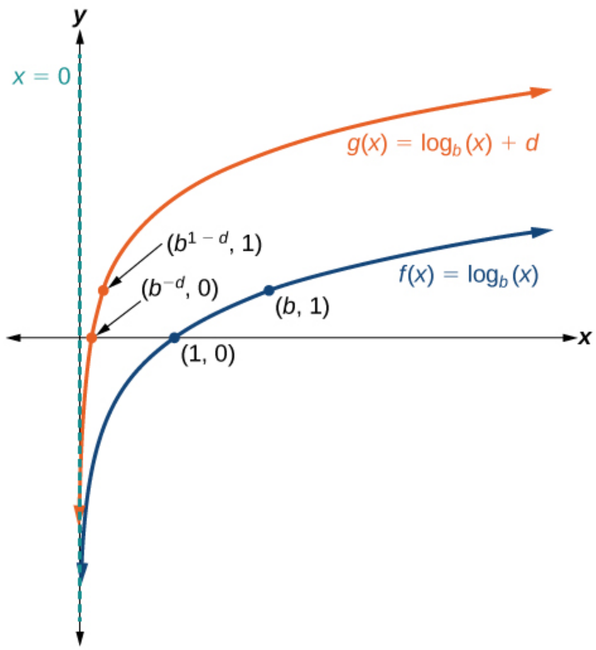
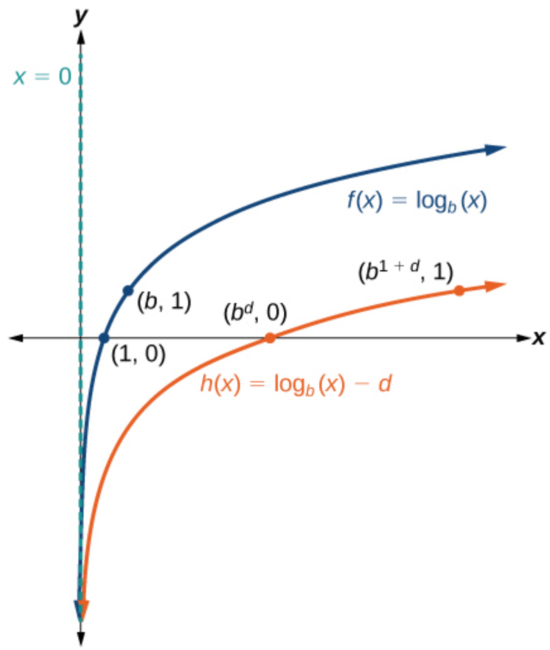
Solution:



The domain is $(-4, \infty)$,the range $(-\infty, \infty)$,and the asymptote $x = -4$.

Graphing a Vertical Shift of *y* = log*b*(*x*)

When a constant d is added to the parent function $f(x) = \log_b(x)$,the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$.See [link].

Shift up $g(x) = \log_b(x) + d$	**Shift down** $h(x) = \log_b(x) - d$
	
•The asymptote remains $x = 0$. •The domain remains to $(0, \infty)$. •The range remains $(-\infty, \infty)$.	•The asymptote remains $x = 0$. •The domain remains to $(0, \infty)$. •The range remains $(-\infty, \infty)$.

Note:
Vertical Shifts of the Parent Function $y = \log_b(x)$
For any constant d,the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up d units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down d units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Note:
Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$down d units.
2. Draw the vertical asymptote $x = 0$.

3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding d to the y coordinate.
4. Label the three points.
5. The domain is $(0,\infty)$,the range is $(-\infty,\infty)$,and the vertical asymptote is $x=0$.

Example:
Exercise:

Problem:
Graphing a Vertical Shift of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x)=\log_3(x)-2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:

Since the function is $f(x)=\log_3(x)-2$,we will notice $d=-2$. Thus $d<0$.

This means we will shift the function $f(x)=\log_3(x)$ down 2 units.

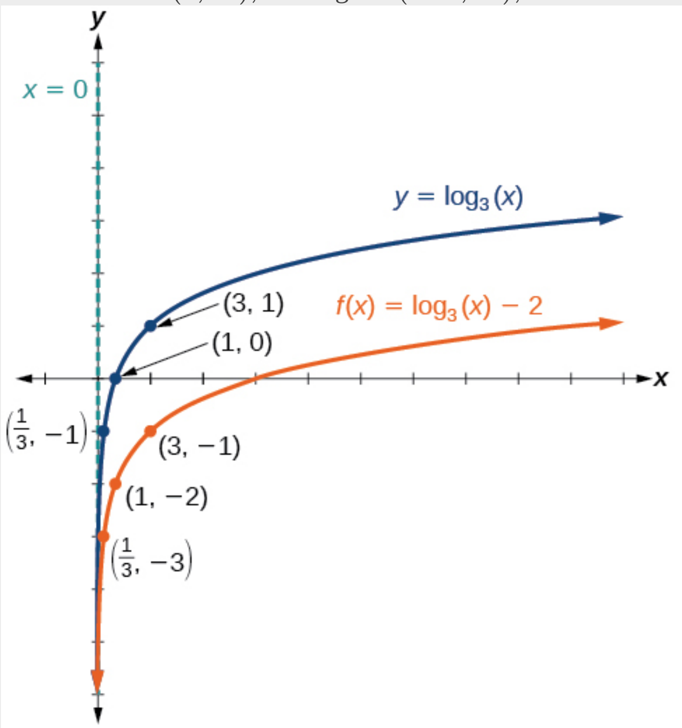
The vertical asymptote is $x=0$.

Consider the three key points from the parent function, $\left(\frac{1}{3},-1\right)$,$(1,0)$,and $(3,1)$.

The new coordinates are found by subtracting 2 from the *y* coordinates.

Label the points $\left(\frac{1}{3},-3\right)$,$(1,-2)$, and $(3,-1)$.

The domain is $(0,\infty)$,the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.



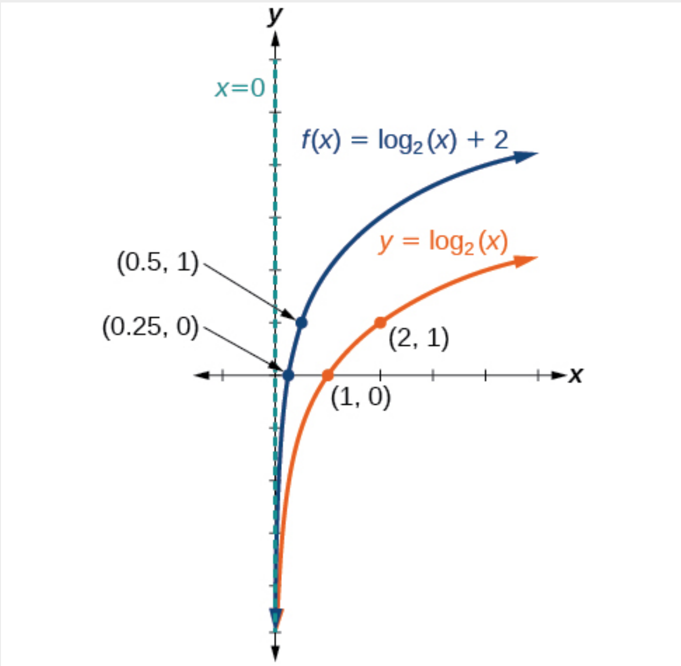
The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem:

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

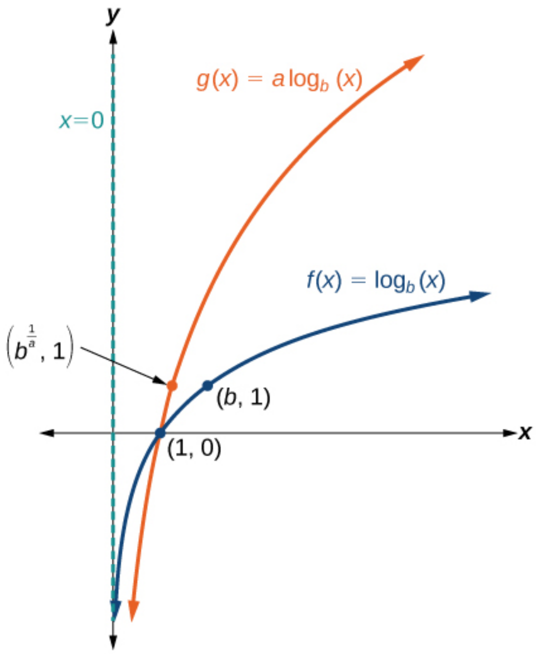
Solution:

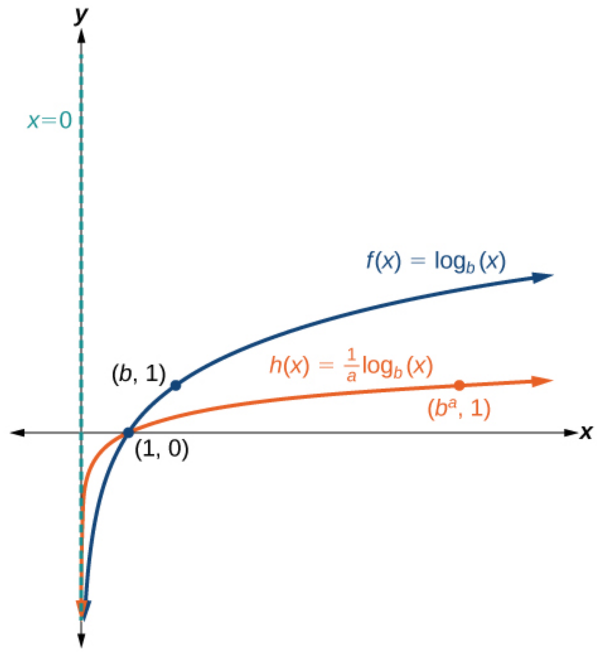


The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Graphing Stretches and Compressions of $y = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$.See [link].

<table>
<tr>
<th>Vertical Stretch
$g(x) = a\log_b(x)$, $a > 1$</th>
<th>Vertical Compression
$h(x) = \frac{1}{a}\log_b(x)$, $a > 1$</th>
</tr>
<tr>
<td>


•The asymptote remains $x = 0$.
•The x-intercept remains (1, 0).
•The domain remains $(0, \infty)$.
•The range remains $(-\infty, \infty)$.
</td>
<td>


•The asymptote remains $x = 0$.
•The x-intercept remains (1, 0).
•The domain remains $(0, \infty)$.
•The range remains $(-\infty, \infty)$.
</td>
</tr>
</table>

Note:
Vertical Stretches and Compressions of the Parent Function $y = \log_b(x)$
For any constant $a > 1$,the function $f(x) = a\log_b(x)$

- stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $a > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the x-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Note:
Given a logarithmic function with the form $f(x) = a\log_b(x)$,$a > 0$,graph the translation.

1. Identify the vertical stretch or compressions:
    - If $|a| > 1$,the graph of $f(x) = \log_b(x)$ is stretched by a factor of a units.
    - If $|a| < 1$,the graph of $f(x) = \log_b(x)$ is compressed by a factor of a units.

2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying the y coordinates by a.
4. Label the three points.
5. The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Example:
Exercise:

Problem:
Graphing a Stretch or Compression of the Parent Function *y* = log*b*(*x*)

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:

Since the function is $f(x) = 2\log_4(x)$,we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

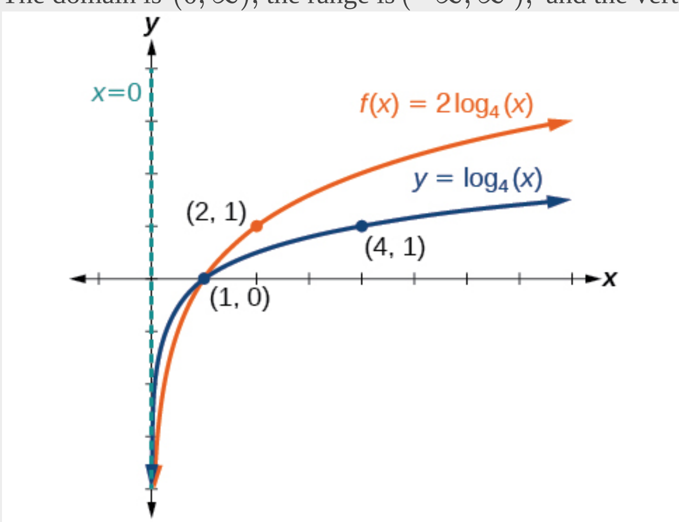
The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{4}, -1\right)$,$(1, 0\ )$, and $(4, 1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4}, -2\right)$,$(1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty\ )$, and the vertical asymptote is $x = 0$. See [link].



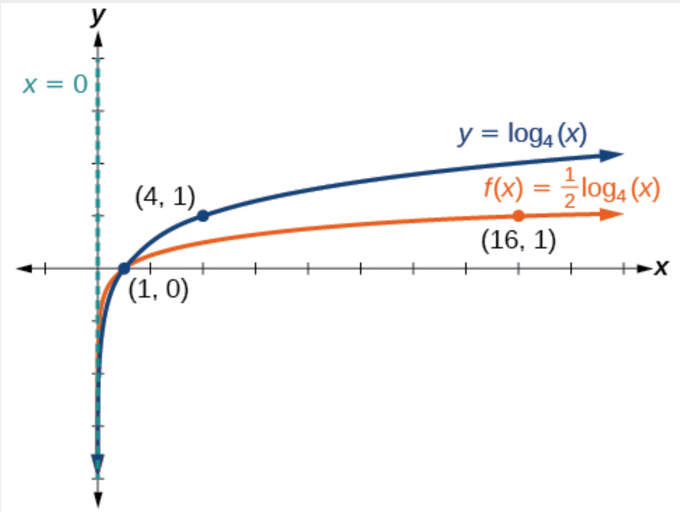
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem:

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:



The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

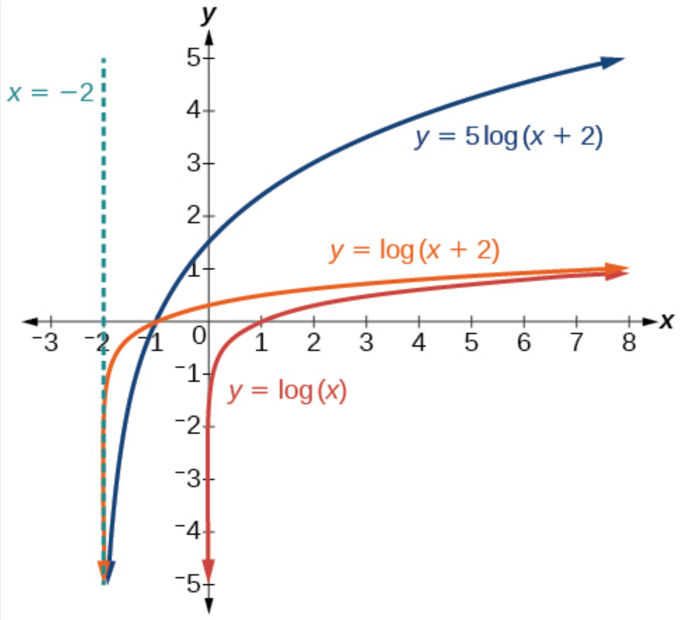
Example:
Exercise:

Problem:
Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x+2)$. State the domain, range, and asymptote.

Solution:

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in [link]. The vertical asymptote will be shifted to $x = -2$. The *x*-intercept will be $(-1,0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5\ )$. We chose $x = 8$ as the *x*-coordinate of one point to graph because when $x = 8,\ x + 2 = 10$, the base of the common logarithm.



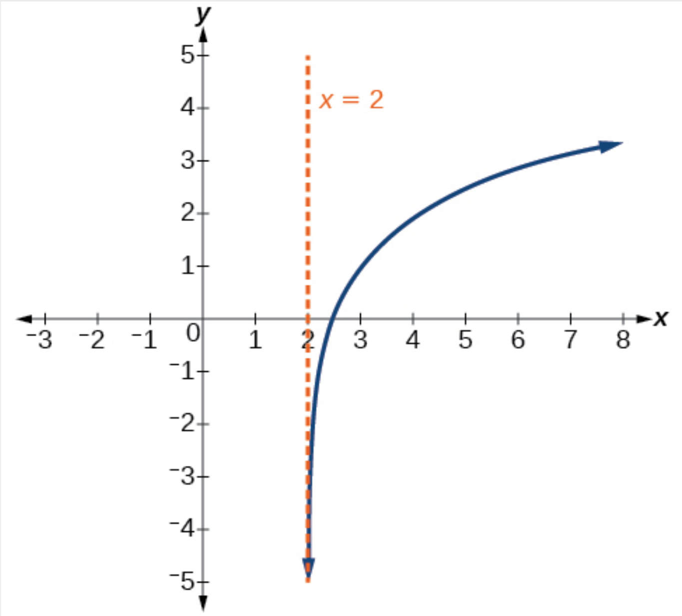
The domain is $(-2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = -2$.

Note:
Exercise:

Problem:

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

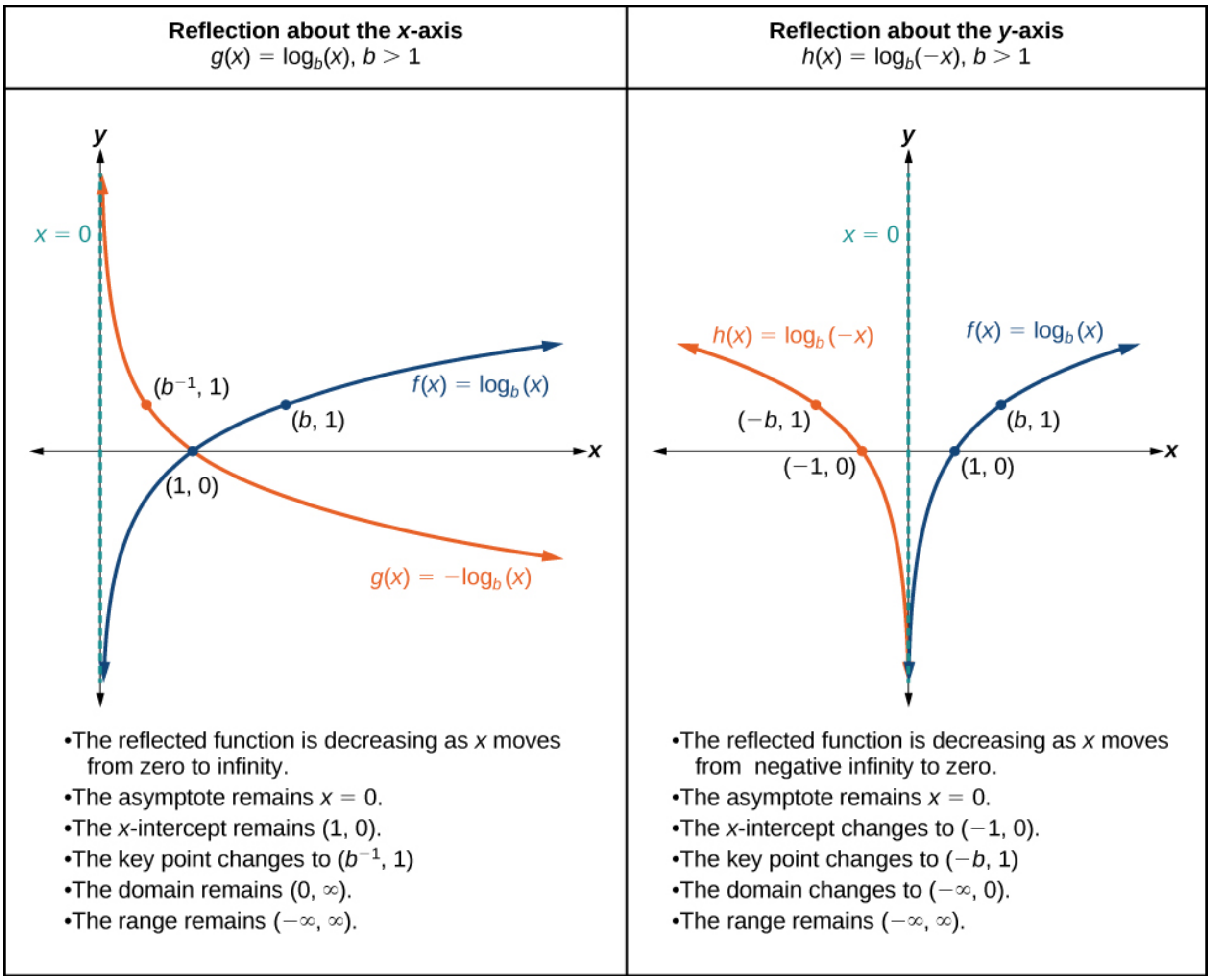
Solution:



The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

Graphing Reflections of $f(x) = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by -1,the result is a reflection about the *x*-axis. When the *input* is multiplied by -1,the result is a reflection about the *y*-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the *x*-axis, $g(x) = -\log_b(x)$ and the reflection about the *y*-axis, $h(x) = \log_b(-x)$.



Note:
Reflections of the Parent Function $y = \log_b(x)$
The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the *x*-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the *y*-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

Note:
Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
1. Draw the vertical asymptote, $x = 0$.	1. Draw the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.	2. Plot the *x*-intercept, $(1, 0)$.
3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *x*-axis.	3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *y*-axis.
4. Draw a smooth curve through the points.	4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.	5. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.

Example:
Exercise:

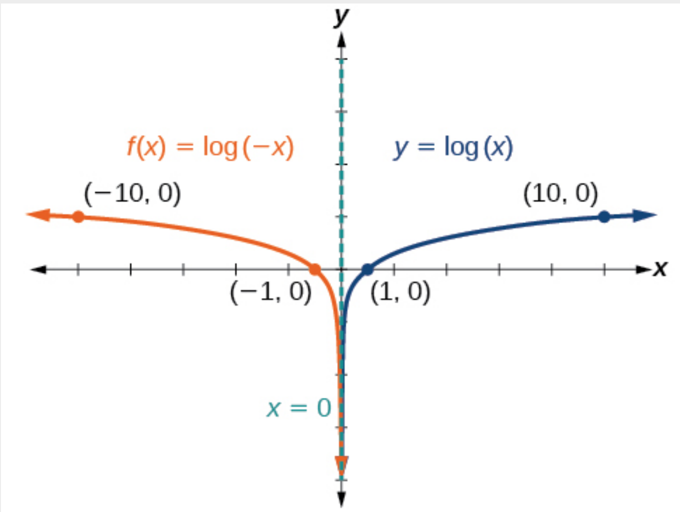
Problem:
Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:

Before graphing $f(x) = \log(-x)$,identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1, f is a reflection of the parent graph about the *y*-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The *x*-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

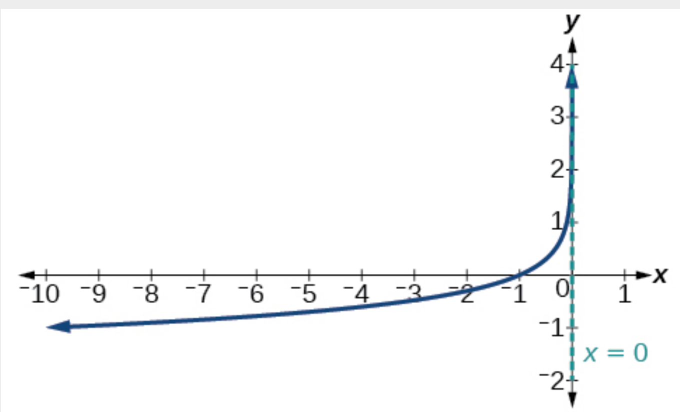


The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem: Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Solution:



The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Note:
Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as **Y_1=** and, if needed, **Y_2=**.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of x,for the point(s) of intersection.

Example:
Exercise:

Problem:
Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x-1)$ graphically. Round to the nearest thousandth.

Solution:

Press **[Y=]** and enter $4\ln(x) + 1$ next to **Y_1**=. Then enter $-2\ln(x-1)$ next to **Y_2=**. For a window, use the values 0 to 5 for x and –10 to 10 for y. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x = 1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 1.3385297. (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x \approx 1.339$.

Note:
Exercise:

Problem: Solve $5\log(x+2) = 4 - \log(x)$ graphically. Round to the nearest thousandth.

Solution:

$x \approx 3.049$

Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in [link] to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $y = \log_b(x)$	
Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$y = \log_b(x + c) + d$
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $\lvert a \rvert < 1$	$y = a\log_b(x)$
Reflect about the *x*-axis	$y = -\log_b(x)$
Reflect about the *y*-axis	$y = \log_b(-x)$
General equation for all translations	$y = a\log_b(x + c) + d$

Note:
Translations of Logarithmic Functions
All translations of the parent logarithmic function, $y = \log_b(x)$, have the form
Equation:

$$f(x) = a\log_b(x + c) + d$$

where the parent function, $y = \log_b(x), b > 1$,is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the *x*-axis when $a < 0$.

For $f(x) = \log(-x)$, the graph of the parent function is reflected about the *y*-axis.

Example:
Exercise:

Problem:
Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x + 4) + 5$?

Solution:

The vertical asymptote is at $x = -4$.

Analysis

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x = -4$.

Note:
Exercise:

Problem: What is the vertical asymptote of $f(x) = 3 + \ln(x - 1)$?

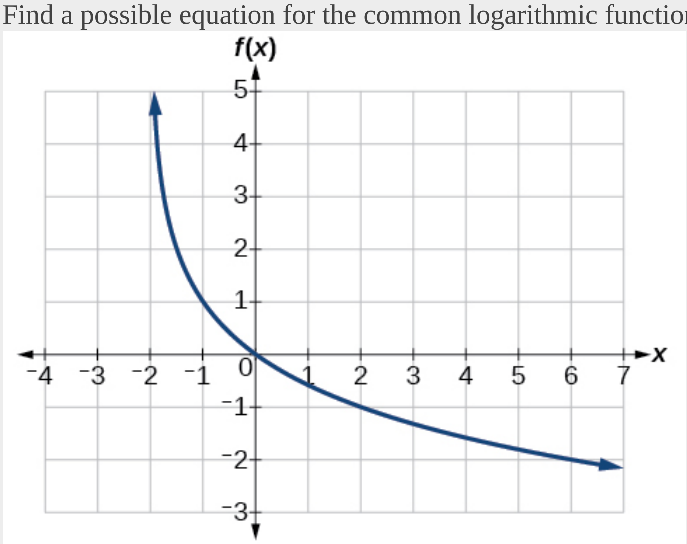
Solution:

$x = 1$

Example:
Exercise:

Problem:
Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in [link].



Solution:

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:
Equation:

$$f(x) = -a \log(x + 2) + k$$

It appears the graph passes through the points $(-1, 1)$ and $(2, -1)$. Substituting $(-1, 1)$,

Equation:

$$\begin{array}{ll} 1 = -a\log(-1+2)+k & \text{Substitute } (-1,1). \\ 1 = -a\log(1)+k & \text{Arithmetic.} \\ 1 = k & \log(1)=0. \end{array}$$

Next, substituting in $(2,–1)$,

Equation:

$$\begin{array}{ll} -1 = -a\log(2+2)+1 & \text{Plug in } (2,-1). \\ -2 = -a\log(4) & \text{Arithmetic.} \\ a = \frac{2}{\log(4)} & \text{Solve for } a. \end{array}$$

This gives us the equation $f(x) =– \frac{2}{\log(4)}\log(x+2)+1.$

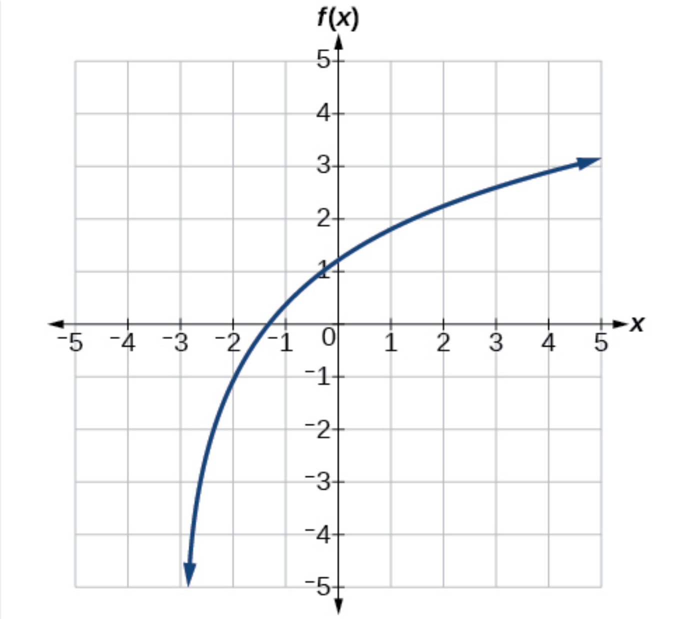
Analysis

We can verify this answer by comparing the function values in [link] with the points on the graph in [link].

x	−1	0	1	2	3
$f(x)$	1	0	−0.58496	−1	−1.3219
x	4	5	6	7	8
$f(x)$	−1.5850	−1.8074	−2	−2.1699	−2.3219

Note:
Exercise:

Problem: Give the equation of the natural logarithm graphed in [link].



Solution:

$f(x) = 2\ln(x+3) - 1$

Note:

Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?

Yes, if we know the function is a general logarithmic function. For example, look at the graph in [link]. The graph approaches $x = -3$ *(or thereabouts) more and more closely, so* $x = -3$ *is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right,* $\{x \mid x > -3\}$. *The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as* $x \to -3^+, f(x) \to -\infty$ *and as* $x \to \infty, f(x) \to \infty$.

Note:

Access these online resources for additional instruction and practice with graphing logarithms.

- Graph an Exponential Function and Logarithmic Function
- Match Graphs with Exponential and Logarithmic Functions
- Find the Domain of Logarithmic Functions

Key Equations

General Form for the Translation of the Parent Logarithmic Function $f(x)=\log_b(x)$	$f(x)=a\log_b(x+c)+d$

Key Concepts

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. See [link] and [link]
- The graph of the parent function $f(x)=\log_b(x)$ has an *x*-intercept at $(1,0)$,domain $(0,\infty)$,range $(-\infty,\infty)$,vertical asymptote $x=0$,and
  - if $b>1$,the function is increasing.
  - if $0<b<1$, the function is decreasing.

  See [link].
- The equation $f(x)=\log_b(x+c)$ shifts the parent function $y=\log_b(x)$ horizontally
  - left c units if $c>0$.
  - right c units if $c<0$.

  See [link].
- The equation $f(x)=\log_b(x)+d$ shifts the parent function $y=\log_b(x)$ vertically
  - up d units if $d>0$.
  - down d units if $d<0$.

  See [link].
- For any constant $a>0$, the equation $f(x)=a\log_b(x)$
  - stretches the parent function $y=\log_b(x)$ vertically by a factor of a if $|a|>1$.
  - compresses the parent function $y=\log_b(x)$ vertically by a factor of a if $|a|<1$.

  See [link] and [link].
- When the parent function $y=\log_b(x)$ is multiplied by -1, the result is a reflection about the *x*-axis. When the input is multiplied by -1, the result is a reflection about the *y*-axis.
  - The equation $f(x)=-\log_b(x)$ represents a reflection of the parent function about the *x*-axis.
  - The equation $f(x)=\log_b(-x)$ represents a reflection of the parent function about the *y*-axis.

  See [link].
  - A graphing calculator may be used to approximate solutions to some logarithmic equations See [link].
- All translations of the logarithmic function can be summarized by the general equation $f(x)=a\log_b(x+c)+d$. See [link].
- Given an equation with the general form $f(x)=a\log_b(x+c)+d$,we can identify the vertical asymptote $x=-c$ for the transformation. See [link].
- Using the general equation $f(x)=a\log_b(x+c)+d$,we can write the equation of a logarithmic function given its graph. See [link].

Section Exercises

Verbal

Exercise:

Problem:

The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

Solution:

Since the functions are inverses, their graphs are mirror images about the line $y = x.$ So for every point (a, b) on the graph of a logarithmic function, there is a corresponding point (b, a) on the graph of its inverse exponential function.

Exercise:

Problem: What type(s) of translation(s), if any, affect the range of a logarithmic function?

Exercise:

Problem: What type(s) of translation(s), if any, affect the domain of a logarithmic function?

Solution:

Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

Exercise:

Problem: Consider the general logarithmic function $f(x) = \log_b(x).$ Why can't x be zero?

Exercise:

Problem: Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Solution:

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

Algebraic

For the following exercises, state the domain and range of the function.

Exercise:

Problem: $f(x) = \log_3(x + 4)$

Exercise:

Problem: $h(x) = \ln\left(\frac{1}{2} - x\right)$

Solution:

Domain: $\left(-\infty, \frac{1}{2}\right)$; Range: $(-\infty, \infty)$

Exercise:

Problem: $g(x) = \log_5 (2x + 9) - 2$

Exercise:

Problem: $h(x) = \ln (4x + 17) - 5$

Solution:

Domain: $\left(-\frac{17}{4}, \infty\right)$; Range: $(-\infty, \infty)$

Exercise:

Problem: $f(x) = \log_2 (12 - 3x) - 3$

For the following exercises, state the domain and the vertical asymptote of the function.

Exercise:

Problem: $f(x) = \log_b(x - 5)$

Solution:

Domain: $(5, \infty)$; Vertical asymptote: $x = 5$

Exercise:

Problem: $g(x) = \ln(3 - x)$

Exercise:

Problem: $f(x) = \log(3x + 1)$

Solution:

Domain: $\left(-\frac{1}{3}, \infty\right)$; Vertical asymptote: $x = -\frac{1}{3}$

Exercise:

Problem: $f(x) = 3\log(-x) + 2$

Exercise:

Problem: $g(x) = -\ln(3x + 9) - 7$

Solution:

Domain: $(-3, \infty)$; Vertical asymptote: $x = -3$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

Exercise:

Problem: $f(x)=\ln(2-x)$

Exercise:

Problem: $f(x)=\log\left(x-\frac{3}{7}\right)$

Solution:

Domain: $\left(\frac{3}{7},\infty\right)$;
Vertical asymptote: $x=\frac{3}{7}$; End behavior: as $x\to\left(\frac{3}{7}\right)^{+}$, $f(x)\to-\infty$ and as $x\to\infty$, $f(x)\to\infty$

Exercise:

Problem: $h(x)=-\log(3x-4)+3$

Exercise:

Problem: $g(x)=\ln(2x+6)-5$

Solution:

Domain: $(-3,\infty)$; Vertical asymptote: $x=-3$;
End behavior: as $x\to-3^{+}$, $f(x)\to-\infty$ and as $x\to\infty$, $f(x)\to\infty$

Exercise:

Problem: $f(x)=\log_3(15-5x)+6$

For the following exercises, state the domain, range, and *x*- and *y*-intercepts, if they exist. If they do not exist, write DNE.

Exercise:

Problem: $h(x)=\log_4(x-1)+1$

Solution:

Domain: $(1,\infty)$; Range: $(-\infty,\infty)$; Vertical asymptote: $x=1$; *x*-intercept: $\left(\frac{5}{4},0\right)$; *y*-intercept: DNE

Exercise:

Problem: $f(x)=\log(5x+10)+3$

Exercise:

Problem: $g(x)=\ln(-x)-2$

Solution:

Domain: $(-\infty,0)$; Range: $(-\infty,\infty)$; Vertical asymptote: $x=0$; *x*-intercept: $\left(-e^{2},0\right)$; *y*-intercept: DNE

Exercise:

Problem: $f(x) = \log_2(x+2) - 5$

Exercise:

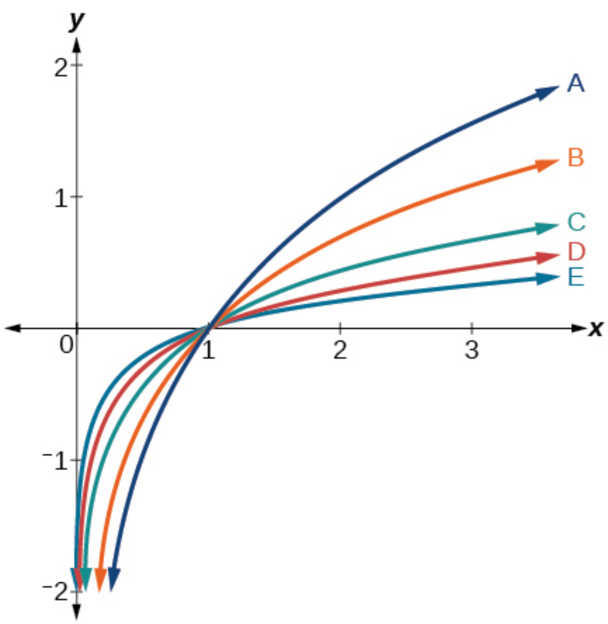
Problem: $h(x) = 3\ln(x) - 9$

Solution:

Domain: $(0, \infty)$; Range: $(-\infty, \infty)$; Vertical asymptote: $x = 0$; x-intercept: $\left(e^3, 0\right)$; y-intercept: DNE

Graphical

For the following exercises, match each function in [link] with the letter corresponding to its graph.



Exercise:

Problem: $d(x) = \log(x)$

Exercise:

Problem: $f(x) = \ln(x)$

Solution:

B

Exercise:

Problem: $g(x)=\log_2(x)$

Exercise:

Problem: $h(x)=\log_5(x)$

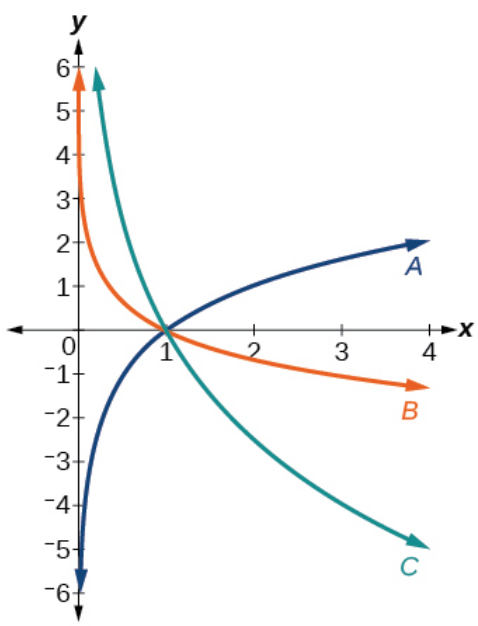
Solution:

C

Exercise:

Problem: $j(x)=\log_{25}(x)$

For the following exercises, match each function in [link] with the letter corresponding to its graph.



Exercise:

Problem: $f(x)=\log_{\frac{1}{3}}(x)$

Solution:

B

Exercise:

Problem: $g(x)=\log_2(x)$

Exercise:

Problem: $h(x)=\log_{\frac{3}{4}}(x)$

Solution:

C

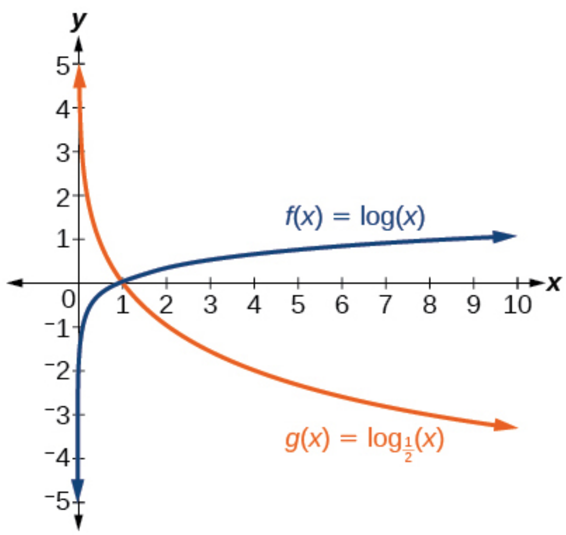
For the following exercises, sketch the graphs of each pair of functions on the same axis.
Exercise:

Problem: $f(x) = \log(x)$ and $g(x) = 10^x$

Exercise:

Problem: $f(x) = \log(x)$ and $g(x) = \log_{\frac{1}{2}}(x)$

Solution:



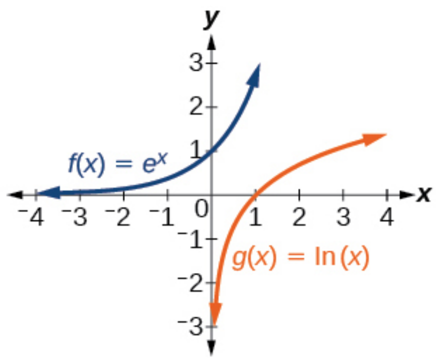
Exercise:

Problem: $f(x) = \log_4(x)$ and $g(x) = \ln(x)$

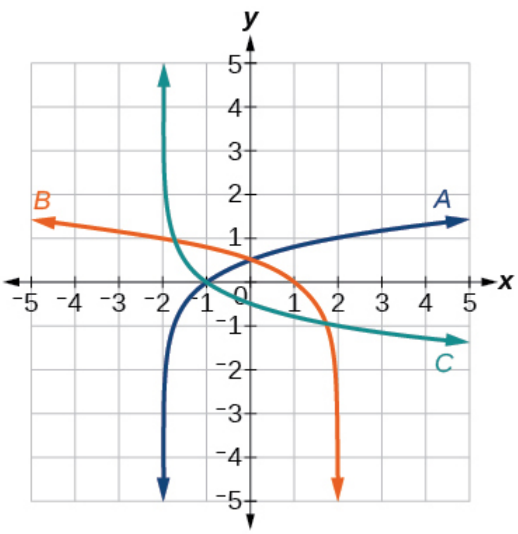
Exercise:

Problem: $f(x) = e^x$ and $g(x) = \ln(x)$

Solution:



For the following exercises, match each function in [link] with the letter corresponding to its graph.



Exercise:

Problem: $f(x) = \log_4(-x + 2)$

Exercise:

Problem: $g(x) = -\log_4(x + 2)$

Solution:

C

Exercise:

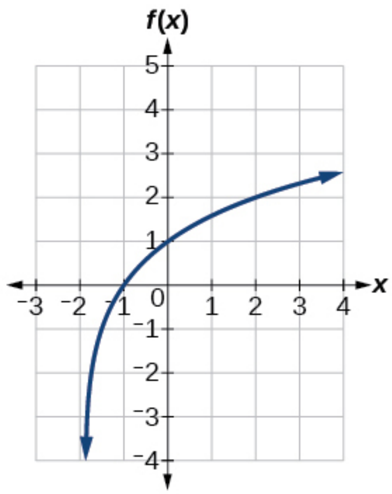
Problem: $h(x) = \log_4(x + 2)$

For the following exercises, sketch the graph of the indicated function.

Exercise:

Problem: $f(x) = \log_2(x+2)$

Solution:



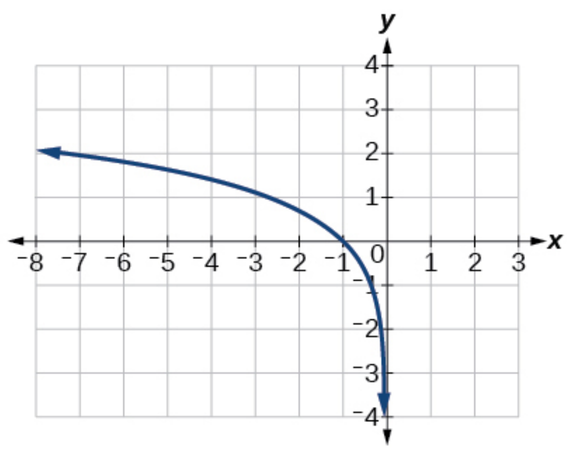
Exercise:

Problem: $f(x) = 2\log(x)$

Exercise:

Problem: $f(x) = \ln(-x)$

Solution:



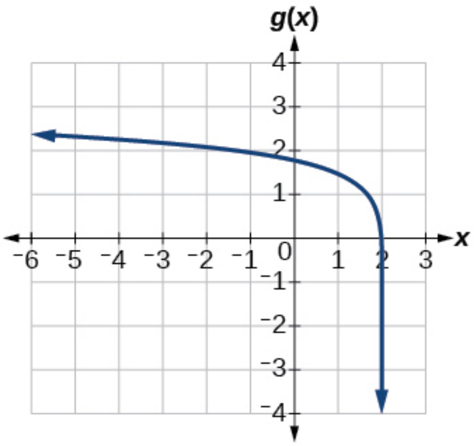
Exercise:

Problem: $g(x) = \log(4x+16) + 4$

Exercise:

Problem: $g(x) = \log(6 - 3x) + 1$

Solution:



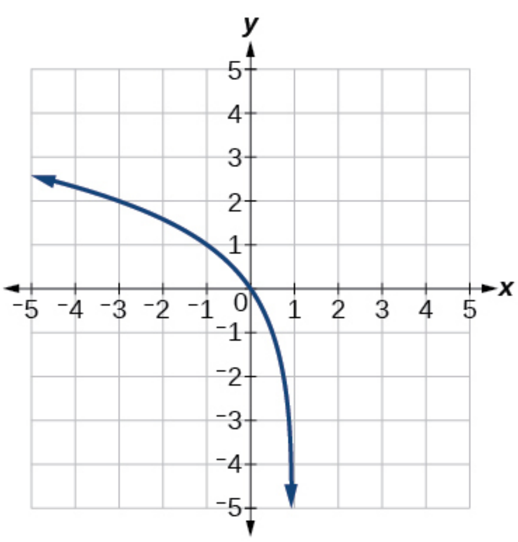
Exercise:

Problem: $h(x) = -\frac{1}{2}\ln(x + 1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

Exercise:

Problem: Use $y = \log_2(x)$ as the parent function.

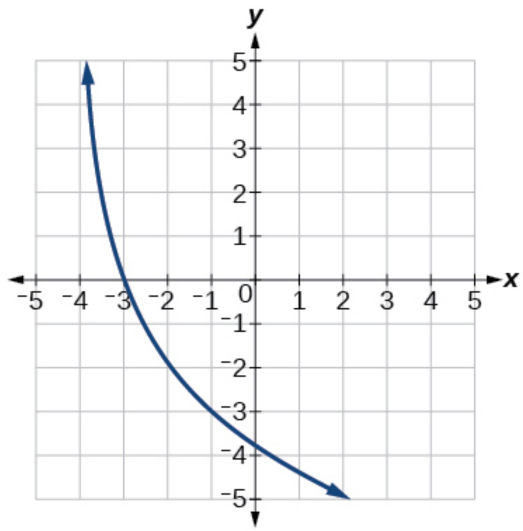


Solution:

$f(x) = \log_2(-(x - 1))$

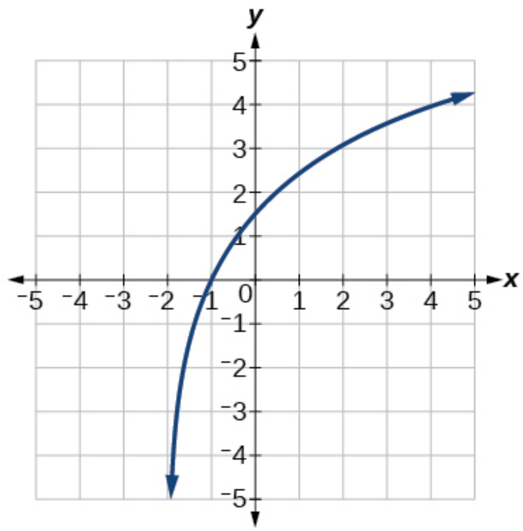
Exercise:

Problem: Use $f(x) = \log_3(x)$ as the parent function.



Exercise:

Problem: Use $f(x) = \log_4(x)$ as the parent function.

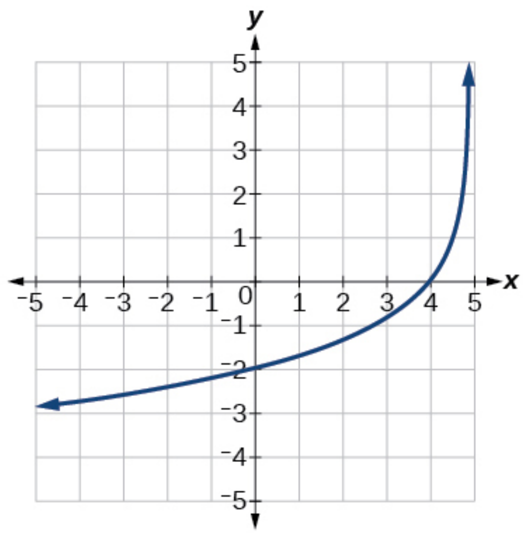


Solution:

$f(x) = 3\log_4(x + 2)$

Exercise:

Problem: Use $f(x) = \log_5(x)$ as the parent function.



Technology

For the following exercises, use a graphing calculator to find approximate solutions to each equation.

Exercise:

Problem: $\log(x-1)+2=\ln(x-1)+2$

Solution:

$x=2$

Exercise:

Problem: $\log(2x-3)+2=-\log(2x-3)+5$

Exercise:

Problem: $\ln(x-2)=-\ln(x+1)$

Solution:

$x\approx 2.303$

Exercise:

Problem: $2\ln(5x+1)=\frac{1}{2}\ln(-5x)+1$

Exercise:

Problem: $\frac{1}{3}\log(1-x)=\log(x+1)+\frac{1}{3}$

Solution:

$x \approx -0.472$

Extensions

Exercise:

Problem:

Let b be any positive real number such that $b \neq 1$. What must $\log_b 1$ be equal to? Verify the result.

Exercise:

Problem:

Explore and discuss the graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$. Make a conjecture based on the result.

Solution:

The graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$ appear to be the same; Conjecture: for any positive base $b \neq 1$, $\log_b(x) = -\log_{\frac{1}{b}}(x)$.

Exercise:

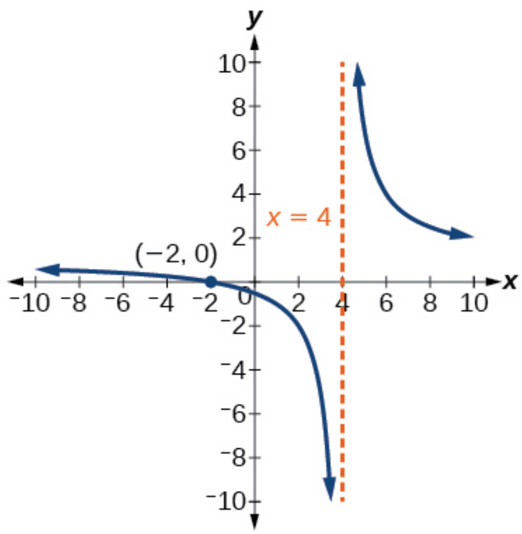
Problem: Prove the conjecture made in the previous exercise.

Exercise:

Problem: What is the domain of the function $f(x) = \ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.

Solution:

Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4} > 0$. From the graph of the function $f(x) = \frac{x+2}{x-4}$, note that the graph lies above the *x*-axis on the interval $(-\infty, -2)$ and again to the right of the vertical asymptote, that is $(4, \infty)$. Therefore, the domain is $(-\infty, -2) \cup (4, \infty)$.



Exercise:

Problem:

Use properties of exponents to find the *x*-intercepts of the function $f(x) = \log\left(x^2 + 4x + 4\right)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

Logarithmic Properties
In this section, you will:

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where a is the concentration of hydrogen ion in the solution
Equation:

$$\begin{aligned}\text{pH} &= -\log([H^+]) \\ &= \log\left(\frac{1}{[H^+]}\right)\end{aligned}$$

The equivalence of $-\log([H^+])$ and $\log\left(\frac{1}{[H^+]}\right)$ is one of the logarithm properties we will examine in this section.

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

Equation:

$$\begin{aligned} \log_b 1 &= 0 \\ \log_b b &= 1 \end{aligned}$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

Equation:

$$\begin{aligned} \log_b(b^x) &= x \\ b^{\log_b x} &= x, x > 0 \end{aligned}$$

For example, to evaluate $\log(100)$,we can rewrite the logarithm as $\log_{10}\left(10^2\right)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}\left(10^2\right) = 2$.

To evaluate $e^{\ln(7)}$,we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

Equation:

$$\log_b M = \log_b N \text{ if and only if } M = N$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x+5)$ for x. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x :

Equation:

$$\begin{aligned} 3x &= 2x + 5 && \text{Set the arguments equal.} \\ x &= 5 && \text{Subtract } 2x. \end{aligned}$$

But what about the equation $\log_3(3x) + \log_3(2x+5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

Equation:

$$\log_b(MN)=\log_b(M)+\log_b(N).$$

Let $m=\log_b M$ and $n=\log_b N.$ In exponential form, these equations are $b^m=M$ and $b^n=N.$ It follows that

Equation:

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for } M \text{ and } N. \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents.} \\ &= m+n && \text{Apply the inverse property of logs.} \\ &= \log_b(M)+\log_b(N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz).$ Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

Equation:

$$\log_b(wxyz)=\log_b w+\log_b x+\log_b y+\log_b z$$

Note:

The Product Rule for Logarithms

The **product rule for logarithms** can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

Equation:

$$\log_b(MN)=\log_b(M)+\log_b(N) \text{ for } b>0$$

Note:

Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Example:

Exercise:

Problem:

Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4)).$

Solution:

We begin by factoring the argument completely, expressing 30 as a product of primes.

Equation:

$$\log_3(30x(3x+4))=\log_3(2\cdot3\cdot5\cdot x\cdot(3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.
Equation:

$$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$$

Note:
Exercise:

Problem: Expand $\log_b(8k)$.

Solution:

$\log_b 2 + \log_b 2 + \log_b 2 + \log_b k = 3\log_b 2 + \log_b k$

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $\frac{x^a}{x^b} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show
Equation:

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that
Equation:

$$\begin{aligned} \log_b\left(\tfrac{M}{N}\right) &= \log_b\left(\tfrac{b^m}{b^n}\right) && \text{Substitute for } M \text{ and } N. \\ &= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents.} \\ &= m - n && \text{Apply the inverse property of logs.} \\ &= \log_b(M) - \log_b(N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,
Equation:

$$\begin{aligned} \log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator.} \\ &= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors.} \end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.
Equation:

$$
\begin{aligned}
\log\left(\tfrac{2x}{3}\right) &= \log(2x) - \log(3) \\
&= \log(2) + \log(x) - \log(3)
\end{aligned}
$$

Note:
The Quotient Rule for Logarithms
The **quotient rule for logarithms** can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.
Equation:

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Note:
Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Example:
Exercise:

Problem:
Using the Quotient Rule for Logarithms

Expand $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Solution:

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.
Equation:

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2\left(15x(x-1)\right) - \log_2\left((3x+4)(2-x)\right)$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.
Equation:

$$
\begin{aligned}
\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) \\
&= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log
\end{aligned}
$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$,$x > 1$,$x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Note:
Exercise:

Problem: Expand $\log_3 \left(\frac{7x^2+21x}{7x(x-1)(x-2)} \right)$.

Solution:

$\log_3 (x+3) - \log_3 (x-1) - \log_3 (x-2)$

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:
Equation:

$$\begin{aligned} \log_b \left(x^2\right) &= \log_b (x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x \end{aligned}$$

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,
Equation:

$$100 = 10^2 \qquad \sqrt{3} = 3^{\frac{1}{2}} \qquad \frac{1}{e} = e^{-1}$$

Note:
The Power Rule for Logarithms
The **power rule for logarithms** can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.
Equation:

$$\log_b (M^n) = n\log_b M$$

Note:

Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Example:
Exercise:

Problem:
Expanding a Logarithm with Powers

Expand $\log_2 x^5$.

Solution:

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.
Equation:

$$\log_2\left(x^5\right) = 5\log_2 x$$

Note:
Exercise:

Problem: Expand $\ln x^2$.

Solution:

$2\ln x$

Example:
Exercise:

Problem:
Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

Solution:

Expressing the argument as a power, we get $\log_3(25) = \log_3\left(5^2\right)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.
Equation:

$$\log_3\left(5^2\right) = 2\log_3(5)$$

Note:
Exercise:

Problem: Expand $\ln\left(\frac{1}{x^2}\right)$.

Solution:

$-2\ln(x)$

Example:
Exercise:

Problem:
Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution:

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$,we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

Note:
Exercise:

Problem: Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution:

$\log_3 16$

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:
Equation:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:
Equation:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b\left(AC^{-1}\right)\\ &= \log_b(A) + \log_b\left(C^{-1}\right)\\ &= \log_b A + (-1)\log_b C\\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example:
Exercise:

Problem:
Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs.

Solution:

First, because we have a quotient of two expressions, we can use the quotient rule:
Equation:

$$\ln\left(\frac{x^4 y}{7}\right) = \ln\left(x^4 y\right) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:
Equation:

$$\ln\left(x^4 y\right) - \ln(7) = \ln\left(x^4\right) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:
Equation:

$$\ln\left(x^4\right) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

Note:
Exercise:

Problem: Expand $\log\left(\frac{x^2 y^3}{z^4}\right)$.

Solution:

$2\log x + 3\log y - 4\log z$

Example:
Exercise:

Problem:
Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log\left(\sqrt{x}\right)$.

Solution:
Equation:

$$\begin{aligned}\log\left(\sqrt{x}\right) &= \log x^{\left(\frac{1}{2}\right)} \\ &= \tfrac{1}{2}\log x\end{aligned}$$

Note:
Exercise:

Problem: Expand $\ln\left(\sqrt[3]{x^2}\right)$.

Solution:

$\frac{2}{3}\ln x$

Note:
Can we expand $\ln\left(x^2+y^2\right)$?
No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Example:
Exercise:

Problem:
Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Solution:

We can expand by applying the Product and Quotient Rules.
Equation:

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Quotient Rule.} \\ &= \log_6 2^6 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) && \text{Simplify by writing } 64 \text{ as } 2^6. \\ &= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Power Rule.}\end{aligned}$$

Note:
Exercise:

Problem: Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

Solution:

$\frac{1}{2}\ln(x-1)+\ln(2x+1)-\ln(x+3)-\ln(x-3)$

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

Note:
Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Example:
Exercise:

Problem:
Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution:

Using the product and quotient rules
Equation:

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to
Equation:

$$\log_3(40)-\log_3(2)$$

Then, using the quotient rule
Equation:

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

Note:
Exercise:

Problem: Condense $\log 3 - \log 4 + \log 5 - \log 6$.

Solution:

$\log\left(\frac{3\cdot 5}{4\cdot 6}\right)$; can also be written $\log\left(\frac{5}{8}\right)$ by reducing the fraction to lowest terms.

Example:
Exercise:

Problem:
Condensing Complex Logarithmic Expressions

Condense $\log_2\left(x^2\right) + \frac{1}{2}\log_2(x-1) - 3\log_2\left((x+3)^2\right)$.

Solution:

We apply the power rule first:
Equation:

$$\log_2\left(x^2\right) + \frac{1}{2}\log_2(x-1) - 3\log_2\left((x+3)^2\right) = \log_2\left(x^2\right) + \log_2\left(\sqrt{x-1}\right) - \log_2\left((x+3)^6\right)$$

Next we apply the product rule to the sum:
Equation:

$$\log_2\left(x^2\right) + \log_2\left(\sqrt{x-1}\right) - \log_2\left((x+3)^6\right) = \log_2\left(x^2\sqrt{x-1}\right) - \log_2\left((x+3)^6\right)$$

Finally, we apply the quotient rule to the difference:
Equation:

$$\log_2\left(x^2\sqrt{x-1}\right) - \log_2\left((x+3)^6\right) = \log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

Note:
Exercise:

Problem: Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

Solution:

$\log\left(\frac{5(x-1)^3\sqrt{x}}{(7x-1)}\right)$

Example:
Exercise:

Problem:
Rewriting as a Single Logarithm

Rewrite $2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5)$ as a single logarithm.

Solution:

We apply the power rule first:
Equation:

$$\log(x+5) + \frac{1}{x}\log(3x+5) = \log\left(x^2\right) - \log\left(x+5\right)^4 + \log\left((3x+5)^{x^{-1}}\right)$$

Next we rearrange and apply the product rule to the sum:
Equation:

$$\log\left(x^2\right) - \log\left(x+5\right)^4 + \log\left((3x+5)^{x^{-1}}\right)$$

Equation:

$$= \log\left(x^2\right) + \log\left((3x+5)^{x^{-1}}\right) - \log\left(x+5\right)^4$$

Equation:

$$= \log\left(x^2(3x+5)^{x^{-1}}\right) - \log\left(x+5\right)^4$$

Finally, we apply the quotient rule to the difference:
Equation:

$$= \log\left(x^2(3x+5)^{x^{-1}}\right) - \log(x+5)^4 = \log\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4}$$

Note:
Exercise:

Problem: Condense $4(3\log(x) + \log(x+5) - \log(2x+3))$.

Solution:

$\log \frac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log \left(\frac{x^3(x+5)}{(2x+3)}\right)^4$.

Example:
Exercise:

Problem:
Applying of the Laws of Logs

Recall that, in chemistry, $\text{pH} = -\log[H^+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution:

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

Equation:

$$\text{pH} = -\log(2C)$$

Using the product rule of logs

Equation:

$$\text{pH} = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$,the new pH is

Equation:

$$\text{pH} = P - \log(2) \approx P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

Note:
Exercise:

Problem: How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Solution:

The pH increases by about 0.301.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e,we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and **power rule for logarithms**.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,we show
Equation:

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M.$ By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M.$ It follows that
Equation:

$$\begin{array}{lll} \log_n(b^y) & = \log_n M & \text{Apply the one-to-one property.} \\ y\log_n b & = \log_n M & \text{Apply the power rule for logarithms.} \\ y & = \frac{\log_n M}{\log_n b} & \text{Isolate } y. \\ \log_b M & = \frac{\log_n M}{\log_n b} & \text{Substitute for } y. \end{array}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.
Equation:

$$\begin{array}{lll} \log_5 36 & = \frac{\log(36)}{\log(5)} & \text{Apply the change of base formula using base 10.} \\ & \approx 2.2266 & \text{Use a calculator to evaluate to 4 decimal places.} \end{array}$$

Note:
The Change-of-Base Formula
The **change-of-base formula** can be used to evaluate a logarithm with any base.
For any positive real numbers $M, b,$ and n, where $n \neq 1$ and $b \neq 1$,
Equation:

$$\log_b M = \frac{\log_n M}{\log_n b}.$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.
Equation:

$$\log_b M = \frac{\ln M}{\ln b}$$

and
Equation:

$$\log_b M = \frac{\log M}{\log b}$$

Note:
Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$,has base e.
2. Rewrite the log as a quotient using the change-of-base formula
   - The numerator of the quotient will be a logarithm with base n and argument M.
   - The denominator of the quotient will be a logarithm with base n and argument b.

Example:
Exercise:

Problem:
Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

Solution:

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.
Equation:

$$\begin{aligned} \log_b M &= \frac{\ln M}{\ln b} \\ \log_5 3 &= \frac{\ln 3}{\ln 5} \end{aligned}$$

Note:
Exercise:

Problem: Change $\log_{0.5} 8$ to a quotient of natural logarithms.

Solution:

$\frac{\ln 8}{\ln 0.5}$

Note:
Can we change common logarithms to natural logarithms?
Yes. Remember that $\log 9$ *means* $\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

Example:
Exercise:

Problem:
Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

Solution:

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

Equation:

$$\begin{array}{lll} \log_2 10 = \frac{\ln 10}{\ln 2} & & \text{Apply the change of base formula using base } e. \\ \approx 3.3219 & & \text{Use a calculator to evaluate to 4 decimal places.} \end{array}$$

Note:
Exercise:

Problem: Evaluate $\log_5(100)$ using the change-of-base formula.

Solution:

$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$

Note:
Access these online resources for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms
- Expand Logarithmic Expressions
- Evaluate a Natural Logarithmic Expression

Key Equations

The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$
The Quotient Rule for Logarithms	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
The Power Rule for Logarithms	$\log_b(M^n) = n\log_b M$
The Change-of-Base Formula	$\log_b M = \frac{\log_n M}{\log_n b} \qquad n > 0, n \neq 1, b \neq 1$

Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See [link].

- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See [link].
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See [link], [link], and [link].
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See [link], [link], and [link].
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See [link], [link], [link], and [link].
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See [link].
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and e as the quotient of natural or common logs. That way a calculator can be used to evaluate. See [link].

Section Exercises

Verbal

Exercise:

Problem: How does the power rule for logarithms help when solving logarithms with the form $\log_b\left(\sqrt[n]{x}\right)$?

Solution:

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b(x)$.

Exercise:

Problem: What does the change-of-base formula do? Why is it useful when using a calculator?

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

Exercise:

Problem: $\log_b\left(7x\cdot 2y\right)$

Solution:

$\log_b\left(2\right) + \log_b\left(7\right) + \log_b\left(x\right) + \log_b\left(y\right)$

Exercise:

Problem: $\ln\left(3ab\cdot 5c\right)$

Exercise:

Problem: $\log_b\left(\frac{13}{17}\right)$

Solution:

$\log_b\left(13\right) - \log_b\left(17\right)$

Exercise:

Problem: $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

Exercise:

Problem: $\ln\left(\frac{1}{4^k}\right)$

Solution:

$-k\ln(4)$

Exercise:

Problem: $\log_2\left(y^x\right)$

For the following exercises, condense to a single logarithm if possible.

Exercise:

Problem: $\ln\left(7\right)+\ln\left(x\right)+\ln\left(y\right)$

Solution:

$\ln\left(7xy\right)$

Exercise:

Problem: $\log_3(2)+\log_3(a)+\log_3(11)+\log_3(b)$

Exercise:

Problem: $\log_b(28)-\log_b(7)$

Solution:

$\log_b(4)$

Exercise:

Problem: $\ln\left(a\right)-\ln\left(d\right)-\ln\left(c\right)$

Exercise:

Problem: $-\log_b\left(\frac{1}{7}\right)$

Solution:

$\log_b\left(7\right)$

Exercise:

Problem: $\frac{1}{3}\ln\left(8\right)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

Exercise:

Problem: $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

Solution:

$15\log(x)+13\log(y)-19\log(z)$

Exercise:

Problem: $\ln\left(\frac{a^{-2}}{b^{-4}c^{5}}\right)$

Exercise:

Problem: $\log\left(\sqrt{x^3y^{-4}}\right)$

Solution:

$\frac{3}{2}\log(x)-2\log(y)$

Exercise:

Problem: $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

Exercise:

Problem: $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

Solution:

$\frac{8}{3}\log(x)+\frac{14}{3}\log(y)$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

Exercise:

Problem: $\log\left(2x^4\right)+\log\left(3x^5\right)$

Exercise:

Problem: $\ln(6x^9)-\ln(3x^2)$

Solution:

$\ln(2x^7)$

Exercise:

Problem: $2\log(x)+3\log(x+1)$

Exercise:

Problem: $\log(x)-\frac{1}{2}\log(y)+3\log(z)$

Solution:

$\log\left(\frac{xz^3}{\sqrt{y}}\right)$

Exercise:

Problem: $4\log_7(c) + \frac{\log_7(a)}{3} + \frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

Exercise:

Problem: $\log_7(15)$ to base e

Solution:

$\log_7(15) = \frac{\ln(15)}{\ln(7)}$

Exercise:

Problem: $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6) = a$ and $\log_5(11) = b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

Exercise:

Problem: $\log_{11}(5)$

Solution:

$\log_{11}(5) = \frac{\log_5(5)}{\log_5(11)} = \frac{1}{b}$

Exercise:

Problem: $\log_6(55)$

Exercise:

Problem: $\log_{11}\left(\frac{6}{11}\right)$

Solution:

$\log_{11}\left(\frac{6}{11}\right) = \frac{\log_5\left(\frac{6}{11}\right)}{\log_5(11)} = \frac{\log_5(6)-\log_5(11)}{\log_5(11)} = \frac{a-b}{b} = \frac{a}{b} - 1$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

Exercise:

Problem: $\log_3\left(\frac{1}{9}\right) - 3\log_3(3)$

Exercise:

Problem: $6\log_8(2) + \frac{\log_8(64)}{3\log_8(4)}$

Solution:

3

Exercise:

Problem: $2\log_9(3) - 4\log_9(3) + \log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

Exercise:

Problem: $\log_3(22)$

Solution:

2.81359

Exercise:

Problem: $\log_8(65)$

Exercise:

Problem: $\log_6(5.38)$

Solution:

0.93913

Exercise:

Problem: $\log_4\left(\frac{15}{2}\right)$

Exercise:

Problem: $\log_{\frac{1}{2}}(4.7)$

Solution:

-2.23266

Extensions

Exercise:

Problem:

Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6) + \log_{12}(x+2) = 2.$ Show the steps for solving.

Exercise:

Problem:

Use the quotient rule for logarithms to find all x values such that $\log_6(x+2) - \log_6(x-3) = 1$. Show the steps for solving.

Solution:

$x = 4;$ By the quotient rule:$\log_6(x+2) - \log_6(x-3) = \log_6\left(\frac{x+2}{x-3}\right) = 1$.

Rewriting as an exponential equation and solving for x :

$$\begin{aligned} 6^1 &= \frac{x+2}{x-3} \\ 0 &= \frac{x+2}{x-3} - 6 \\ 0 &= \frac{x+2}{x-3} - \frac{6(x-3)}{(x-3)} \\ 0 &= \frac{x+2-6x+18}{x-3} \\ 0 &= \frac{x-4}{x-3} \\ x &= 4 \end{aligned}$$

Checking, we find that $\log_6(4+2) - \log_6(4-3) = \log_6(6) - \log_6(1)$ is defined, so $x = 4$.

Exercise:

Problem:

Can the power property of logarithms be derived from the power property of exponents using the equation $b^x = m$? If not, explain why. If so, show the derivation.

Exercise:

Problem: Prove that $\log_b(n) = \frac{1}{\log_n(b)}$ for any positive integers $b > 1$ and $n > 1$.

Solution:

Let b and n be positive integers greater than 1. Then, by the change-of-base formula, $\log_b(n) = \frac{\log_n(n)}{\log_n(b)} = \frac{1}{\log_n(b)}$.

Exercise:

Problem: Does $\log_{81}(2401) = \log_3(7)$? Verify the claim algebraically.

Glossary

change-of-base formula
: a formula for converting a logarithm with any base to a quotient of logarithms with any other base.

power rule for logarithms
: a rule of logarithms that states that the log of a power is equal to the product of the exponent and the log of its base

product rule for logarithms
: a rule of logarithms that states that the log of a product is equal to a sum of logarithms

quotient rule for logarithms
: a rule of logarithms that states that the log of a quotient is equal to a difference of logarithms

Introduction to Systems of Equations and Inequalities class="introduction"

Enigma machines like this one, once owned by Italian dictator Benito Mussolini, were used by government and military officials for enciphering and deciphering top-secret communications during World War II. (credit: Dave Addey, Flickr)

By 1943, it was obvious to the Nazi regime that defeat was imminent unless it could build a weapon with unlimited destructive power, one that had never been seen before in the history of the world. In September, Adolf Hitler ordered German scientists to begin building an atomic bomb. Rumors and whispers began to spread from across the ocean. Refugees and diplomats told of the experiments happening in Norway. However, Franklin D. Roosevelt wasn't sold, and even doubted British Prime Minister Winston Churchill's warning. Roosevelt wanted undeniable proof. Fortunately, he soon received the proof he wanted when a group of mathematicians cracked the "Enigma" code, proving beyond a doubt that Hitler was building an atomic bomb. The next day, Roosevelt gave the order that the United States begin work on the same.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of

the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

Systems of Linear Equations: Two Variables
In this section, you will:

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

(credit: Thomas Sørenes)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. A **system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

Equation:

$$\begin{aligned} 2x + y &= \ 15 \\ 3x - y &= \ 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair (4, 7) is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

Equation:

$$\begin{array}{ll} 2(4) + (7) = 15 & \text{True} \\ 3(4) - (7) = 5 & \text{True} \end{array}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A consistent

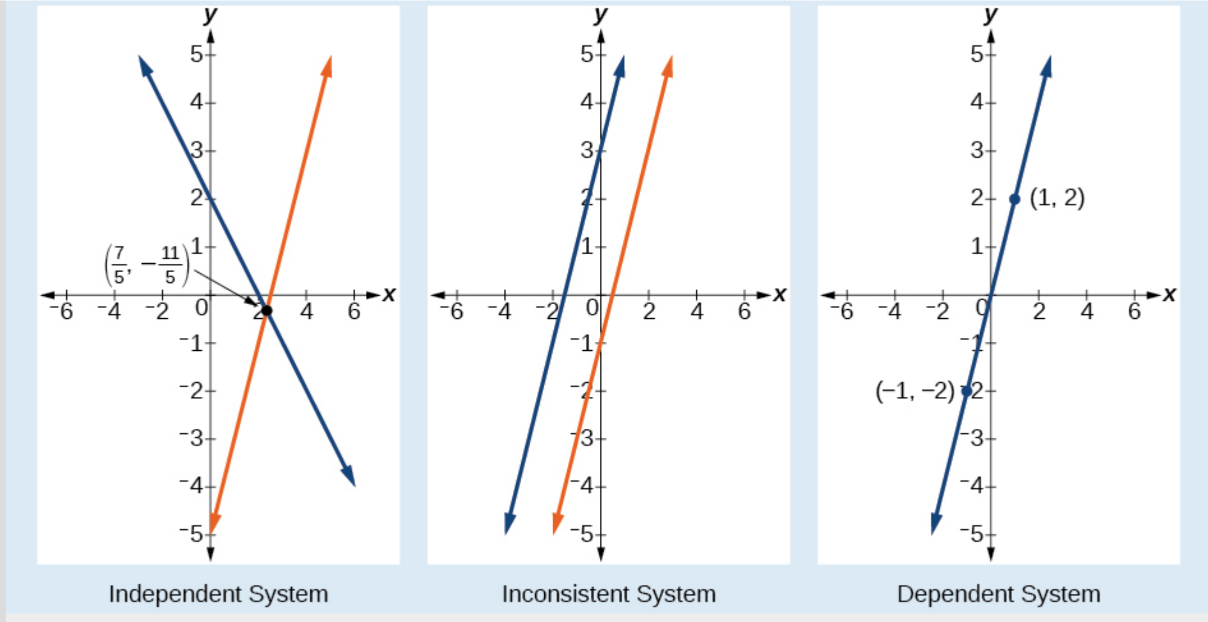
system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same *y*-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different *y*-intercepts. There are no points common to both lines; hence, there is no solution to the system.

Note:
Types of Linear Systems
There are three types of systems of linear equations in two variables, and three types of solutions.

- An **independent system** has exactly one solution pair $(x, y).$ The point where the two lines intersect is the only solution.
- An **inconsistent system** has no solution. Notice that the two lines are parallel and will never intersect.
- A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

[link] compares graphical representations of each type of system.



Note:
Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution.

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example:
Exercise:

Problem:
Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.
Equation:

$$x + 3y = 8$$
$$2x - 9 = y$$

Solution:

Substitute the ordered pair $(5, 1)$ into both equations.

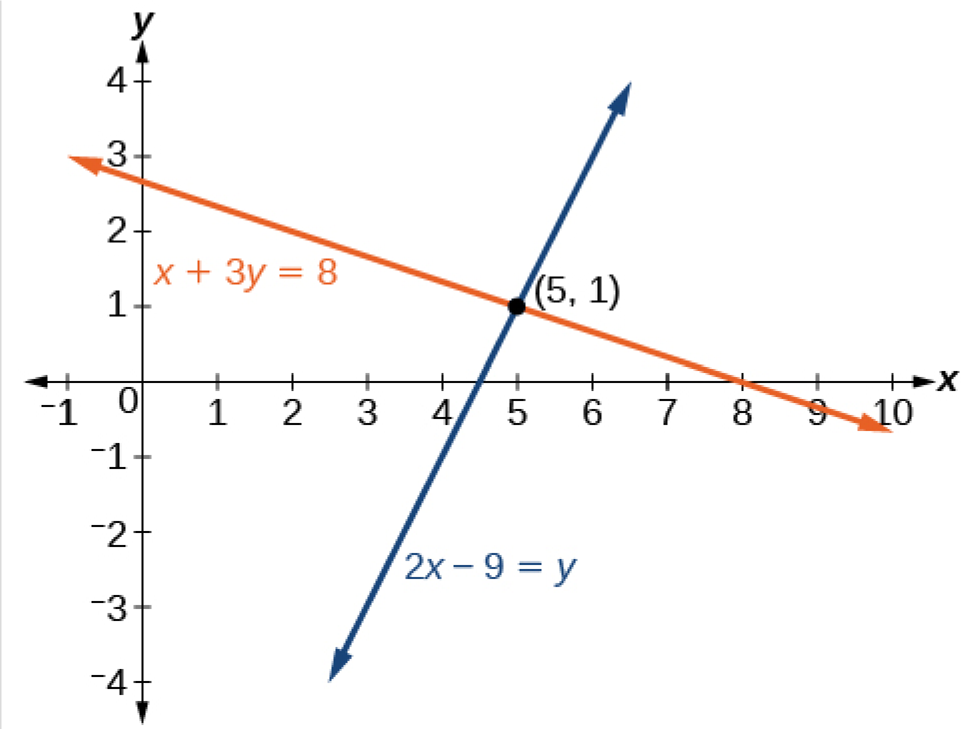
Equation:

$$\begin{aligned}(5) + 3(1) &= 8 \\ 8 &= 8 \qquad \text{True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \qquad \text{True}\end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See [link].



Note:
Exercise:

Problem:

Determine whether the ordered pair $(8,5)$ is a solution to the following system.

Equation:

$$5x - 4y = 20$$
$$2x + 1 = 3y$$

Solution:

Not a solution.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

Example:
Exercise:

Problem:
Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.
Equation:

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Solution:

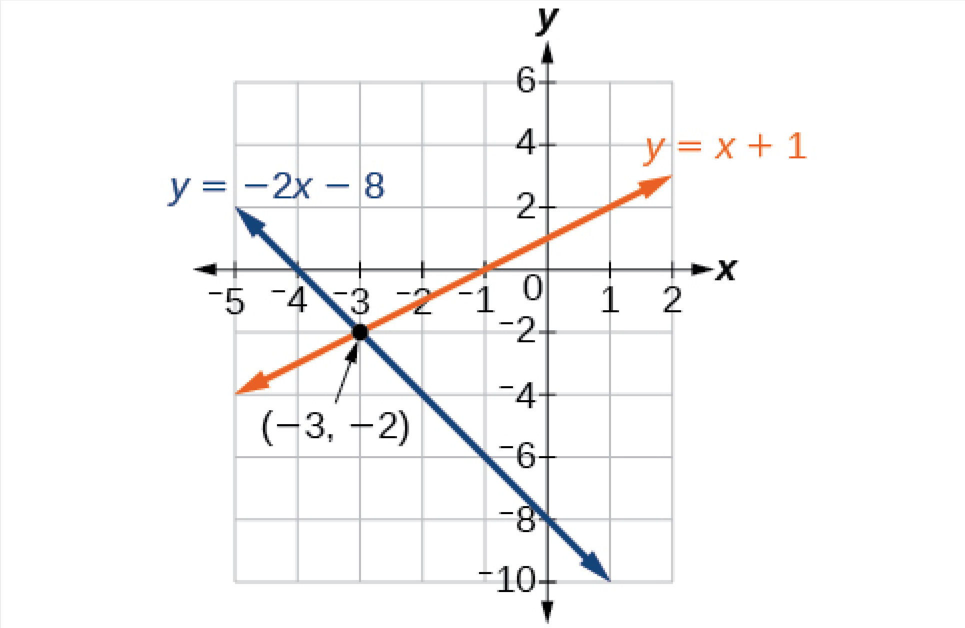
Solve the first equation for y.
Equation:

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for y.
Equation:

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in [link].



The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

Equation:

$$\begin{aligned} 2(-3)+(-2) &= -8 \\ -8 &= -8 \quad \text{True} \\ (-3)-(-2) &= -1 \\ -1 &= -1 \quad \text{True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$,so the system is independent.

Note:
Exercise:

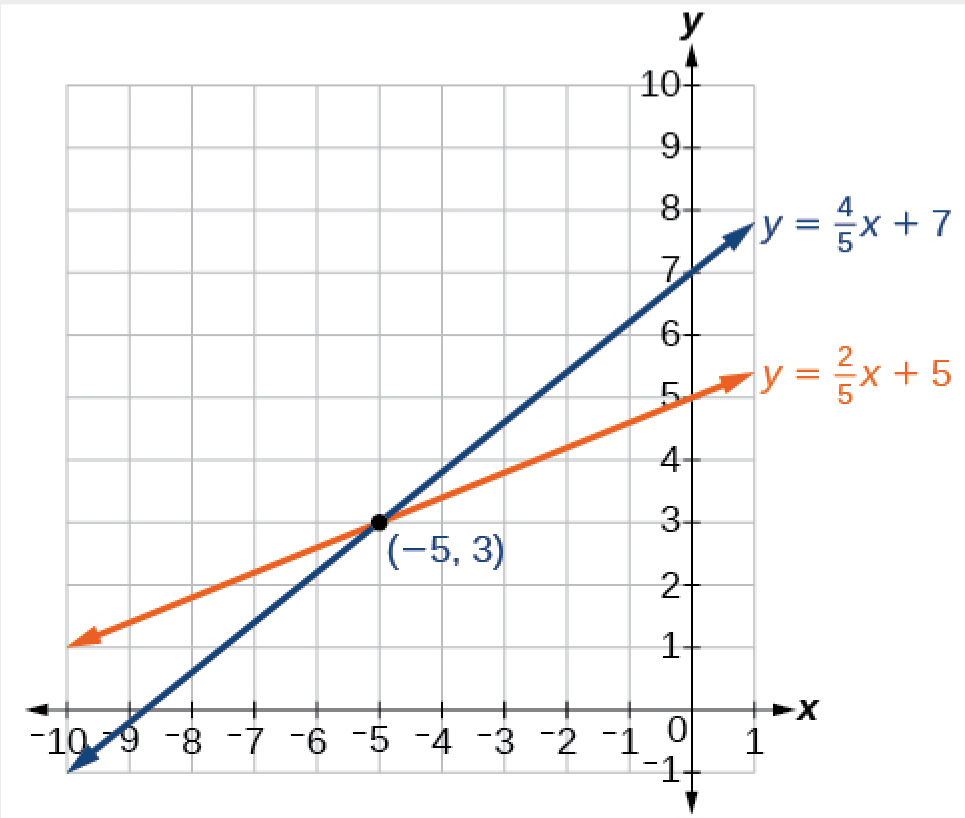
Problem: Solve the following system of equations by graphing.

Equation:

$$\begin{aligned} 2x - 5y &= -25 \\ -4x + 5y &= 35 \end{aligned}$$

Solution:

The solution to the system is the ordered pair $(-5, 3)$.



Note:
Can graphing be used if the system is inconsistent or dependent?
Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no

solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

Note:
Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example:
Exercise:

Problem:

Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

Equation:

$$\begin{aligned} -x+y&=-5 \\ 2x-5y&=1 \end{aligned}$$

Solution:

First, we will solve the first equation for y.

Equation:

$$\begin{aligned} -x+y&=-5 \\ y&=x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

Equation:

$$\begin{aligned} 2x-5y&=1 \\ 2x-5(x-5)&=1 \\ 2x-5x+25&=1 \\ -3x&=-24 \\ x&=8 \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for y.

Equation:

$$\begin{aligned} -(8)+y&=-5 \\ y&=3 \end{aligned}$$

Our solution is $(8,3)$.

Check the solution by substituting $(8, 3)$ into both equations.
Equation:

$$\begin{aligned} -x + y &= -5 \\ -(8) + (3) &= -5 \qquad \text{True} \\ 2x - 5y &= 1 \\ 2(8) - 5(3) &= 1 \qquad \text{True} \end{aligned}$$

Note:
Exercise:

Problem: Solve the following system of equations by substitution.
Equation:

$$\begin{aligned} x &= y + 3 \\ 4 &= 3x - 2y \end{aligned}$$

Solution:

$(-2, -5)$

Note:
Can the substitution method be used to solve any linear system in two variables?
Yes, but the method works best if one of the equations contains a coefficient of 1 or –1 so that we do not have to deal with fractions.

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the **addition method**. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

Note:
Given a system of equations, solve using the addition method.

1. Write both equations with x- and y-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example:
Exercise:

Problem:
Solving a System by the Addition Method

Solve the given system of equations by addition.
Equation:

$$\begin{aligned} x + 2y &= -1 \\ -x + y &= 3 \end{aligned}$$

Solution:

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, –1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.
Equation:

$$\begin{array}{r} x + 2y = -1 \\ -x + y = 3 \\ \hline 3y = 2 \end{array}$$

Now that we have eliminated x, we can solve the resulting equation for y.
Equation:

$$\begin{aligned} 3y &= 2 \\ y &= \tfrac{2}{3} \end{aligned}$$

Then, we substitute this value for y into one of the original equations and solve for x.
Equation:

$$
\begin{aligned}
-x+y&=3\\
-x+\tfrac{2}{3}&=3\\
-x&=3-\tfrac{2}{3}\\
-x&=\tfrac{7}{3}\\
x&=-\tfrac{7}{3}
\end{aligned}
$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

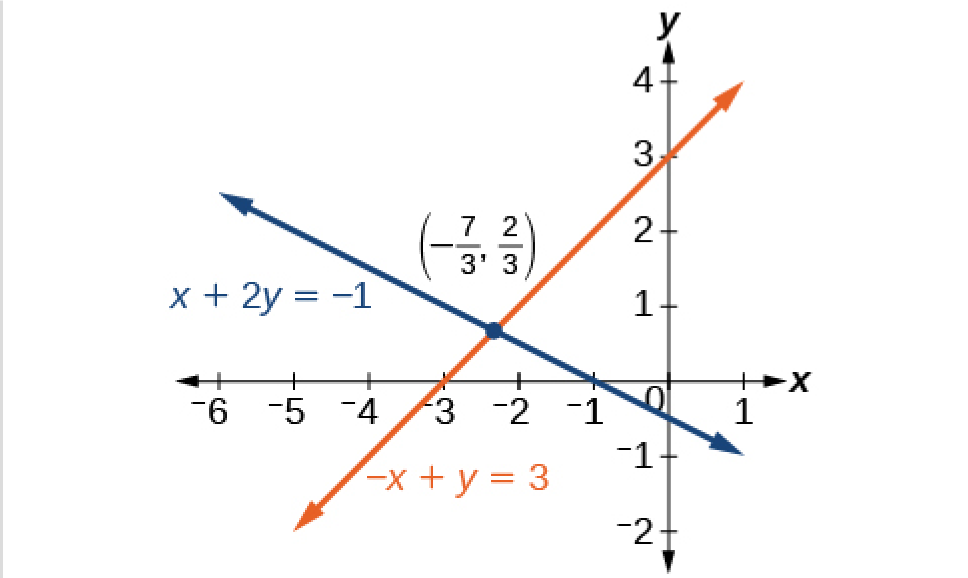
Check the solution in the first equation.

Equation:

$$
\begin{aligned}
x+2y&=-1\\
\left(-\tfrac{7}{3}\right)+2\left(\tfrac{2}{3}\right)&=\\
-\tfrac{7}{3}+\tfrac{4}{3}&=\\
-\tfrac{3}{3}&=\\
-1&=-1 \qquad \text{True}
\end{aligned}
$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See [link] to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.



Example:
Exercise:

Problem:
Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.
Equation:

$$3x + 5y = -11$$

$$x - 2y = 11$$

Solution:

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3, the *x*-terms will add to zero.
Equation:

$$\begin{aligned} x-2y &= 11 \\ -3(x-2y) &= -3(11) && \text{Multiply both sides by } -3. \\ -3x+6y &= -33 && \text{Use the distributive property.} \end{aligned}$$

Now, let’s add them.
Equation:

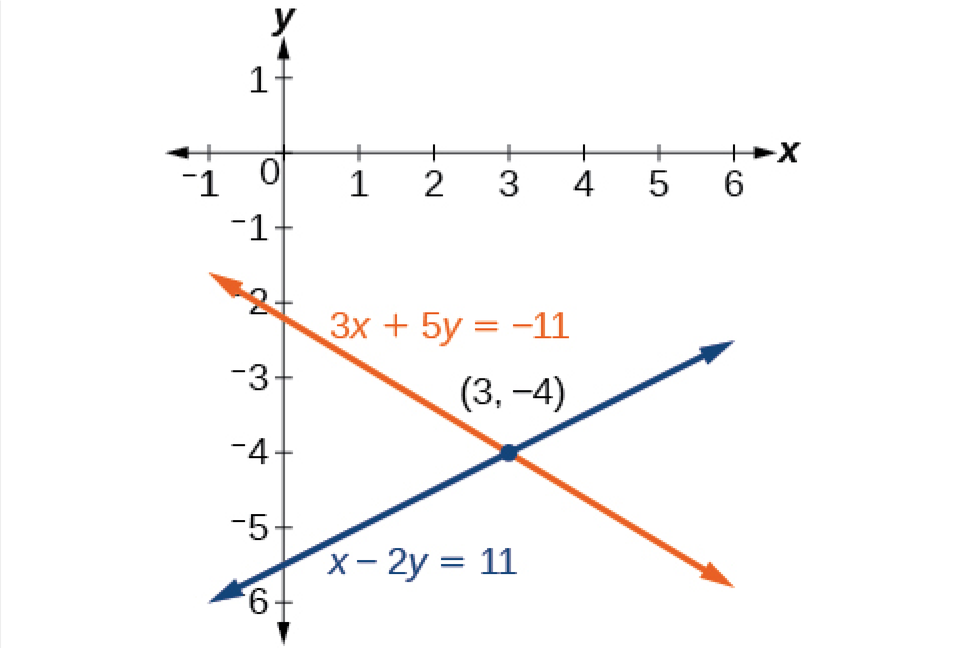
$$\begin{array}{r} 3x+5y=-11 \\ -3x+6y=-33 \\ \hline 11y=-44 \\ y=-4 \end{array}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.
Equation:

$$\begin{aligned} 3x+5y &= -11 \\ 3x+5(-4) &= -11 \\ 3x-20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See [link]. Check the solution in the original second equation.
Equation:

$$\begin{aligned} x-2y &= 11 \\ (3)-2(-4) &= 3+8 \\ 11 &= 11 && \text{True} \end{aligned}$$



Note:
Exercise:

Problem: Solve the system of equations by addition.
Equation:

$$\begin{aligned} 2x - 7y &= 2 \\ 3x + y &= -20 \end{aligned}$$

Solution:

$(-6, -2)$

Example:
Exercise:

Problem:
Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.
Equation:

$$2x + 3y = -16$$
$$5x - 10y = 30$$

Solution:

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.
Equation:

$$-5(2x + 3y) = -5(-16)$$
$$-10x - 15y = 80$$
$$2(5x - 10y) = 2(30)$$
$$10x - 20y = 60$$

Then, we add the two equations together.
Equation:

$$\begin{array}{r} -10x - 15y = 80 \\ 10x - 20y = 60 \\ \hline -35y = 140 \\ y = -4 \end{array}$$

Substitute $y = -4$ into the original first equation.

Equation:

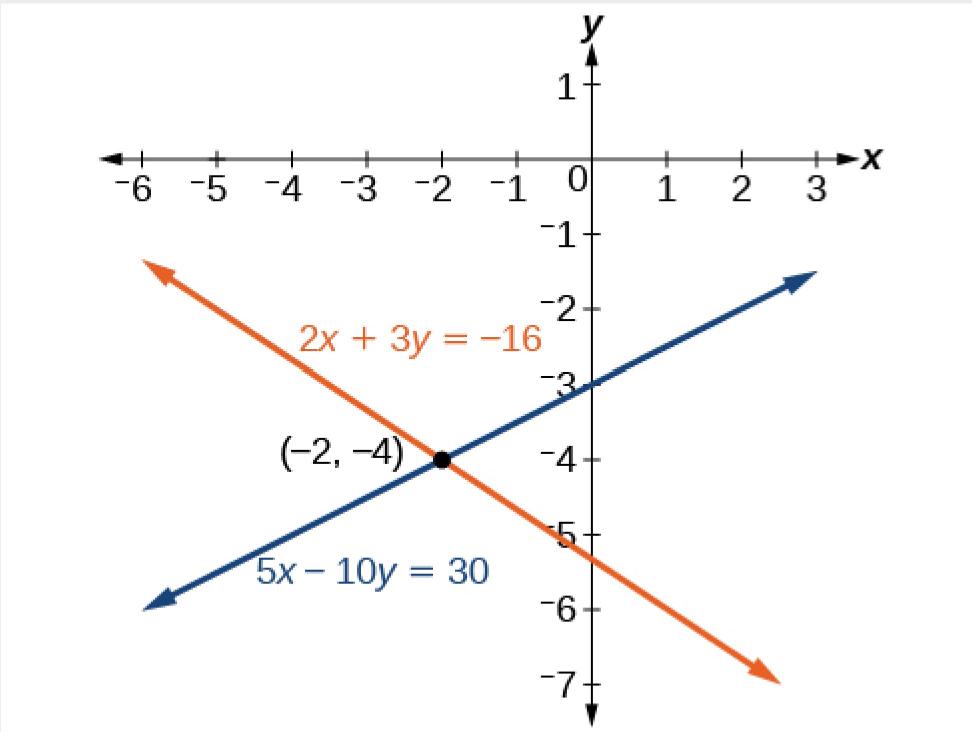
$$\begin{aligned} 2x + 3(-4) &= -16 \\ 2x - 12 &= -16 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The solution is $(-2, -4)$. Check it in the other equation.

Equation:

$$\begin{aligned} 5x - 10y &= 30 \\ 5(-2) - 10(-4) &= 30 \\ -10 + 40 &= 30 \\ 30 &= 30 \end{aligned}$$

See [link].



Example:
Exercise:

Problem:
Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.
Equation:

$$\begin{aligned} \frac{x}{3} + \frac{y}{6} &= 3 \\ \frac{x}{2} - \frac{y}{4} &= \ 1 \end{aligned}$$

Solution:

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.
Equation:

$$\begin{aligned} 6\left(\frac{x}{3} + \frac{y}{6}\right) &= 6(3) \\ 2x + y &= 18 \\ 4\left(\frac{x}{2} - \frac{y}{4}\right) &= 4(1) \\ 2x - y &= 4 \end{aligned}$$

Now multiply the second equation by -1 so that we can eliminate the *x*-variable.
Equation:

$$\begin{aligned} -1(2x - y) &= -1(4) \\ -2x + y &= -4 \end{aligned}$$

Add the two equations to eliminate the *x*-variable and solve the resulting equation.
Equation:

$$\begin{array}{r} 2x + y = 18 \\ \underline{-2x + y = -4} \\ 2y = 14 \\ y = 7 \end{array}$$

Substitute $y = 7$ into the first equation.
Equation:

$$\begin{aligned} 2x + (7) &= 18 \\ 2x &= 11 \\ x &= \frac{11}{2} \\ &= 5.5 \end{aligned}$$

The solution is $\left(\frac{11}{2}, 7\right)$. Check it in the other equation.
Equation:

$$\begin{aligned} \frac{x}{2} - \frac{y}{4} &= 1 \\ \frac{\frac{11}{2}}{2} - \frac{7}{4} &= 1 \\ \frac{11}{4} - \frac{7}{4} &= 1 \\ \frac{4}{4} &= 1 \end{aligned}$$

Note:
Exercise:

Problem: Solve the system of equations by addition.
Equation:

$$\begin{aligned} 2x + 3y &= 8 \\ 3x + 5y &= 10 \end{aligned}$$

Solution:

$$(10, -4)$$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

Example:
Exercise:

Problem:
Solving an Inconsistent System of Equations

Solve the following system of equations.
Equation:

$$\begin{aligned} x &= 9 - 2y \\ x + 2y &= 13 \end{aligned}$$

Solution:

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

Equation:

$$\begin{aligned} x + 2y &= 13 \\ (9 - 2y) + 2y &= 13 \\ 9 + 0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

Equation:

$$\begin{aligned} x &= 9 - 2y \\ 2y &= -x + 9 \\ y &= -\frac{1}{2}x + \frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

Equation:

$$\begin{aligned} x + 2y &= 13 \\ 2y &= -x + 13 \\ y &= -\frac{1}{2}x + \frac{13}{2} \end{aligned}$$

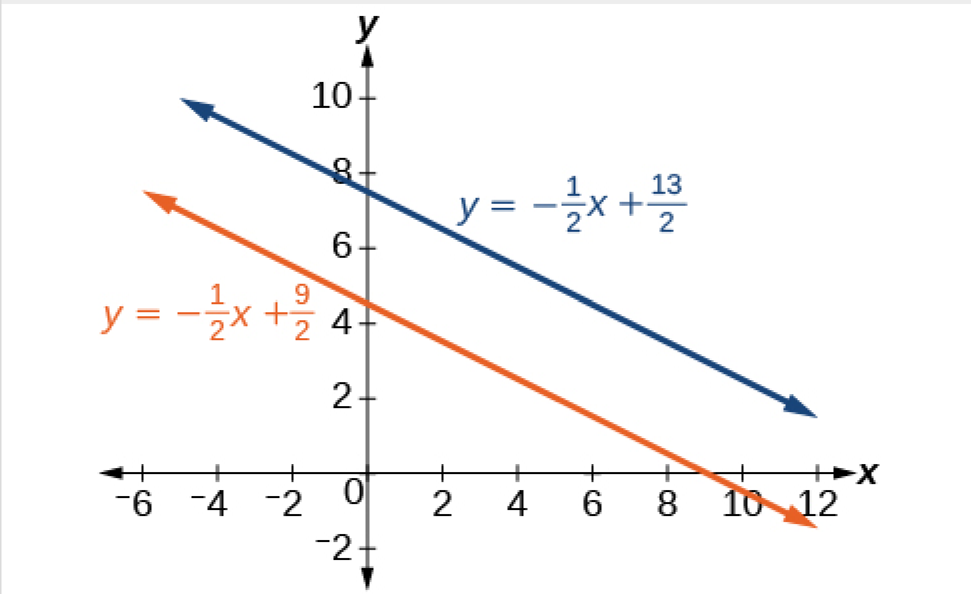
Comparing the equations, we see that they have the same slope but different *y*-intercepts. Therefore, the lines are parallel and do not intersect.

Equation:

$$y = -\frac{1}{2}x + \frac{9}{2}$$
$$y = -\frac{1}{2}x + \frac{13}{2}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in [link].



Note:
Exercise:

Problem: Solve the following system of equations in two variables.
Equation:

$$2y - 2x = 2$$
$$2y - 2x = 6$$

Solution:

No solution. It is an inconsistent system.

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0 = 0$.

Example:
Exercise:

Problem:
Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the addition method.
Equation:

$$\begin{aligned} x + 3y &= 2 \\ 3x + 9y &= 6 \end{aligned}$$

Solution:

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.
Equation:

$$\begin{aligned} x+3y&=2 \\ (-3)(x+3y)&=(-3)(2) \\ -3x-9y&=-6 \end{aligned}$$

Now add the equations.
Equation:

$$\begin{array}{rrl} & -3x-9y & =-6 \\ + & 3x+9y & =6 \\ \hline & 0 & =0 \end{array}$$

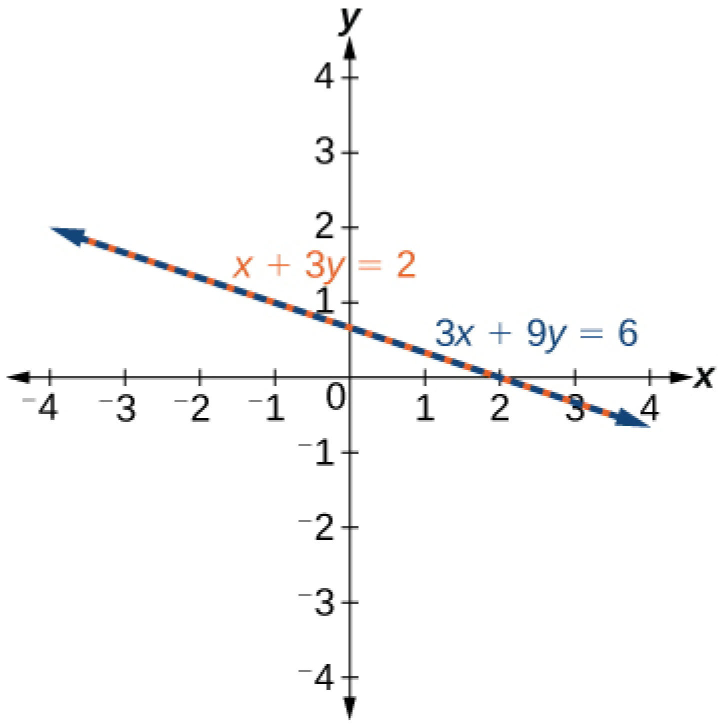
We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.
Equation:

$$\begin{aligned} x+3y&=2 \\ 3y&=-x+2 \\ y&=-\frac{1}{3}x+\frac{2}{3} \\ 3x+9y&=6 \\ 9y&=-3x+6 \\ y&=-\frac{3}{9}x+\frac{6}{9} \\ y&=-\frac{1}{3}x+\frac{2}{3} \end{aligned}$$

See [link]. Notice the results are the same. The general solution to the system is $\left(x\ ,-\ \frac{1}{3}x+\frac{2}{3}\right)$.



Note:
Exercise:

Problem: Solve the following system of equations in two variables.
Equation:

$$\begin{aligned} y – 2x &= 5 \\ -3y + 6x &= -15 \end{aligned}$$

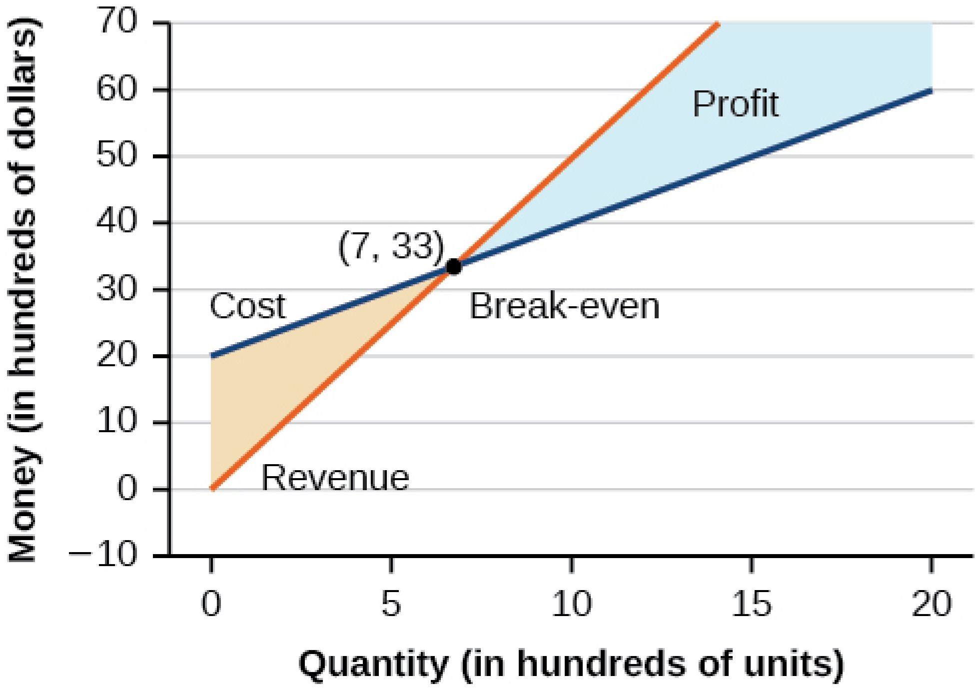
Solution:

The system is dependent so there are infinite solutions of the form $(x, 2x + 5)$.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer’s **revenue function** is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where x =quantity and p =price. The revenue function is shown in orange in [link].

The **cost function** is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in [link]. The x-axis represents quantity in hundreds of units. The *y*-axis represents either cost or revenue in hundreds of dollars.



The point at which the two lines intersect is called the **break-even point**. We can see from the graph that if 700 units are produced, the cost is $3,300 and the revenue is also $3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The **profit function** is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example:
Exercise:

Problem:
Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35{,}000$ and the revenue function $R(x) = 1.55x$,find the break-even point and the profit function.

Solution:

Write the system of equations using y to replace function notation.
Equation:

$$\begin{aligned} y &= 0.85x + 35{,}000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35{,}000$ from the first equation into the second equation and solve for x.
Equation:

$$\begin{aligned} 0.85x + 35{,}000 &= 1.55x \\ 35{,}000 &= 0.7x \\ 50{,}000 &= x \end{aligned}$$

Then, we substitute $x = 50{,}000$ into either the cost function or the revenue function.

Equation:

$$1.55\,(50{,}000) = 77{,}500$$

The break-even point is $(50{,}000, 77{,}500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

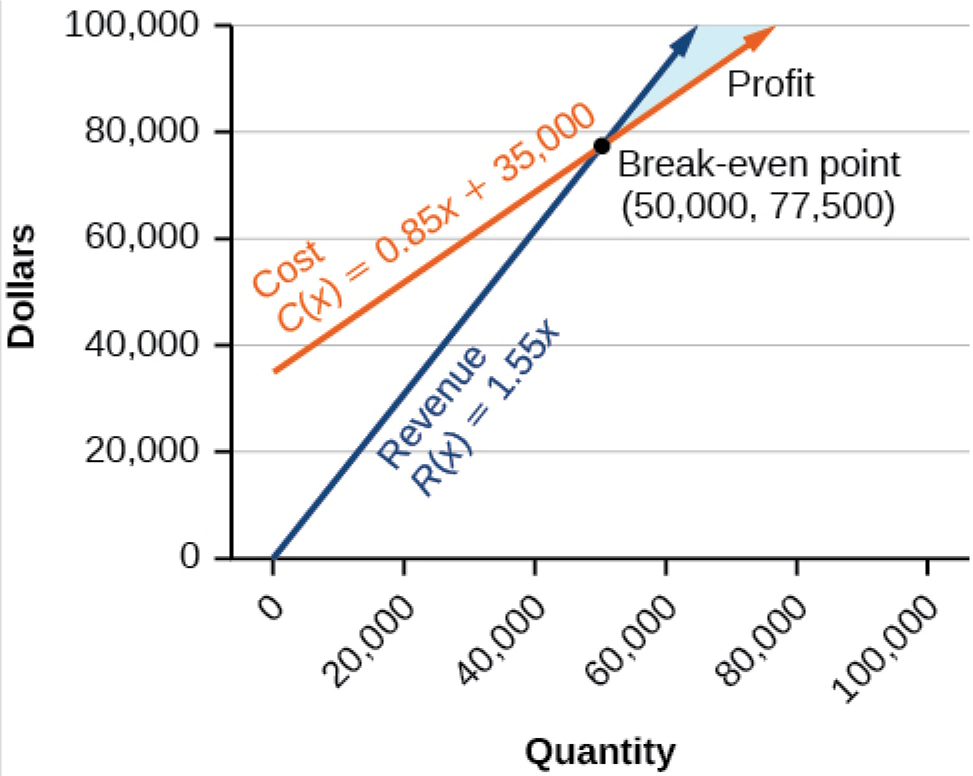
Equation:

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

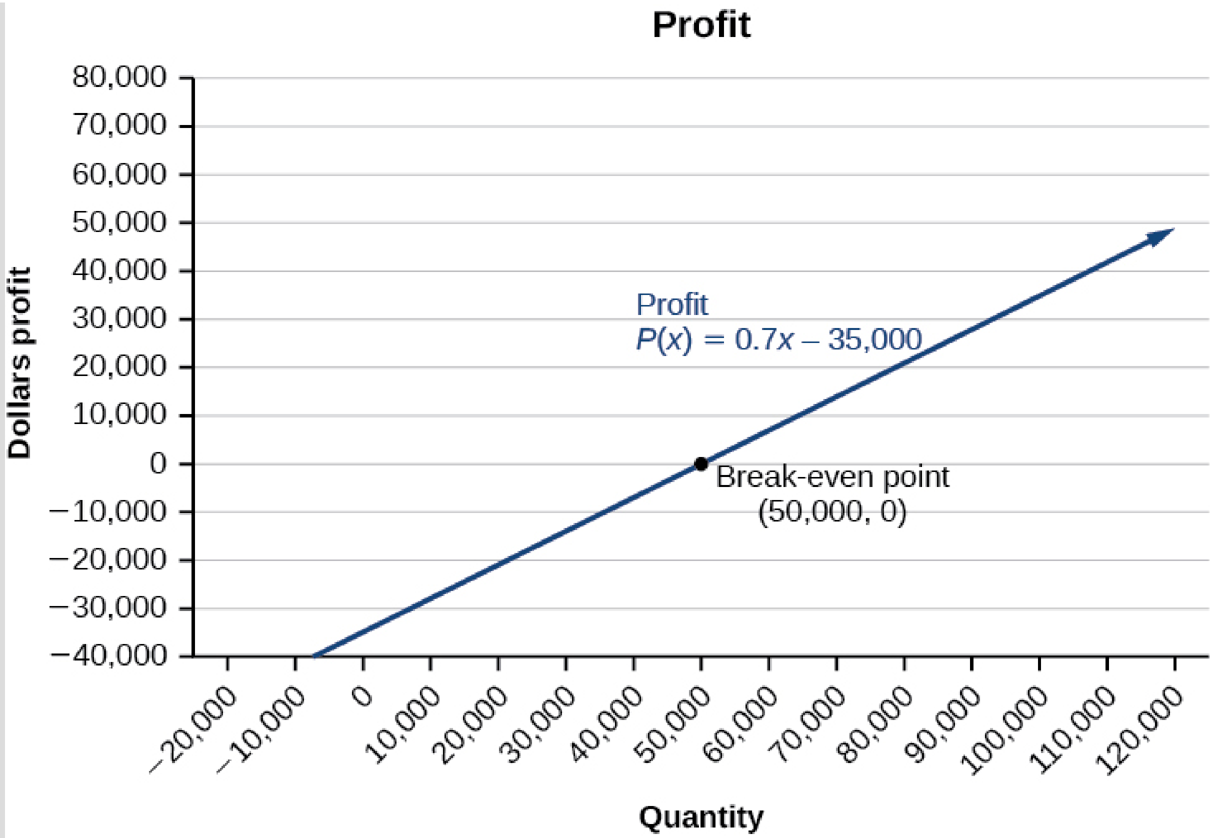
The profit function is $P(x) = 0.7x{-}35{,}000$.

Analysis

The cost to produce 50,000 units is $77,500, and the revenue from the sales of 50,000 units is also $77,500. To make a profit, the business must produce and sell more than 50,000 units. See [link].



We see from the graph in [link] that the profit function has a negative value until $x = 50{,}000$, when the graph crosses the *x*-axis. Then, the graph emerges into positive *y*-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.



Example:
Exercise:

Problem:
Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000$. How many children and how many adults bought tickets?

Solution:

Let *c* = the number of children and *a* = the number of adults in attendance.

The total number of people is $2{,}000$. We can use this to write an equation for the number of people at the circus that day.
Equation:

$$c + a = 2{,}000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is$\$70{,}000$. We can use this to write an equation for the revenue.
Equation:

$$25c + 50a = 70{,}000$$

We now have a system of linear equations in two variables.
Equation:

$$\begin{aligned} c + a &= 2{,}000 \\ 25c + 50a &= 70{,}000 \end{aligned}$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.
Equation:

$$\begin{aligned} c + a &= 2{,}000 \\ a &= 2{,}000 - c \end{aligned}$$

Substitute the expression $2{,}000 - c$ in the second equation for a and solve for c.
Equation:

$$\begin{aligned} 25c + 50(2{,}000 - c) &= 70{,}000 \\ 25c + 100{,}000 - 50c &= 70{,}000 \\ -25c &= -30{,}000 \\ c &= 1{,}200 \end{aligned}$$

Substitute $c = 1{,}200$ into the first equation to solve for a.

Equation:

$$\begin{aligned} 1{,}200 + a &= 2{,}000 \\ a &= 800 \end{aligned}$$

We find that $1{,}200$ children and 800 adults bought tickets to the circus that day.

Note:

Exercise:

Problem:

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1{,}650$ meal tickets were bought for a total of $\$14{,}200$, how many children and how many adults bought meal tickets?

Solution:

700 children, 950 adults

Note:

Access these online resources for additional instruction and practice with systems of linear equations.

Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See [link].
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See [link].
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See [link].
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See [link].
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See [link], [link], and [link].
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See [link].
- The solution to a system of dependent equations will always be true because both equations describe the same line. See [link].
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Can a system of linear equations have exactly two solutions? Explain why or why not.

Solution:

No, you can either have zero, one, or infinitely many. Examine graphs.

Exercise:

Problem:

If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company’s profit margins.

Exercise:

Problem:

If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

Solution:

This means there is no realistic break-even point. By the time the company produces one unit they are already making profit.

Exercise:

Problem:

If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

Exercise:

Problem:

Given a system of equations, explain at least two different methods of solving that system.

Solution:

You can solve by substitution (isolating x or y), graphically, or by addition.

Algebraic

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

Exercise:

Problem: $\begin{array}{c} 5x - y = 4 \\ x + 6y = 2 \end{array}$ and $(4, 0)$

Exercise:

Problem: $\begin{array}{c} -3x - 5y = 13 \\ -x + 4y = 10 \end{array}$ and $(-6, 1)$

Solution:

Yes

Exercise:

Problem: $\begin{array}{c} 3x + 7y = 1 \\ 2x + 4y = 0 \end{array}$ and $(2, 3)$

Exercise:

Problem: $\begin{array}{c} -2x + 5y = 7 \\ 2x + 9y = 7 \end{array}$ and$(-1, 1)$

Solution:

Yes

Exercise:

Problem: $\begin{array}{c} x + 8y = 43 \\ 3x - 2y = -1 \end{array}$ and $(3, 5)$

For the following exercises, solve each system by substitution.

Exercise:

Problem: $\begin{array}{c} x + 3y = 5 \\ 2x + 3y = 4 \end{array}$

Solution:

$(-1, 2)$

Exercise:

Problem: $\begin{array}{c} 3x - 2y = 18 \\ 5x + 10y = -10 \end{array}$

Exercise:

Problem: $\begin{aligned} 4x + 2y &= -10 \\ 3x + 9y &= 0 \end{aligned}$

Solution:

$(-3, 1)$

Exercise:

Problem: $\begin{aligned} 2x + 4y &= -3.8 \\ 9x - 5y &= 1.3 \end{aligned}$

Exercise:

Problem:
$\begin{aligned} -2x + 3y &= 1.2 \\ -3x - 6y &= 1.8 \end{aligned}$

Solution:

$\left(-\frac{3}{5}, 0\right)$

Exercise:

Problem: $\begin{aligned} x - 0.2y &= 1 \\ -10x + 2y &= 5 \end{aligned}$

Exercise:

Problem: $\begin{aligned} 3x + 5y &= 9 \\ 30x + 50y &= -90 \end{aligned}$

Solution:

No solutions exist.

Exercise:

Problem: $-3x + y = 2$

$12x - 4y = -8$

Exercise:

Problem: $\frac{1}{2}x + \frac{1}{3}y = 16$

$\frac{1}{6}x + \frac{1}{4}y = 9$

Solution:

$\left(\frac{72}{5}, \frac{132}{5}\right)$

Exercise:

Problem: $-\frac{1}{4}x + \frac{3}{2}y = 11$

$-\frac{1}{8}x + \frac{1}{3}y = 3$

For the following exercises, solve each system by addition.

Exercise:

Problem: $-2x + 5y = -42$

$7x + 2y = 30$

Solution:

$(6, -6)$

Exercise:

Problem: $\begin{array}{l} 6x - 5y = -34 \\ 2x + 6y = 4 \end{array}$

Exercise:

Problem: $\begin{array}{r} 5x - y = -2.6 \\ -4x - 6y = 1.4 \end{array}$

Solution:

$\left(-\frac{1}{2}, \frac{1}{10}\right)$

Exercise:

Problem: $\begin{array}{l} 7x - 2y = 3 \\ 4x + 5y = 3.25 \end{array}$

Exercise:

Problem: $\begin{array}{r} -\text{x} + 2y = -1 \\ 5x - 10y = 6 \end{array}$

Solution:

No solutions exist.

Exercise:

Problem: $\begin{array}{r} 7x + 6y = 2 \\ -28x - 24y = -8 \end{array}$

Exercise:

Problem: $$\begin{aligned}\frac{5}{6}x + \frac{1}{4}y &= 0\\ \frac{1}{8}x - \frac{1}{2}y &= -\frac{43}{120}\end{aligned}$$

Solution:

$\left(-\frac{1}{5}, \frac{2}{3}\right)$

Exercise:

Problem: $$\begin{aligned}\frac{1}{3}x + \frac{1}{9}y &= \frac{2}{9}\\ -\frac{1}{2}x + \frac{4}{5}y &= -\frac{1}{3}\end{aligned}$$

Exercise:

Problem: $$\begin{aligned}-0.2x + 0.4y &= 0.6\\ x - 2y &= -3\end{aligned}$$

Solution:

$\left(x, \frac{x+3}{2}\right)$

Exercise:

Problem: $$\begin{aligned}-0.1x + 0.2y &= 0.6\\ 5x - 10y &= 1\end{aligned}$$

For the following exercises, solve each system by any method.

Exercise:

Problem: $$\begin{aligned}5x + 9y &= 16\\ x + 2y &= 4\end{aligned}$$

Solution:

$(-4,4)$

Exercise:

Problem: $\begin{array}{l} 6x-8y=-0.6 \\ 3x+2y=0.9 \end{array}$

Exercise:

Problem: $\begin{array}{l} 5x-2y=2.25 \\ 7x-4y=3 \end{array}$

Solution:

$\left(\frac{1}{2},\frac{1}{8}\right)$

Exercise:

Problem: $\begin{array}{l} x-\frac{5}{12}y=-\frac{55}{12} \\ -6x+\frac{5}{2}y=\frac{55}{2} \end{array}$

Exercise:

Problem: $\begin{array}{l} 7x-4y=\frac{7}{6} \\ 2x+4y=\frac{1}{3} \end{array}$

Solution:

$\left(\frac{1}{6},0\right)$

Exercise:

Problem: $$\begin{array}{c} 3x + 6y = 11 \\ 2x + 4y = 9 \end{array}$$

Exercise:

Problem: $$\begin{array}{c} \frac{7}{3}x - \frac{1}{6}y = 2 \\ -\frac{21}{6}x + \frac{3}{12}y = -3 \end{array}$$

Solution:

$(x, 2(7x - 6))$

Exercise:

Problem: $$\begin{array}{c} \frac{1}{2}x + \frac{1}{3}y = \frac{1}{3} \\ \frac{3}{2}x + \frac{1}{4}y = -\frac{1}{8} \end{array}$$

Exercise:

Problem: $$\begin{array}{c} 2.2x + 1.3y = -0.1 \\ 4.2x + 4.2y = 2.1 \end{array}$$

Solution:

$\left(-\frac{5}{6}, \frac{4}{3}\right)$

Exercise:

Problem: $$\begin{array}{c} 0.1x + 0.2y = 2 \\ 0.35x - 0.3y = 0 \end{array}$$

Graphical

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

Exercise:

Problem:
$$\begin{aligned} 3x - y &= 0.6 \\ x - 2y &= 1.3 \end{aligned}$$

Solution:

Consistent with one solution

Exercise:

Problem:
$$\begin{aligned} -x + 2y &= 4 \\ 2x - 4y &= 1 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} x + 2y &= 7 \\ 2x + 6y &= 12 \end{aligned}$$

Solution:

Consistent with one solution

Exercise:

Problem:
$$\begin{aligned} 3x - 5y &= 7 \\ x - 2y &= 3 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 3x - 2y &= 5 \\ -9x + 6y &= -15 \end{aligned}$$

Solution:

Dependent with infinitely many solutions

Technology

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

Exercise:

Problem:
$$\begin{aligned} 0.1x + 0.2y &= 0.3 \\ -0.3x + 0.5y &= 1 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} -0.01x + 0.12y &= 0.62 \\ 0.15x + 0.20y &= 0.52 \end{aligned}$$

Solution:

$(-3.08, 4.91)$

Exercise:

Problem:
$$\begin{aligned} 0.5x + 0.3y &= 4 \\ 0.25x - 0.9y &= 0.46 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 0.15x + 0.27y &= 0.39 \\ -0.34x + 0.56y &= 1.8 \end{aligned}$$

Solution:

$(-1.52, 2.29)$

Exercise:

Problem: $$\begin{aligned} -0.71x + 0.92y &= 0.13 \\ 0.83x + 0.05y &= 2.1 \end{aligned}$$

Extensions

For the following exercises, solve each system in terms of $A, B, C, D, E,$ and F where $A–F$ are nonzero numbers. Note that $A \neq B$ and $AE \neq BD$.

Exercise:

Problem: $$\begin{aligned} x + y &= A \\ x - y &= B \end{aligned}$$

Solution:

$\left(\frac{A+B}{2}, \frac{A-B}{2}\right)$

Exercise:

Problem: $$\begin{aligned} x + Ay &= 1 \\ x + By &= 1 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} Ax + y &= 0 \\ Bx + y &= 1 \end{aligned}$$

Solution:

$\left(\frac{-1}{A-B}, \frac{A}{A-B}\right)$

Exercise:

Problem: $$\begin{aligned} Ax + By &= C \\ x + y &= 1 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} Ax + By &= C \\ Dx + Ey &= F \end{aligned}$$

Solution:

$\left(\frac{CE-BF}{BD-AE}, \frac{AF-CD}{BD-AE}\right)$

Real-World Applications

For the following exercises, solve for the desired quantity.

Exercise:

Problem:

A stuffed animal business has a total cost of production $C = 12x + 30$ and a revenue function $R = 20x.$ Find the break-even point.

Exercise:

Problem:

A fast-food restaurant has a cost of production $C(x) = 11x + 120$ and a revenue function $R(x) = 5x.$ When does the company start to turn a profit?

Solution:

They never turn a profit.

Exercise:

Problem:

A cell phone factory has a cost of production $C(x) = 150x + 10,000$ and a revenue function $R(x) = 200x.$ What is the break-even point?

Exercise:

Problem:

A musician charges $C(x) = 64x + 20{,}000,$ where x is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

Solution:

$(1,250,100,000)$

Exercise:

Problem:

A guitar factory has a cost of production $C(x) = 75x + 50{,}000.$ If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

Exercise:

Problem: Find two numbers whose sum is 28 and difference is 13.

Solution:

The numbers are 7.5 and 20.5.

Exercise:

Problem:

A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.

Exercise:

Problem:

The startup cost for a restaurant is $120,000, and each meal costs $10 for the restaurant to make. If each meal is then sold for $15, after how many meals does the restaurant break even?

Solution:

24,000

Exercise:

Problem:

A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

Exercise:

Problem:

A total of 1,595 first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?

Solution:

790 sophomores, 805 freshman

Exercise:

Problem:

276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

Exercise:

Problem:

There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?

Solution:

56 men, 74 women

Exercise:

Problem:

A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.

Exercise:

Problem:

If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?

Solution:

10 gallons of 10% solution, 15 gallons of 60% solution

Exercise:

Problem:

An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?

Exercise:

Problem:

An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

Solution:

Swan Peak: $750,000, Riverside: $350,000

Exercise:

Problem:

If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays $2\frac{7}{8}\%$ interest, and the investor earns $737.50 annual interest, how much was invested in each account?

Exercise:

Problem:

If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?

Solution:

$12,500 in the first account, $10,500 in the second account.

Exercise:

Problem:

CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?

Exercise:

Problem:

A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?

Solution:

High-tops: 45, Low-tops: 15

Exercise:

Problem:

A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?

Exercise:

Problem:

Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child’s ticket and the price of the adult ticket?

Solution:

Infinitely many solutions. We need more information.

Glossary

addition method
: an algebraic technique used to solve systems of linear equations in which the equations are added in a way that eliminates one variable, allowing the resulting equation to be solved for the remaining variable; substitution is then used to solve for the first variable

break-even point
: the point at which a cost function intersects a revenue function; where profit is zero

consistent system
: a system for which there is a single solution to all equations in the system and it is an independent system, or if there are an infinite number of solutions and it is a dependent system

cost function
: the function used to calculate the costs of doing business; it usually has two parts, fixed costs and variable costs

dependent system
: a system of linear equations in which the two equations represent the same line; there are an infinite number of solutions to a dependent system

inconsistent system
: a system of linear equations with no common solution because they represent parallel lines, which have no point or line in common

independent system
: a system of linear equations with exactly one solution pair (x, y)

profit function
: the profit function is written as $P(x) = R(x) - C(x)$,revenue minus cost

revenue function
: the function that is used to calculate revenue, simply written as $R = xp$,where $x =$ quantity and $p =$ price

substitution method
: an algebraic technique used to solve systems of linear equations in which one of the two equations is solved for one variable and then substituted into the second equation to solve for the second variable

system of linear equations
: a set of two or more equations in two or more variables that must be considered simultaneously.